EC821: Time Series Econometrics, Spring 2003

Notes Section 4

Nonstationary univariate time series

Let us now consider the definition of an **integrated** series. A series which is stationary after being differenced once is said to be integrated of order one, and is denoted $I(1)$. In general a series which is stationary after being differenced d times is said to be integrated of order d, denoted $I(d)$. A series which is stationary without differencing is said to be $I(0)$. This definition assumes that d is an integer; it is possible to extended the definition to the case of fractional values of d. Note that this does not imply that all nonstationary series are $I(1)$, or more generally $I(d)$ with $d \geq 1$. It may not be possible to difference a nonstationary series to stationarity (see Leybourne et al., JBES 1996). Also, we will find that a series with fractional d, $0.5 \leq d < 1$ is nonstationary, even though the value of d falls short of unity.

A series which is $I(1)$ is said to have a unit root, and a series which is $I(d)$, d a positive integer, is said to have d unit roots. This may be seen for $I(1)$ by writing the $AR(1)$ model in lag operator notation:

$$(1 - \phi_1 L)\, Y_t = \mu + \epsilon_t$$

when $\phi_1 = 1$, the root of the polynomial is unity, and the notion that Y is $I(1)$ corresponds to the presence of the unit root in its AR representation. In a higher–order model, there may be multiple unit roots; for instance, in the $AR(2)$ model

$$\left(1 - \phi_1 L - \phi_2 L^2\right) Y_t = \mu + \epsilon_t$$

the lag polynomial may be factored into $(1 - a_1L)(1 - a_2L) = 0$, where $a_1 + a_2 = \phi_1$ and $a_1a_2 = -\phi_2$, .and the roots of the quadratic polynomial will be the inverses of the a coefficients. If $\phi_1 = 2$ and $\phi_2 = -1$, there will be two unit roots in the AR representation, and the $I(2)$ series must be differenced twice to yield a stationary process. An $AR(2)$ model could also generate an $I(1)$ series; e.g. if $\phi_1 = 1.25$ and $\phi_2 = -0.25$, the polynomial factors into $(1 - L)(1 - 0.25L)$. The once–differenced process will be stationary. For the general $AR(p)$ representation, if one unit root is present, the AR polynomial can be factored into $(1 - L)$ and a $(p - 1)^{th} -$ order polynomial with stable roots. It could be the case that there are multiple unit roots in the $AR(p)$. This gives rise to the $ARIMA(p, d, q)$ model: a process which has a standard $ARMA(p, q)$ after differencing d times to achieve stationarity.

Difference stationary vs trend stationary series

A trend stationary series is one in which shocks have transitory effects. The series will not be covariance stationary, but its second moments may satisfy the conditions for CS; it only fails to be CS due to its time–varying mean. If the variation in the mean can be adequately explained by a linear, polynomial or logarithmic trend, then the detrended series is CS, and shocks to the series will be of a transitory nature. In contrast, a series possessing one or more unit roots will only be stationary after differencing, and shocks to its level will have a permanent effect on the series. It may also possess a trend—that is, it may be a random walk with drift. The simplest trend stationary model is

$$y_t = \mu + \beta t + \epsilon_t$$

where ϵ_t is assumed to be white noise (i.e. $I(0)$). y is clearly $I(0)$, without differencing, since its stochastic properties are entirely determined by those of ϵ_t. Adding an $AR(1)$ component, the model becomes

$$y_t = \mu + \phi_1 y_{t-1} + \beta t + \epsilon_t$$

This model includes several interesting special cases: for instance, when $\beta = 0$, $\mu \neq 0$ and $\phi_1 = 1$, the model is that of a random walk with drift, and is $I(1)$. If on the other hand $\mu = 0$, then a pure random walk model results. Alternatively, if $\beta = 0$, $\mu \neq 0$ and $|\phi_1| < 1$, the model is of a deterministic trend with a stationary $AR(1)$ component: that is, a trend–stationary series. In the latter case, how may we distinguish this model from that of the random walk with drift? The only difference between them lies in the parameter ϕ_1, and our ability to distinguish that estimated parameter from unity. The two series may mimic each other: that is, for a finite sample of data, the characterization of the timeseries as trend stationary may be equally plausible to its identification as a unit root process. But the distinction is crucial, in that applying the appropriate transformation to the series will depend upon our ability to distinguish the two models.

If a series is DS, for instance, then detrending the series—which in reality contains a unit root—will not remove its random walk properties; at best, it will merely change a random walk with drift to a pure random walk. Treating the resulting series as if it is now covariance stationary will be misleading. On the other hand, if a series is TS, then the appropriate transformation to CS is detrending, and differencing the series is not warranted—and since differencing may be considered as an approximation to applying the filter $(1 - \phi_1 L)$, that approximation will be worse the farther is ϕ_1 from unity. Therefore, it is essential that we have a methodology to distinguish the TS from the DS series on empirical grounds.

We should also note that considering linear models of timeseries with either trend or unit root is not restrictive. Many economic and financial timeseries—for instance, that for gross domestic product (GDP)—appear to be better characterised by an exponential trend (constant percentage growth) than by a linear trend (constant growth in its level).For this reason, many models of economic and financial timeseries are applied to the logarithms of the original series. If we consider $y = \log(x)$, then a trend-stationary model of y is actually a constant percentage growth model of the underlying x. Likewise, making use of the approximation that for small changes, the first difference of the logarithm of a variable is approximately the same as its percentage change, we often use $\Delta \log(x)$ to construct a percentage change series. If $y = \log(x)$ possesses a unit root, so that we should be taking its first difference Δy, we are then arguing that the percentage changes of the underlying series x are stationary.

How do trend-stationary and unit root representations of a timeseries differ in practice? For a trend-stationary series, an optimal τ-period-ahead forecast may be constructed by merely adding $\beta\tau$ to the series' current value. For the unit root process $\Delta y_t = \delta + \psi(L)\,\epsilon_t$, the series will be expected to change by δ units per period, so that the accumulated drift after τ periods will just be $\delta\tau$. The difference in these forecasts, if repeated period by period, are in the intercept terms. For the trend-stationary process, the intercept α and slope β define a forecast trajectory to which forecasts will revert. For the unit root process, the intercept will change with each period's shock to the series; the forecast adds the drift δ to whatever value the y process has attained. In terms of interval estimates, the forecast errors for a trend-stationary process will converge to a fixed value as τ increases, while the forecast errors from a unit root process increase without bound, given that the variance of the unit root process increases linearly with the forecast horizon.

To develop this methodology, we now consider testing for a unit root.