EC821: Time Series Econometrics
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Notes Section 10 Part 1

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1. Spectral analysis[1]

Our focus thus far has been on considering how the timeseries Y reacts to a sequence of innovations ϵ_t, and the implications of the relationship between Y and ϵ for the covariance (or correlation) between Y_t and Y_τ at distinct dates t and τ. We speak of this as analysing the properties of Y in the time domain. We might also consider the timeseries behavior in the frequency domain, where Y_t is described as a weighted sum of periodic (sine and cosine) functions:

$$Y_t = \mu + \int_0^\pi \alpha(\omega) \cos(\omega t)\, d\omega + \int_0^\pi \delta(\omega) \sin(\omega t)\, d\omega \tag{1.1}$$

This is known as the spectral representation of the timeseries, and its study is known as spectral analysis. The goal will be to determine how important cycles of different frequencies are in accounting for the behavior of Y. The two types of analysis (time-domain and frequency-domain modelling) are not mutually exclusive. Any CS process has both a time-domain representation and a frequency-domain representation, and any feature of the data that can be described by one representation can be equally well described by the other representation. For some features, the time-domain description may be simpler or more convenient to work with, while for other features the frequency-domain representation may be simpler.

[1] This presentation draws heavily from Ch. 6 of Hamilton (1994).

1.1. The population spectrum

Let Y be a CS process with mean $E(Y_t) = \mu$ and j^{th} autocovariance

$$E\left(Y_t - \mu\right)\left(Y_{t-j} - \mu\right) = \gamma_j.$$

If the autocovariances are absolutely summable, as previously defined, then we may define the autocovariance generating function as

$$g_Y\left(z\right) = \sum_{j=-\infty}^{\infty} \gamma_j z^j \tag{1.2}$$

where j denotes a complex scalar. If (1.2) is divided by 2π, z is represented by $z = e^{-i\omega}$ for $i = \sqrt{-1}$ and ω a real scalar, then we define the population spectrum of Y :

$$s_Y = \frac{1}{2\pi} g_Y\left(e^{-i\omega}\right) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \gamma_j e^{-i\omega j} \tag{1.3}$$

Note that the spectrum is a function of the frequency ω : given any particular value of ω and a sequence of autocovariances $\left\{\gamma_j\right\}$ we could in principle calculate the value of $s_Y\left(\omega\right)$. By De Moivre's theorem,

$$e^{-i\omega j} = \cos\left(\omega j\right) - i \sin\left(\omega j\right)$$

so

$$s_Y = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} \left[\cos\left(\omega j\right) - i \sin\left(\omega j\right)\right]$$

For a CS process, $\gamma_j = \gamma_{-j}$, so that we can consider only the positive half line. Using some elementary results from trigonometry, we may simplify the expression to

$$s_Y\left(\omega\right) = \frac{1}{2\pi}\left\{\gamma_0 + 2\sum_{j=1}^{\infty} \gamma_j \cos\left(\omega j\right)\right\}.$$

This implies that the population spectrum $s_Y\left(\omega\right)$ exists, and is a continuous, real-valued function of ω. If the γ_j are the autocovariances of a CS process, then $s_Y\left(\omega\right)$ will be nonnegative for all ω, and given symmetry of the cosine function, the spectrum is symmetric around $\omega = 0$. Since $\cos[(\omega + 2\pi k)j] = \cos\left[\omega j\right]$ for any integers k and j, it follows that $s_Y\left(\omega + 2\pi k\right) = s_Y\left(\omega\right)$ for any integer k. Thus,. the spectrum is a periodic function of ω. If we know the value of $s_Y\left(\omega\right)$ for all ω

between 0 and π, we can infer the value of $s_Y(\omega)$ for any ω. We routinely express the empirical spectrum as a function defined over $(0,\pi)$, with the understanding that it is symmetric around 0.

We may define the spectrum, or spectral density function, for a number of $ARMA$ processes analytically. For instance, a white noise process has a flat spectrum, with $s_Y(\omega) = \sigma^2/2\pi$, and the meaning of white noise reflects the equiprobability of all frequencies in the representation of the process. A $MA(1)$ process $Y_t = \epsilon_t + \theta\epsilon_{t-1}$ will have a monotonic spectrum, decreasing over $(0,\pi)$ for $\theta > 0$, increasing for $\theta < 0$. An $AR(1)$ process $Y_t = c + \phi Y_{t-1} + \epsilon_t$ with $\phi > 0$ will have a monotonic spectrum, decreasing over $(0,\pi)$, while for $\phi < 0$, $s_Y(\omega)$ will be a monotone increasing function of ω.

How may we interpret the spectrum? We may demonstrate that

$$\begin{aligned}\int_{-\pi}^{\pi} s_Y(\omega)\, e^{iwk} d\omega &= \gamma_k, or \\ \int_{-\pi}^{\pi} s_Y(\omega)\cos(wk)\, d\omega &= \gamma_k.\end{aligned}$$

This implies, for $k = 0$, that the integral of the spectrum over its range equals γ_0 : the variance of the series. Thus the spectrum is an alternative representation of the variance, or variability, of a series, with the interpretation that the total variance arises from those components defined over the various frequencies contained in the range $(0,\pi)$. Since $s_Y(\omega)$ is nonnegative, if we calculate

$$\int_{-\omega_1}^{\omega_1} s_Y(\omega)\, d\omega$$

for any $\omega_1 \in (0,\pi)$, the result would be a positive number that we could interpret as the portion of the variance of Y associated with frequencies ω that are less than ω_1 in absolute value. Since $s_Y(\omega)$ is symmetric, we may express this quantity as

$$2\int_{0}^{\omega_1} s_Y(\omega)\, d\omega,$$

the portion of the variance of Y attributed to periodic random components with frequencies less than or equal to ω_1. Thus, the variance of a timeseries may be decomposed into those portions due to low-frequency variation (or long waves); medium-frequency variations (e.g. business cycle frequencies); and high-frequency variations (such as seasonal variations in quarterly or monthly data).

1.2. The sample periodogram

Since the population spectrum is defined in terms of the sequence $\left\{\gamma_j\right\}$, the population second moments of the timeseries, we may compute a sample counterpart from the empirical autocorrelations of the series measured over T observations,

$$\hat{\gamma}_j = T^{-1} \sum_{t=j+1}^{T} (y_t - \bar{y})(y_{t-j} - \bar{y}), j = 0, 1, 2, \ldots, T-1 \tag{1.4}$$

where $\bar{y}$ is the sample mean of the timeseries. For any given ω we may then construct the sample periodogram,

$$\begin{aligned} \hat{s}_Y(\omega) &= \frac{1}{2\pi} \sum_{j=-T+1}^{T-1} \hat{\gamma}_j e^{-i\omega j}, \mathit{or} \\ &= \frac{1}{2\pi} \left\{ \hat{\gamma}_0 + 2 \sum_{j=1}^{T-1} \hat{\gamma}_j \cos(\omega j) \right\} \end{aligned} \tag{1.5}$$

which may constructed from any timeseries of sufficient length, using the estimates of the sample autocovariance function. As we discussed above, the sample periodogram allows the timeseries to be represented by components of different frequencies, and the empirical variance of the timeseries to be decomposed into that emanating from certain frequency bands. The highest frequency that may be considered in this representation is $\omega = \pi$, known as the Nyquist frequency. Since a frequency ω is associated with a period of $2\pi/\omega$ periods, the Nyquist frequency is associated with a cycle of 2 periods' length (two months for monthly data, two days for daily data...) and our analysis cannot detect a cycle of higher frequency (for instance, a set of daily stock prices cannot be used to analyse intra-daily movements in stock prices). The lowest frequency that can be represented in a finite sample of T observations is $\omega_1 = 2\pi/T$, corresponding to a period of T. We cannot infer anything about cycles that last longer than that (e.g. 50-year swings in economic activity cannot be analysed using 40 years of postwar data).

1.3. Estimating the population spectrum

Although it might seem natural to estimate the population spectrum $s_Y(\omega)$ with the sample periodogram $\hat{s}_Y(\omega)$, this approach has its limitations. Unless the sample is very large, estimates from the periodogram may be very imprecise.

Just as the sample autocorrelation function may be not very smooth, the sample periodogram is likely to be something other than the smooth function that theory would suggest for the population spectrum. Most approaches to estimation of the spectrum involve the use of a window, or kernel estimator: a technique that averages adjacent estimates of the periodogram to create a smooth function. Such an estimator reduces the variance of the resulting spectrum at the cost of introducing some bias. A certain amount of subjective judgment is often employed to consider the appropriate bandwidth for a spectral window, or kernel estimator.

References

[1] Hamilton, James D., 1994. Time series analysis. Princeton: Princeton University Press.