BOSTON COLLEGE
Department of Economics
EC771: Econometrics
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SOLUTION KEY FOR PROBLEM SET 4

1. The expression for the restricted coefficient vector in (6-14) may be written in the form $\mathbf{b}_* = [\mathbf{I} - \mathbf{CR}]\mathbf{b} + \mathbf{w}$, where $\mathbf{w}$ does not involve $\mathbf{b}$. What is $\mathbf{C}$? Show that the covariance matrix of the restricted least squares estimator is $\sigma^2(\mathbf{X'X})^{-1} - \sigma^2(\mathbf{X'X})^{-1}\mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}$ and that this matrix may be written as,

$$var[\mathbf{b}|\mathbf{X}]\{[var(\mathbf{b}|\mathbf{X})]^{-1} - \mathbf{R'}[var(\mathbf{Rb}|\mathbf{X})]^{-1}R\}var[\mathbf{b}|\mathbf{X}].$$

By factoring the result in (6-14), we obtain $\mathbf{b}_* = [\mathbf{I} - \mathbf{CR}]\mathbf{b} + \mathbf{w}$, where $\mathbf{C} = (\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}$ and $\mathbf{w} = \mathbf{Cq}$. The covariance matrix of the least squares estimator is,

$$\begin{aligned} var[\mathbf{b}_*] &= [\mathbf{I} - \mathbf{CR}]\sigma^2(\mathbf{X'X})^{\mathbf{-1}}[\mathbf{I} - \mathbf{CR}]' \\ &= \sigma^2(\mathbf{X'X})^{-1} + \sigma^2\mathbf{CR}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'C'} - \sigma^2\mathbf{CR}(\mathbf{X'X})^{\mathbf{-1}} - \sigma^2(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'C'}. \end{aligned}$$

By multiplying it out, we find $\mathbf{CR}(\mathbf{X'X})^{\mathbf{-1}} = (\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}} = \mathbf{CR}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'C'}$ so $var[\mathbf{b}_*] = \sigma^2(\mathbf{X'X})^{-1} - \sigma^2\mathbf{CR}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'C'} = \sigma^2(\mathbf{X'X})^{-1} - \sigma^2(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}$. This may also be written as

$$\begin{aligned} var[\mathbf{b}_*] &= \sigma^2(\mathbf{X'X})^{-1}\{\mathbf{I} - \mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\} \\ &= \sigma^2(\mathbf{X'X})^{-1}\{[\sigma^2(\mathbf{X'X})^{-1}]^{-1} - \mathbf{R'}[\mathbf{R}\sigma^{\mathbf{2}}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R}\}\sigma^2(\mathbf{X'X})^{-1} \end{aligned}$$

Since $var[\mathbf{Rb}] = \mathbf{R}\sigma^{\mathbf{2}}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}$ this is the answer we seek.

2. Prove the result that the restricted least squares estimator never has a larger variance matrix than the unrestricted least squares estimator.

The variance of the restricted least squares estimator is given in the previous exercise by,

$$var[\mathbf{b}_*] = \sigma^2(\mathbf{X'X})^{-1} - \sigma^2(\mathbf{X'X})^{-1}\mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}.$$

We know that this matrix is positive definite, since it is derived in the form $var[\mathbf{b}_*] = \mathbf{B}\sigma^{\mathbf{2}}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{B'}$, and $\sigma^2(\mathbf{X'X})^{\mathbf{-1}}$ is positive definite. Therefore, it remains to show only that the matrix subtracted from $var[\mathbf{b}]$ to obtain $var[\mathbf{b}_*]$ is positive definite. Consider, then, a quadratic form in $var[\mathbf{b}_*]$,

$$\begin{aligned} \mathbf{z'}var[\mathbf{b}_*]\mathbf{z} &= \mathbf{z'}var[\mathbf{b}]\mathbf{z} - \sigma^2\mathbf{z'}(\mathbf{X'X})^{\mathbf{-1}}(\mathbf{R'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]^{\mathbf{-1}}\mathbf{R})(\mathbf{X'X})^{\mathbf{-1}}\mathbf{z} \\ &= \mathbf{z'}var[\mathbf{b}]\mathbf{z} - \mathbf{w'}[\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{R'}]\mathbf{w} \text{ where } \mathbf{w} = \sigma\mathbf{R}(\mathbf{X'X})^{\mathbf{-1}}\mathbf{z}. \end{aligned}$$

It remains only to show, therefore, that the inverse matrix in brackets is positive definite. This is obvious since its inverse is positive definite. This shows that every quadratic form in $var[\mathbf{b}_*]$ is less than a quadratic form in $var[\mathbf{b}]$ in the same vector.

3. Prove that under the hypothesis that $\mathbf{R}\beta = q$, the estimator $s = (\mathbf{y} - \mathbf{X}\mathbf{b}_*)'(\mathbf{y} - \mathbf{X}\mathbf{b}_*)/(n - K + J)$, where J is the number of restrictions, is unbiased for σ^2.

First, use (6-19) to write $\mathbf{e}_*'\mathbf{e}_* = \mathbf{e}'\mathbf{e} + (\mathbf{Rb} - \mathbf{q})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']^{-\mathbf{1}}(\mathbf{Rb} - \mathbf{q})$. Now, the result that $E[\mathbf{e}'\mathbf{e}] = (n - K)\sigma^2$ obtained in Chapter 6 must hold here, so $E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + E[(\mathbf{Rb} - \mathbf{q})'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']^{-\mathbf{1}}(\mathbf{Rb} - \mathbf{q})]$. Now, $\mathbf{b} = \beta + (\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\epsilon$, so $\mathbf{Rb} - \mathbf{q} = \mathbf{R}\beta - \mathbf{q} + \mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\epsilon$. But $\mathbf{R}\beta - \mathbf{q} = 0$, so under the hypothesis $\mathbf{Rb} - \mathbf{q} = \mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\epsilon$. Insert this in the result above to obtain

$$E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + E[\epsilon'\mathbf{X}'(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}'[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']^{-\mathbf{1}}\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\epsilon]$$

The quantity in square brackets is a scalar, so it is equal to its trace. Permute $\epsilon'\mathbf{X}'(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}'$ to obtain

$$E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + E[\mathrm{tr}\{[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']^{-\mathbf{1}}\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\epsilon\epsilon'\mathbf{X}'(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']\}$$

We may carry out the expectation inside the trace and use $E[\epsilon'\epsilon] = \sigma^{\mathbf{2}}\mathbf{I}$ to obtain

$$E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + \mathrm{tr}\{[\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']^{-\mathbf{1}}\mathbf{R}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{X}'\sigma^{\mathbf{2}}\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-\mathbf{1}}\mathbf{R}']\}$$

Carry out the σ^2 outside the trace operator, and after cancellation of the products of matrices times their inverses, we obtain

$$E[\mathbf{e}_*'\mathbf{e}_*] = (n - K)\sigma^2 + \sigma^2\mathrm{tr}[\mathbf{I}_J] = (n - K + J)\sigma^2.$$

4. Show that in the multiple regression of $\mathbf{y}$ on a constant and $\mathbf{x_1}$, and $\mathbf{x_2}$, while imposing the restriction $\beta_1 + \beta_2 = 1$ leads to the regression of $\mathbf{y} - \mathbf{x_1}$ on a constant and $\mathbf{x_2} - \mathbf{x_1}$.
For convenience, we put the constant term last instead of first in the parameter vector. The constraint is $\mathbf{Rb} - \mathbf{q} = 0$ where $\mathbf{R} = [\ 1 \quad 1 \quad 0\ ]$ so $\mathbf{R_1} = [1]$ and $\mathbf{R_2} = [1, 0]$. Then, $\beta_1 = [1]^{-1}[1 - \beta_2] = 1 - \beta_2$. Thus, $\mathbf{y} = (\mathbf{1} - \beta_{\mathbf{2}})\mathbf{x_1} + \beta_{\mathbf{2}}\mathbf{x_2} + \alpha\mathbf{i} + \epsilon$ or $\mathbf{y} - \mathbf{x_1} = \beta_{\mathbf{2}}(\mathbf{x_2} - \mathbf{x_1}) + \alpha\mathbf{i} + \epsilon$.