

Profit-Maximizing Matchmaker*

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Abstract

This paper considers a matchmaker game in the Shapley-Shubik (1971) (one-to-one) assignment problem. Each firm purposes how much it is willing to pay each worker if they are matched. Each worker also purposes for which salary she is willing to work for each firm if they are matched. The monopolistic matchmaker chooses a matching to maximize profit (the sum of difference between the offering and asking salaries from each matched firm-worker). First, we show that Nash equilibrium may generate inefficient outcomes, but the matchmaker's profit is always zero in every Nash equilibrium. Second, we show that the sets of stable assignments and strong Nash equilibria are equivalent. These results extend to the Kelso-Crawford (1982) many-to-one matching problem. Interestingly, in the one-to-one matching case, our results have a close relationship with common agency game by Bernheim and Whinston (1986), while in the many-to-one matching problem, such relationships break down completely.

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1 Introduction

In their influential paper, Shapley and Shubik (1971) introduce an assignment problem which is a transferrable utility (cooperative) game in a two-sided matching model. Each firm's output level is affected by which worker it is matched with, and salaries are determined endogenously by considering stability of assignment. They show that a stable assignment generates an efficient matching, and the core and the set of stable assignments are equivalent. Moreover, the set of market equilibria is equivalent to the core.

Kelso and Crawford (1982) generalize the assignment model by allowing firms to choose how many workers to hire, and analyzing the market equilibrium and the core. They also consider a central planning authority that matches up firms and workers, and analyze a price adjustment mechanism by generalizing the Gale-Shapley deferred acceptance algorithm (Gale and Shapley 1962, and Roth and Sotomayor 1990). Their algorithm finds the firm-optimal stable assignment which is a market equilibrium and a core allocation. As in many centralized market clearing mechanisms successfully used in the real world, such as entry-level medical markets and school choice problems, Kelso and Crawford (1982) assume that the matchmaker is a benevolent central planner who tries to achieve a desirable allocation—a market equilibrium.

By contrast, in this paper, we consider a self-interest-motivated monopolistic matchmaker. She chooses a matching of firms and workers that maximizes profit. In this setup, we want to ask the following questions: How much profit can the matchmaker make? What properties would the resulting allocation have? Are there any differences between one-to-one and many-to-one markets?

Specifically, we will first consider a two-stage noncooperative game in the Shapley-Shubik assignment problem with a matchmaker. Each firm proposes how much it is willing to pay for each worker if they are matched, and each worker proposes for which salary she is willing to accept for each firm if they are matched. These proposals are made simultaneously. Then, in the second stage, the monopolistic matchmaker matches up firms and workers in order to maximize profits (the sum of the difference between the offering and asking

salaries from each matched firm-worker).¹ We will use Nash equilibrium and strong Nash equilibrium (Aumann,1959) as solution concepts.²

This simple game generates a few interesting results. First, in every (sub-game perfect) Nash equilibrium, the matchmaker makes zero rent³ (Theorem 1). Second, the set of stable assignments (the core) is equivalent to the set of strong Nash equilibrium outcomes (Theorem 2). Also, this set is equivalent to the set of Nash equilibrium outcomes when we restrict players' strategies to truthful strategies. In this one-to-one setup, our matchmaker game can be reinterpreted as a common agency game.

Our results extend to a general Kelso-Crawford many-to-one matching problem (Kelso and Crawford 1982). Even without the gross substitute condition, the matchmaker's rent is zero in every Nash equilibrium (Theorem 3), and the sets of stable assignments and strong Nash equilibrium outcomes are equivalent (Theorem 4).

Our matchmaker game in one-to-one matching problems has a close relationship with the *common agency game* introduced by Bernheim and Whinston (1986). In a common agency game, there are multiple principals (players) and an agent, and a set of agendas. All players and the agent have preferences over agendas, and each player offers a contribution schedule to the agent, which is a function from the agenda set to monetary contribution. The agent sees the players' contribution schemes, and chooses the agenda that has the highest total payoff. Defining truthful strategies, Bernheim and Whinston (1986) show that the set of Nash equilibria with truthful strategies is equivalent to the set of coalition-proof Nash equilibrium, which is a consistent refinement of Nash equilibria based on communications among players⁴. Laussel and Le

¹This matchmaker game can be considered as a common agency game introduced by Bernheim and Whinston (1986), and we will discuss the relationship between this paper in detail below.

²Although, strictly speaking, the game is a two-stage game, the second stage is a mere maximization problem by the matchmaker, we can regard the game as a static one (see Bernheim and Whinston 1986, and Laussel and Le Breton 2001).

³We will use rent and profit interchangeably in this paper.

⁴They show that a truthful strategy is always in the set of player's best responses, and argue that a Nash equilibrium with truthful strategies (truthful Nash equilibrium) may be quite *focal*.

Breton (2001) show that the set of truthful equilibrium outcomes is the core in a cooperative game that is generated from the common agency game, if and only if the agent earns zero rent in *all truthful Nash equilibria (the no-rent property)*. Konishi, Le Breton, and Weber (1999) show that coalition-proof Nash equilibrium and strong Nash equilibrium are equivalent if and only if a common agency game has the no-rent property. Our one-to-one problem can be reinterpreted as a common agency game. Thus, given our zero rent result *in all Nash equilibria* (Theorem 1), our equivalence result between the sets of strong Nash equilibrium and stable assignments (Theorem 2) can be shown by using the results in common agency games.⁵ However, interestingly, although our matchmaker game can be interpreted as a common agency game if the assignment problem is one-to-one, it does not allow such an interpretation when the assignment problem is many-to-one. Truthful Nash equilibrium may give the matchmaker a positive rent in a common agency game generated from many-to-one matching problem (Example 4). Even if we assume an additive environment (Alcalde et al. 1998) and we decompose each firm into multiple positions, we cannot restore the equivalence of the sets of stable assignments and truthful Nash equilibrium outcomes (Example 5).

Implementation results are obtained in the Kelso-Crawford many-to-one matching problem with monetary transfers. Alcalde et al. (1998) show that the stable correspondence (competitive equilibrium correspondence) is subgame-perfect-Nash-implementable by a simple two-stage game. Hayashi and Sakai (2009) characterize the stable correspondence by Nash implementation. Note that their results cannot treat the one-to-one problem or a many-to-one problem with quotas. By noting that all Nash equilibrium outcomes are individually rational, we can show that the individually rational correspondence is Nash-implementable by our matchmaker game in a general domain of one-to-one and many-to-one assignment problems by utilizing Theorems 1 and 3 (Corollaries 1 and 3). Theorems 2 and 4 directly show that a stable assignment correspondence is strong-Nash-implementable in the one-to-one problem

⁵However, our proof is much simpler than an indirect proof using the results in common agency games due to the power of Theorem 1.

by our matchmaker game (Corollaries 2 and 4). These results are not dependent on the presence of quotas.

The rest of the paper is organized as follows. In Section 2, the (one-to-one) Shapley-Shubik assignment problem and our matchmaker game are introduced with a few examples. Section 3 presents our results. Section 4 generalizes our results to many-to-one matching case. Section 5 concludes with discussions on the relationship with common agency games.

2 The Model

2.1 The Shapley-Shubik Model

In this section, we define the assignment problem by Shapley and Shubik (1971).⁶ There are two groups of players: the set of firms F with $|F| = I$ and the set of workers W with $|W| = J$. Let $N = F \cup W$. In the basic model, we assume that each firm can hire at most one worker. If $f \in F$ hires $w \in W$, f can earn y_{fw} as its output (gross profit). After paying salary x to w , firm f 's payoff is $y_{fw} - x$. If f does not hire any worker, its output and profit are zero ($y_{ff} = 0$). Let $Y = (y_{fw})_{f \in F, w \in W}$ be an output matrix, and let \mathcal{Y} be the set of all possible output matrices. If w is hired by f , w receives disutility d_{wf} . With salary x , her payoff is $x - d_{wf}$. If w is unemployed, then she receives zero disutility ($d_{ww} = 0$). We assume that $d_{wf} \geq 0$ for all $f \in F$ and all $w \in W$. Let $D = (d_{wf})_{w \in W, f \in F}$ be a disutility matrix, and let \mathcal{D} be the set of all possible disutility matrices.

A **matching** $\mu : W \cup F \rightarrow W \cup F$ is a bijection (a one-to-one function) such that (i) $\mu(f) = w$ iff $\mu(w) = f$, and (ii) $\mu(f) \in F$ implies $\mu(f) = f$ and $\mu(w) \in W$ implies $\mu(w) = w$. Let \mathcal{M} be the set of all matchings. Similarly, for $S \subset W \cup F$, a **matching in S** , $\mu_S : S \rightarrow S$ is a bijection such that (i) $\mu(f) = w \in W \cap S$ iff $\mu(w) = f \in F \cap S$, and (ii) $\mu(f) \in F \cap S$ implies $\mu(f) = f$ and $\mu(w) \in W \cap S$ implies $\mu(w) = w$. Let $\mathcal{M}(S)$ be the set of all matchings in S .

⁶See Roth and Sotomayor (1990) for the detailed discussions.

An **efficient matching** is defined by

$$\mu^* \in \arg \max_{\mu \in \mathcal{M}} \sum_{f \in F} (y_{f\mu(f)} - d_{\mu(f)f}),$$

where $d_{ff} = 0$ for all $f \in F$. An **allocation** is a list $(v, u, \mu) \in \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$ such that (i) $v_f = 0$ for all $f \in F$ with $\mu(f) = f$, (ii) $u_w = 0$ for all $w \in W$ with $\mu(w) = w$, and (iii) $v_f + u_{\mu(f)} = y_{f\mu(f)} - d_{\mu(f)f}$ for all $f \in F$ with $\mu(f) \in W$. An allocation (v, u, μ) is **efficient** if μ is an efficient matching. Similarly, we can define an allocation and an efficient allocation for $S \subset W \cup F$. An allocation (v, u, μ) is **individually rational** if $v \geq 0$ and $u \geq 0$. An allocation is a **stable assignment** if it is individually rational and there is no pair $(f, w) \in F \times W$ such that $y_{fw} - d_{wf} > v_f + u_w$. Shapley and Shubik (1972) showed that stable allocations are efficient and the set of stable assignments is equivalent to the core.

A mapping $\varphi : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is a **social choice correspondence** iff $\varphi(Y, D) \neq \emptyset$ for all $(Y, D) \in \mathcal{Y} \times \mathcal{D}$. An **individually rational correspondence** $IR : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is a social choice correspondence such that $IR(Y, D) \subset \mathcal{M}$ is the set of all individually rational allocations (v, u, μ) for (Y, D) . An **stable assignment correspondence** $SA : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is a social choice correspondence such that $SA(Y, D) \subset \mathcal{M}$ is the set of all stable assignments (v, u, μ) for (Y, D) .

2.2 The Matchmaker Game

In this paper, we consider a mechanism in which a matchmaker match up firms and workers. This matchmaker can be regarded as a monopolistic private firm, or can be regarded as a central planning authority who chooses a matching based on submitted information by firms and workers. A matchmaker asks each firm how much it is willing to offer to each worker if the pair is matched up, and asks each worker how much salary she demands from each firm's position. That is, each firm $f \in F$ submits $\sigma_f : W \rightarrow \mathbb{R}_+$ (or $\sigma_f = (\sigma_f(w))_{w \in W}$), and each worker w submits $s_w : F \rightarrow \mathbb{R}_+$ (or $s_w = (s_w(f))_{f \in F}$). These are

their strategies. The matchmaker is allowed to take the difference between $\sigma_f(w)$ and $s_w(f)$ if she matches f and w . Needless to say, the matchmaker would not match a pair (f, w) if $\sigma_f(w) < s_w(f)$: it would rather let them left unmatched.

With these submitted information $\sigma = (\sigma_f)_{f \in F}$ and $s = (s_w)_{w \in W}$, the matchmaker maximizes its profit $U : \mathcal{M} \times \mathbb{R}_+^{I \times J} \times \mathbb{R}_+^{J \times I} \rightarrow \mathbb{R}$ with $U(\mu, \sigma, s) = \sum_{f \in F} (\sigma_f(\mu(f)) - s_{\mu(f)}(f))$. Let the set $M(\sigma, s) \subset \mathcal{M}$ be

$$M(\sigma, s) \equiv \arg \max_{\mu \in \mathcal{M}} U(\mu, \sigma, s).$$

Each firm f , worker w and the matchmaker obtain the following payoffs under $\mu \in M(\sigma, s)$:

$$v_f(\mu, \sigma, s) = y_{f\mu(f)} - \sigma_f(\mu(f)),$$

$$u_w(\mu, \sigma, s) = s_w(\mu(w)) - d_{w\mu(w)},$$

and

$$U(\mu, \sigma, s) = \sum_{f \in F} (\sigma_f(\mu(f)) - s_{\mu(f)}(f)),$$

respectively. We call this game a **matchmaker game**. Note that each firm f only cares about $\mu(f)$, and the rest of the matching is irrelevant to it. Similarly, each worker w_j only cares about $\mu(w)$. An outcome of a matchmaker game is $(\mu, (\sigma, s))$. An outcome $(\mu^*, (\sigma^*, s^*))$ is **Nash equilibrium** if (i) $\mu^* \in M(\sigma^*, s^*)$, (ii) there is no $f \in F$ such that $\sigma_f : W \rightarrow \mathbb{R}$ and $\mu \in M(\sigma_f, \sigma_{-f}^*, s^*)$ such that $v_f(\mu, \sigma_f, \sigma_{-f}^*, s^*) > v_f(\mu^*, \sigma^*, s^*)$, and (iii) there is no $w \in W$ such that $s_w : F \rightarrow \mathbb{R}$ and $\mu \in M(\sigma^*, s_w, s_{-w}^*)$ such that $u_w(\mu, \sigma^*, s_w, s_{-w}^*) > u_w(\mu^*, \sigma^*, s^*)$. In a matchmaker game, an outcome $(\mu^*, (\sigma^*, s^*))$ is a **(strictly) strong Nash equilibrium (SNE)** if $\mu^* \in M(\sigma^*, s^*)$, and there is no coalition $S \subset N$, their strategies $(\sigma_{S \cap F}, s_{S \cap W}) = ((\sigma_f)_{f \in S \cap F}, (s_w)_{w \in S \cap W})$ and a matching $\mu \in M(\sigma_{S \cap F}, s_{S \cap W}, \sigma_{-S \cap F}^*, s_{-S \cap W}^*)$ such that $v_f(\mu, \sigma_{S \cap F}, s_{S \cap W}, \sigma_{-S \cap F}^*, s_{-S \cap W}^*) \geq v_f(\mu^*, \sigma^*, s^*)$ for all $f \in S \cap F$, and $u_w(\mu, \sigma_{S \cap F}, s_{S \cap W}, \sigma_{-S \cap F}^*, s_{-S \cap W}^*) \geq u_w(\mu^*, \sigma^*, s^*)$ for all $w \in S \cap W$ with

at least one strict inequality.

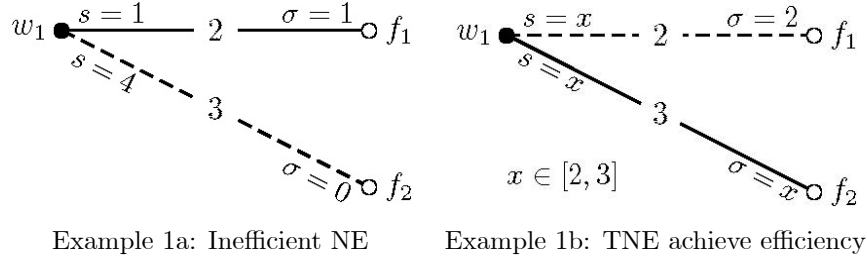
In the one-to-one matching problem, another refinement of Nash equilibrium turns out to be useful. Following Bernheim and Whinston (1986), we define “truthful strategies” as follows. For firm f , a strategy σ_f is said to be **truthful relative to** $\bar{w} \in W \cup \{f\}$ if and only if for all $w \in W \cup \{f\}$ either (i) $y_{fw} - \sigma_f(w) = y_{f\bar{w}} - \sigma_f(\bar{w})$ or (ii) $y_{fw} - \sigma_f(w) < y_{f\bar{w}} - \sigma_f(\bar{w})$ and $\sigma_f(w) = 0$. For worker w , a strategy s_w is said to be **truthful relative to** $\bar{f} \in F \cup \{w\}$ if and only if for all $f \in F \cup \{w\}$, $s_w(f) - d_{wf} = s_w(\bar{f}) - d_{w\bar{f}}$. An outcome $(\mu^*, (\sigma^*, s^*))$ is a **truthful Nash equilibrium (TNE)** if and only if it is a Nash equilibrium, and all players’ strategies are truthful relative to μ^* .

2.3 Examples

In this subsection, we will illustrate how Nash and truthful Nash equilibria look like. We start with a very simple example.

Example 1. There are two firms $\{f_1, f_2\}$ and one worker $\{w_1\}$. Let $y_{f_1 w_1} = 2$, $y_{f_2 w_1} = 3$, and $d_{w_1 f_1} = d_{w_1 f_2} = 0$. Even in this simple example, there are many Nash equilibria with different matching of a firm and a worker. Let $\sigma_{f_1}(w_1) = 1$ and $\sigma_{f_2}(w_1) = 0$, and $s_{w_1}(f_1) = 1$ and $s_{w_1}(f_2) = 4$. Under this strategy profile, the matchmaker chooses $\mu(f_1) = w_1$ and $\mu(f_2) = f_2$, and makes no profit. This is a Nash equilibrium, but the resulting matching is inefficient. The reason for inefficiency is a coordination problem. Firm f_2 wants to hire w_1 , but somehow w_1 is asking unreasonable salary, so f_1 does not have an incentive to try. So, it offers zero salary to w_1 . Worker w_1 sees f_2 offers zero salary, so has no incentive to reduce her salary demand for f_2 .

In contrast, let $\sigma_{f_2}(w_1) = x$ and $\sigma_{f_1}(w_1) = 2$, and $s_{w_1}(f_1) = x$ and $s_{w_1}(f_2) = x$, where $x \in [2, 3]$. If the matchmaker chooses $\mu'(f_2) = w_1$ and $\mu'(f_1) = f_1$ (indeed, unless $x = 2$, she must choose μ'), this is a truthful strategy, since f_1 is indifferent between hiring w_1 or not. In contrast, f_2 is strictly better off by hiring w_1 , unless $x = 3$. Thus, any salary $x \in [2, 3]$ can be supported by a truthful Nash equilibrium, and efficiency is achieved. Since there is no profitable deviations from such an allocation, it is also strong Nash



equilibrium. Note that each of these allocations is a stable assignment.

Example 1 shows that Nash equilibria in matchmaker games can generate inefficient matchings. However, if truthful strategies are used by players, then the resulting matching is efficient, and the set of truthful Nash equilibria coincide with the set of strong Nash equilibria and the set of stable assignments. Moreover, somewhat surprisingly, the matchmaker's rent is zero in all Nash equilibria. In the next two examples, we consider slightly more general situations, and show that the matchmaker's rent is still zero.

Example 2. There are two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$. Let $y_{f_1 w_1} = 3$ and $y_{f_1 w_2} = y_{f_2 w_1} = y_{f_2 w_2} = 0$, and let $d_{w_j f_i} = 0$ for all $i, j = 1, 2$. Clearly, the efficient matching involves $\mu(f_1) = w_1$. Suppose that the matchmaker is earning a positive rent in a Nash equilibrium at least from the pair $\{f_1, w_1\}$ by choosing μ . If it were the case, $\sigma_{f_1}(w_1) > s_{w_1}(f_1)$ holds. This can happen only when both f_1 and w_1 have reasons that they cannot ask more. In the former case, f_1 is afraid about reducing $\sigma_{f_1}(w_1)$, since otherwise f_2 would take w_1 : for this to happen, $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_2}(w_1) - s_{w_1}(f_2)$ must hold. In the latter case, w_1 is afraid about increasing $s_{w_1}(f_1)$ since otherwise f_1 would choose w_2 . For this to happen, $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_1}(w_2) - s_{w_2}(f_1)$ must hold. However, if these two hold, then the matchmaker will choose matching μ' such that $\mu'(f_1) = w_2$ and $\mu'(f_2) = w_1$. This matching does not do any good to firms, so it cannot be a Nash equilibrium. Thus, in this example, Nash equilibrium rent must be zero. By the same logic as in Example 1, the set of outcomes of truthful Nash equilibria coincides with the set of strong Nash equilibria and the set of stable assignments.

Example 3. There are two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$. Let $y_{f_1 w_1} = y_{f_2 w_2} = 3$ and $y_{f_1 w_2} = y_{f_2 w_1} = 0$, and let $d_{w_j f_i} = 0$ for all $i, j = 1, 2$. Clearly, the efficient matching is $\mu(f_1) = w_1$, and $\mu(f_2) = w_2$. Suppose that the matchmaker is earning a positive rent in a Nash equilibrium at least from the pair $\{f_1, w_1\}$ by choosing μ . Again, let us assume $\sigma_{f_1}(w_1) > s_{w_1}(f_1)$. Repeating the same logic in Example 2, $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_1}(w_2) - s_{w_2}(f_1) = \sigma_{f_2}(w_1) - s_{w_1}(f_2)$ must hold. Thus, for μ to be chosen by the matchmaker, $\sigma_{f_2}(w_2) - s_{w_2}(f_2) > 0$ also need to hold. However, then, this pair also need to be threatened. As the result, we conclude that $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_2}(w_2) - s_{w_2}(f_2) = \sigma_{f_1}(w_2) - s_{w_2}(f_1) = \sigma_{f_2}(w_1) - s_{w_1}(f_2) > 0$ must hold. However, if it were the case, f_1 can reduce $\sigma_{f_1}(w_1)$ and $\sigma_{f_1}(w_2)$ simultaneously by $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_1}(w_2) - s_{w_2}(f_1)$, f_1 can earn more profit. It is because the matchmaker is made indifferent between μ and μ' by this change. As a result, this cannot be a Nash equilibrium. Thus, in this example again, Nash equilibrium rent must be zero. Repeating the same logic, the set of outcomes of truthful Nash equilibria coincides with the set of strong Nash equilibria and the set of stable assignments.

In the next section, we will investigate whether the above observations are not a coincidence.

3 The Main Results

Before stating our theorem, we introduce a few notations. For all $S \subseteq N$, $\mu \in \mathcal{M}(N)$, let $C(S, \mu) \equiv \{k \in S : \mu(k) \in S \text{ and } \mu(k) \neq k\}$. That is, set $C(S, \mu)$ is the set of members of S , who have partners in S under matching μ (coupled). Given a strategy profile $(\sigma, s) \in \mathbb{R}_+^{I \times J} \times \mathbb{R}_+^{J \times I}$, let $R(S, (\sigma, s), \mu)$ be the rent generated in S under μ such that $R(S, (\sigma, s), \mu) = \sum_{f \in C(S, \mu) \cap F} (\sigma_f(\mu(f)) - s_{\mu(f)}(f))$. Let $R^*(S, (\sigma, s)) \equiv \max_{\mu \in \mathcal{M}(S)} R(S, (\sigma, s), \mu)$, and let $A^*(S, (\sigma, s)) \equiv \arg \max_{\mu \in \mathcal{M}(S)} R(S, (\sigma, s), \mu)$ be the maximum rent generated in coalition S given firms' strategies σ and workers strategies s , and its associated matching respectively. We can characterize Nash equilibrium pay-

offs in an interesting way.

Proposition 1. In the one-to-one matching problem, $R^*(N, (\sigma, s)) = R^*(N \setminus \{k\}, (\sigma, s))$ for all $k \in N$ in every Nash equilibrium $((\sigma, s), \mu)$.

Proof. Since worker's deviation incentive can be treated in a similar manner, we will focus on firm's deviation incentive. It is easy to see the statement is trivial when $\mu(f) = f$ or $\sigma_f(\mu(f)) = s_{\mu(f)}(f)$. Let us focus on $\mu(f) \in W$ and $\sigma_f(\mu(f)) > s_{\mu(f)}(f)$. Suppose the firm reduces the salary offers uniformly by $\epsilon > 0$: i.e., $\sigma'_f(w) = \max\{\sigma_f(w) - \epsilon, 0\}$. Then matchmaker have to choose a different matching μ' where $\mu'(f) \neq w$, Otherwise, the firm would have deviated with improved profit.

Suppose $\mu'(f) \in W$. Since μ' maximizes the total rent under $(\sigma'_f, \sigma_{-f}, s)$, then $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(N, (\sigma, s), \mu') - \epsilon \geq R(N, (\sigma, s), \mu) - \epsilon = R(N, (\sigma'_f, \sigma_{-f}, s), \mu)$ holds. However, $\mu \in A^*(N, (\sigma, s))$ implies $R(N, (\sigma, s), \mu') \leq R(N, (\sigma, s), \mu)$ so that $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(N, (\sigma'_f, \sigma_{-f}, s), \mu)$. But then, firm f can improve its payoff by $\epsilon > 0$ by choosing σ''_f such that $\sigma''_f(\mu(f)) = \sigma'_f(\mu(f)) = \sigma_f(\mu(f)) - \epsilon$, and $\sigma''_f(w) = 0$ for all $w \notin \mu(f)$ as the matchmaker is forced to choose μ . Thus, $R^*(N \setminus \{f\}, (\sigma, s)) = R(N, (\sigma, s), \mu')$.

By optimality of μ , $R(N, (\sigma, s), \mu) \geq R(N, (\sigma, s), \mu')$. We claim that the equality must be binding. Suppose not. Define $\delta \equiv R(N, (\sigma, s), \mu) - R(N, (\sigma, s), \mu') > 0$. Consider $\epsilon < \delta$. Then $R(N, (\sigma, s), \mu) - \epsilon > R(N, (\sigma, s), \mu) - \delta = R(N, (\sigma, s), \mu')$ which is contradiction.

Since $R^*(N, (\sigma, s)) = R(N, (\sigma, s), \mu)$ and $R^*(N \setminus \{f\}, (\sigma, s)) = R(N, (\sigma, s), \mu')$, we have the desired result. \square

Let $\mu \in \mathcal{M}(N \setminus \{k\})$ be such that $R^*(N \setminus \{k\}, (\sigma, s)) = R(N \setminus \{k\}, (\sigma, s), \mu)$, and let $S_k = \{k' \in N \setminus \{k\} : \mu(k') \neq k'\}$. This implies that $R^*(S_k, (\sigma, s)) = R(S_k, (\sigma, s), \mu) = R^*(N \setminus \{k\}, (\sigma, s))$. Proposition 1 says that in every Nash equilibrium $((\sigma, s), \mu)$, for all $k \in N$, there exists $S_k \subseteq N \setminus \{k\}$ such that the following equation holds

$$R^*(N, (\sigma, s)) = R^*(S_k, (\sigma, s)).$$

We call the above a system of **fundamental equations**.⁷ The first main result of this section is the following theorem.

Theorem 1. In the one-to-one matching problem, the matchmaker's rent is zero in every Nash equilibrium.

Proof. We will prove the theorem by contradiction. To the contrary, assume that there is a Nash equilibrium allocation $((\sigma, s), \mu)$ with positive rent ($R(N, (\sigma, s), \mu) = R^*(N, (\sigma, s)) > 0$), and we will reach a contradiction.

First, note that $R(N, (\sigma, s), \mu) = \sum_{f \in C(N, \mu) \cap F} R(\{f, \mu(f)\}, (\sigma, s), \mu)$. Pick up a pair $(f_1, w_1) \subset N$ that generates the *highest* positive rent under (σ, s) and μ : $R(\{f_1, w_1\}, (\sigma, s), \mu) > 0$. The relevant fundamental equations for f_1 and w_1 can be written as

$$\sum_{f \in C(S_{f_1}, \mu') \cap F} R(\{f, \mu'(f)\}, (\sigma, s), \mu') = \sum_{f \in C(N, \mu) \cap F} R(\{f, \mu(f)\}, (\sigma, s), \mu)$$

and

$$\sum_{f \in C(S_{w_1}, \mu'') \cap F} R(\{f, \mu''(f)\}, (\sigma, s), \mu'') = \sum_{f \in C(N, \mu) \cap F} R(\{f, \mu(f)\}, (\sigma, s), \mu),$$

where $\mu' \in \mathcal{M}(S_{f_1})$ and $\mu'' \in \mathcal{M}(S_{w_1})$ are such that $R(S_{f_1}, (\sigma, s), \mu') = R^*(S_{f_1}, (\sigma, s))$ and $R(S_{w_1}, (\sigma, s), \mu'') = R^*(S_{w_1}, (\sigma, s))$, respectively.

Our first lemma is the following.⁸

Lemma 1. We have $w_1 \in S_{f_1}$ and $R(\{\mu'(w_1), w_1\}, (\sigma, s), \mu') > 0$. Similarly, $f_1 \in S_{w_1}$ and $R(\{f_1, \mu''(f_1)\}, (\sigma, s), \mu'') > 0$.

⁷Our system of fundamental equations is inspired by a system of fundamental equations in Laussel and Le Breton (2001). However, these two systems of equations are very different from each other. The one by Laussel and Le Breton is generated from maximizing each coalition's value, while ours is generated for each Nash equilibrium strategy profile (σ, s) . This difference reflects the difference of the notions of no-rent properties in these two papers. The no-rent property by Laussel and Le Breton (2001) concerns about all truthful Nash equilibria, but our no-rent result is on all Nash equilibria.

⁸Technically speaking, Lemma 1 is not needed for the proof of Theorem 1. However, we keep it here in order to make the exposition easier.

Proof of Lemma 1. We will prove the first half (the second half follows by a symmetric argument). Suppose $w_1 \notin S_{f_1}$ or $\mu'(w_1) = w_1$. Then, we can construct a new matching μ^* such that $\mu^*(k) = \mu'(k)$ for all $k \in S_{f_1}$, $\mu^*(f_1) = w_1$, and $\mu^*(k) = k$ for all $k \in N \setminus (S_{f_1} \cup \{w_1, f_1\})$. Then, we have

$$\begin{aligned} & R(N, (\sigma, s), \mu^*) \\ &= R(S_{f_1}, (\sigma, s), \mu') + R(\{w_1, f_1\}, (\sigma, s), \mu) \\ &> R(S_{f_1}, (\sigma, s), \mu') = R(N, (\sigma, s), \mu). \end{aligned}$$

Note that the last equality comes from fundamental equation. This is a contradiction with $\mu \in A_{\Gamma}^*(N)$.

Now, suppose $R(\{\mu'(w_1), w_1\}, (\sigma, s), \mu') = 0$ (if it is negative, unmatched the pair improves the value). Then, we have

$$\begin{aligned} & R(S_{f_1}, (\sigma, s), \mu') \\ &= R(S_{f_1} \setminus \{w_1, \mu'(w_1)\}, (\sigma, s), \mu') + R(\{\mu'(w_1), w_1\}, (\sigma, s), \mu') \\ &= R(S_{f_1} \setminus \{w_1, \mu'(w_1)\}, \mu'). \end{aligned}$$

Then we could construct μ^* such that $\mu^*(k) = \mu'(k)$ for all $k \in S_{f_1}$, $\mu^*(f_1) = w_1$, and $\mu^*(k) = k$ for all $k \in N \setminus (S_{f_1} \cup \{w_1, \mu'(w_1)\})$. Then we have

$$\begin{aligned} & R(N, (\sigma, s), \mu^*) \\ &= R(S_{f_1} \setminus \{w_1, \mu'(w_1)\}, (\sigma, s), \mu') + R(\{w_1, f_1\}, (\sigma, s), \mu') \\ &= R(S_{f_1}, (\sigma, s), \mu') + R(\{w_1, f_1\}) > R(N, (\sigma, s), \mu). \end{aligned}$$

This violates $\mu \in A_{\Gamma}^*(N)$. \square

Recall μ' and μ'' are matchings that achieve values $R^*(S_{f_1}, (\sigma, s))$ and $R^*(S_{w_1}, (\sigma, s))$, respectively. By using Lemma 1, we will construct chains of pairs from matchings μ, μ' and μ'' . Let $f_{\ell+1} \equiv \mu'(w_{\ell})$ and $w_{\ell+1} = \mu(f_{\ell+1})$ for $\ell = 1, 2, \dots, L$, where L is such that $\mu(f_{\ell}) \in C(N, \mu) \cap W$ and $\mu'(w_{\ell}) \in C(N, \mu) \cap F$ for all $\ell < L$ and $\mu(f_L) \notin C(N, \mu) \cap W$. Similarly, let $\tilde{w}_{\ell+1} \equiv \mu''(\tilde{f}_{\ell})$ and $\tilde{f}_{\ell+1} = \mu(\tilde{w}_{\ell+1})$ for $\ell = 1, 2, \dots, \tilde{L}$, where \tilde{L} is such that $\mu(\tilde{w}_{\ell}) \in C(N, \mu) \cap F$

and $\mu'(f_\ell) \in C(N, \mu) \cap W$ for all $\ell < \tilde{L}$ and $\mu(\tilde{w}_{\tilde{L}}) \notin C(N, \mu) \cap F$. The following is our key lemma.

Lemma 2. $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ or $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$ holds.

Proof of Lemma 2. First, note that $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) \geq \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) \geq \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$. Otherwise, we conclude that μ does not maximize the value. Thus, suppose to the contrary that $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$. There are two cases: (Case 1) $(\cup_{\ell=1}^L \{w_\ell, f_\ell\}) \cap (\cup_{\ell=1}^{\tilde{L}} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) = \emptyset$, and (Case 2) $(\cup_{\ell=1}^L \{w_\ell, f_\ell\}) \cap (\cup_{\ell=1}^{\tilde{L}} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) \neq \emptyset$. We will analyze the two cases by noting $\{w_1, f_1\} = \{\tilde{w}_1, \tilde{f}_1\}$.

Let us start with a simpler case.

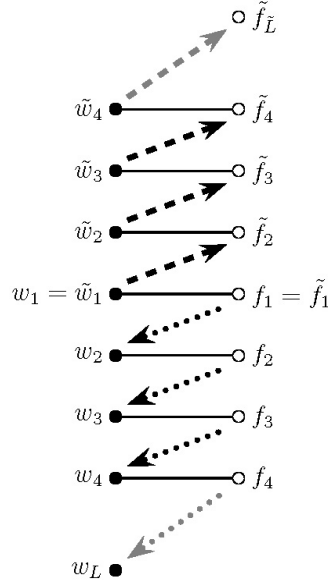


Figure 1: Solid, dotted and dashed line represents matching μ , μ' and μ'' respectively. Arrow represents μ^* .

(Case 1) Suppose $(\cup_{\ell=1}^L \{w_\ell, f_\ell\}) \cap (\cup_{\ell=1}^{\tilde{L}} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) = \emptyset$. See Figure 1. By assumption, $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$. Summing them up, we have

$$\begin{aligned}
& \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu') + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \\
&= \sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) + \sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) \\
&= \left(\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) + \sum_{\ell=2}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) \right) + R(\{w_1, f_1\}, (\sigma, s), \mu),
\end{aligned}$$

since $\{\tilde{w}_1, \tilde{f}_1\} = \{w_1, f_1\}$. Let $A \equiv (\cup_{\ell=1}^L \{w_\ell, f_\ell\}) \cup (\cup_{\ell=2}^{\tilde{L}-1} \{\tilde{w}_\ell, \tilde{f}_\ell\})$. Note there is no double counting of players in it. Note that in the RHS of the above formula, the set of players appears in the formula is $A \setminus \{w_L, \tilde{f}_L\}$, and each appears only once. Let $\mu^* \in \mathcal{M}(A)$ be such that $\mu^*(w_\ell) = f_{\ell+1}$ for $\ell = 1, \dots, L-1$, $\mu^*(\tilde{f}_\ell) = \tilde{w}_{\ell+1}$ for $\ell = 1, \dots, \tilde{L}-1$, $\mu^*(w_L) = w_L$, and $\mu^*(f_{\tilde{L}}) = f_{\tilde{L}}$. Replacing μ by μ^* , the total value in A increases by $R(\{w_1, f_1\}, (\sigma, s), \mu)$. By the prevailing assumption, $R(\{w_1, f_1\}, (\sigma, s), \mu) > 0$. This is a contradiction.

(Case 2) $(\cup_{\ell=1}^L \{w_\ell, f_\ell\}) \cap (\cup_{\ell=1}^{\tilde{L}} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) \neq \emptyset$. See figure 2a. In this case, let $\bar{\ell}$ be such that for all $1 < \ell < \bar{\ell}$, $\tilde{w}_\ell, \tilde{f}_\ell \notin \cup_{\ell=1}^L \{w_\ell, f_\ell\}$, and $\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}} \in \cup_{\ell=1}^L \{w_\ell, f_\ell\}$. In the latter case, more specifically, we can let $\{\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}}\} = \{w_{\ell'}, f_{\ell'}\}$ for some $\ell' \in \{2, \dots, L\}$. We consider the set of players $B_1 \equiv (\cup_{\ell=1}^{\ell'} \{w_\ell, f_\ell\}) \cup (\cup_{\ell=2}^{\bar{\ell}-1} \{\tilde{w}_\ell, \tilde{f}_\ell\})$ as in figure 2b. There is no double counting in B_1 . Now, consider two matchings in B_1 : μ and μ^* such that $\mu^*(w_\ell) = \mu'(w_\ell)$ for $\ell = 1, \dots, \ell' - 1$, and $\mu^*(\tilde{f}_\ell) = \mu''(\tilde{f}_\ell)$ for $\ell = 1, \dots, \bar{\ell} - 1$ (note $\tilde{f}_1 = f_1$ and

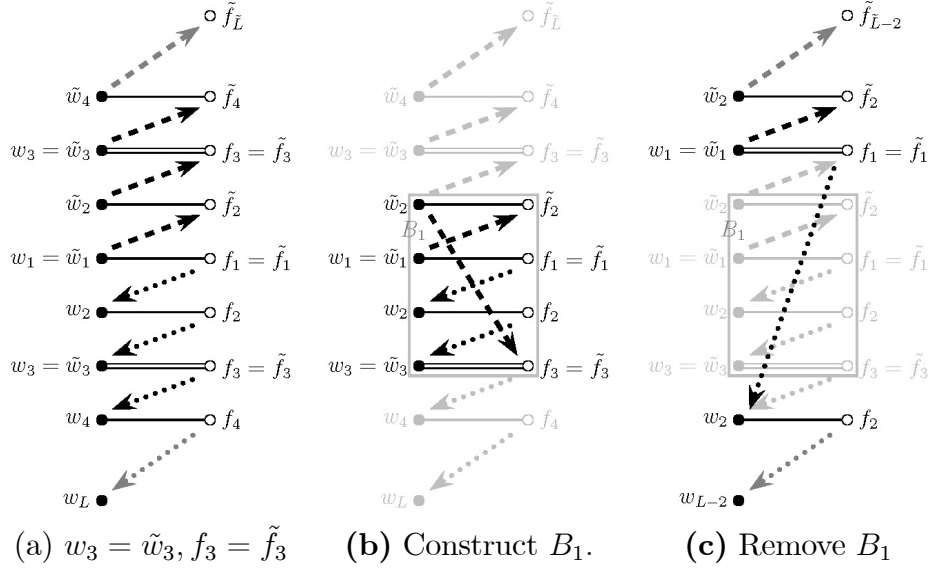


Figure 2: Examples when $\ell' = \bar{\ell} = 3$.

$\tilde{w}_{\bar{\ell}} = w_{\ell'}$). We will compare the values of these two. First,

$$\begin{aligned}
& R(B_1, (\sigma, s), \mu) \\
&= \sum_{\ell=1}^{\ell'} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu') + \sum_{\ell=2}^{\bar{\ell}-1} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu'') \\
&= \left[\sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu') + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \right] \\
&\quad - \left[\sum_{\ell=\ell'+1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) + \sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) + R(\{w_1, f_1\}, (\sigma, s), \mu) \right].
\end{aligned}$$

and

$$\begin{aligned}
& R(B_1, (\sigma, s), \mu^*) \\
&= \sum_{\ell=1}^{\ell'-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu') + \sum_{\ell=1}^{\bar{\ell}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \\
&= \left[\sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu') + \sum_{\ell=1}^{\bar{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \right] \\
&\quad - \left[\sum_{\ell=\ell'}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu') + \sum_{\ell=\bar{\ell}}^{\bar{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \right].
\end{aligned}$$

Thus, we have

$$\begin{aligned}
& R(B_1, (\sigma, s), \mu^*) - R(B_1, (\sigma, s), \mu) \\
&= R(\{w_1, f_1\}, (\sigma, s), \mu) - R(\{w_{\ell'}, f_{\ell'}\}, (\sigma, s), \mu) \\
&\quad + \left[\sum_{\ell=\ell'}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) - \sum_{\ell=\ell'}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu') \right] \\
&\quad + \left[\sum_{\ell=\bar{\ell}}^{\bar{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) - \sum_{\ell=\bar{\ell}}^{\bar{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \right].
\end{aligned}$$

Note that the contents in the both brackets must be nonnegative since μ maximizes the total value in N . Since $\{w_1, f_1\}$ generates the highest rent under (σ, s) and μ , $R(\{w_1, f_1\}, (\sigma, s), \mu) \geq R(\{w_{\ell'}, f_{\ell'}\}, (\sigma, s), \mu)$ must hold. Thus, $R(B_1, (\sigma, s), \mu^*) \geq R(B_1, (\sigma, s), \mu)$ must hold. If $R(B_1, (\sigma, s), \mu^*) > R(B_1, (\sigma, s), \mu)$, it is a contradiction, so assume that $R(B_1, (\sigma, s), \mu^*) = R(B_1, (\sigma, s), \mu)$. For this to happen, the following three conditions must hold:

1. $R(\{w_1, f_1\}, (\sigma, s), \mu) = R(\{w_{\ell'}, f_{\ell'}\}, (\sigma, s), \mu)$.
2. $\sum_{\ell=\ell'}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) = \sum_{\ell=\ell'}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$.
3. $\sum_{\ell=\bar{\ell}}^{\bar{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) = \sum_{\ell=\bar{\ell}}^{\bar{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$.

Note that $\{w_{\ell'}, f_{\ell'}\} = \{\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}}\}$. Rename $w_\ell, f_\ell, \tilde{w}_\ell, \tilde{f}_\ell, L$, and \bar{L} by $w_{\ell-\ell'+1}, f_{\ell-\ell'+1}, \tilde{w}_{\ell-\ell'+1}, \tilde{f}_{\ell-\ell'+1}, L$, and \bar{L} respectively.

$f_{\ell-\ell'+1}$, $\tilde{w}_{\ell-\bar{\ell}+1}$, $\tilde{f}_{\ell-\bar{\ell}+1}$, $L - \ell' + 1$, and $\tilde{L} - \bar{\ell} + 1$, respectively. Then, we again have exactly the same problem as before: $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$ as in figure 2c. If (case 1) applies, then we have a contradiction. If (case 2) applies, then we again find $\{w_{\ell'}, f_{\ell'}\} = \{\tilde{w}_{\ell'}, \tilde{f}_{\ell'}\}$, and we can again find a cycle set B_2 . If the cycle achieves a strict improvement, we reach a contradiction. So assuming equalities, the leftover players after taking B_2 out still satisfies the above three conditions. Applying this procedure repeatedly, eventually, (case 1) applies (finite number of players). Hence, we conclude that $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ or $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$ holds. ■

The last part of the proof of Theorem 1. Now, we will complete the proof of Theorem 1. Suppose, without loss of generality, that $\sum_{\ell=1}^L R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ holds. Then, there are two possibilities: (1) $S_{f_1} = \cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}$, and (2) $S_{f_1} \supsetneq \cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}$. In the former case 1, we have $R(S_{f_1}, (\sigma, s), \mu') < R(N, (\sigma, s), \mu)$. This is a contradiction. In the latter case 2, the chain created from matchings μ and μ' is disconnected in the middle. There are two subcases: (i) $\mu(f_L) = f_L$, and (ii) $\mu'(w_L) = w_L$. In either case, we have $R(N \setminus \sum_{\ell=1}^L \{w_\ell, f_\ell\}, (\sigma, s), \mu) = R(N \setminus \sum_{\ell=1}^L \{w_\ell, f_\ell\}, (\sigma, s), \mu')$ (in case (i), w_L does not exist, since f_L is single: this is a slight abuse of notation). This again implies $R(S_{f_1}, (\sigma, s), \mu') < R(N, (\sigma, s), \mu)$. This is a contradiction. Hence, in either way, we conclude that there is no pair that generates positive rent. □

Even though there are many Nash equilibria, and many of them generate inefficient outcomes, still the matchmaker receives zero rent in every equilibrium. This result heavily depends on the structure of two-sided matching market. Note that every Nash equilibrium generates an individually rational assignment given that the matchmaker generates zero rent. The converse is easy to show, too.

Corollary 1. In the one-to-one matching problem, the individually rational

correspondence $IR : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is implemented by the matchmaker game in Nash equilibria.

Proof. It is clear that every Nash equilibrium is individually rational. So, we prove the converse. Pick up an individually rational assignment (v, u, μ) . For each matched pair (f, w) with $\mu(f) = w$, let $\sigma_f(w) = y_{fw} - v$, $\sigma_f(w') = 0$ for all $w' \neq w$, and let $s_w(f) = u + d_{wf}$, and set her salary demand $s_w(f')$ at a prohibitively high level for all $f' \neq f$. For each single firm, let its salary offer be zero for all workers, and for each single worker, let her salary demand at a prohibitively high level. This strategy profile generates the outcome (v, u, μ) . \square

Although Theorem 1 is somewhat surprising by itself, it turns out quite useful when we consider refinement of Nash equilibrium. We will use truthful strategies for the refinement. As we have seen in Example 1, matchmaker games involve a lot of coordination problems. We also noticed that if Nash equilibrium is refined by truthful Nash equilibrium, then a stable matching is achieved. The next theorem shows that it is not a coincidence by using Theorem 1.

Theorem 2. In the one-to-one matching problem, the sets of outcomes of TNE and SNE in our matchmaker game, and the set of stable assignments are all equivalent.

Proof. We will prove $\text{TNE} \implies \text{SNE} \implies \text{stable assignment} \implies \text{TNE}$.

(1a. $\text{TNE} \implies \text{stable assignment}$) Suppose not. Then, there is a TNE (σ, s, μ) and pair (f, w) with $\mu(f) \neq w$ and equilibrium utilities (v_f, u_w) such that $y_{fw} - d_{fw} > v_f + u_w$. Since both f and w use truthful strategies, $\sigma_f(w) = y_{fw} - v_f$ and $s_w(f) = u_w + d_{fw}$, $\sigma_f(w) - s_w(f) = y_{fw} - v_f - u_w - d_{fw} > 0$. This implies that the matchmaker can make a positive rent by matching f and w . Since every Nash equilibrium generates a zero rent by Theorem 1, so does the TNE. This is a contradiction.

(1b. $\text{TNE} \implies \text{SNE}$) Suppose not. If there is an improving coalitional deviation, there is at least one pair of players whose payoffs improve. This

implies that the truthful Nash equilibrium is not a stable assignment, which cannot happen from 1a.

(2. SNE \implies stable assignment) From theorem 1, all Nash equilibrium has a zero rent, so does SNE. Let (v, u, μ) be an SNE outcome, and suppose that it is not a stable assignment. Then, there is a pair $(f, w) \in F \times W$ such that $y_{fw} - d_{wf} > v_f + u_w$. Consider $\sigma'_f(w) = u_w + d_{wf} - \frac{\epsilon}{2}$ and $\sigma'_f(w') = 0$ for other $w' \in W$, and $s'_w(f) = u_w + d_{wf} + \frac{\epsilon}{2}$ and $s'_{w'}(f)$ is prohibitively high for other $f' \in F$, where $\epsilon > 0$ satisfies $\epsilon < y_{fw} - d_{wf} - (v_f + u_w)$. Since the matchmaker gets no rent in the SNE, she is happy to match up f and w to make a positive rent. This is a contradiction. Thus, an SNE outcome is a stable matching.

(3. stable assignment \implies TNE) Let (v, u, μ) be a stable assignment. Then, $v \geq 0$ and $u \geq 0$, and for all pairs $(f, w) \in F \times W$ such that $y_{fw} - d_{wf} \leq v_f + u_w$. Let $\sigma_f(w) = \max\{y_{fw} - v_f, 0\}$ and $s_w(f) = u_w + d_{wf}$ for all f and w . Note that (σ, s) is a truthful strategy profile. By Shapley and Shubik (1971), a stable assignment maximizes the total value: $\mu \in \arg \max_{\mu'} \sum_{f \in F} (y_{f\mu(f)} - d_{\mu(f)f})$. That is, $y_{f\mu(f)} - d_{\mu(f)f} = v_f + u_{\mu(f)}$ holds for all $f \in F$ with $\mu(f) \neq f$. Since there is no pair that generates positive rent $\sigma_f(w) - s_w(f) \leq 0$, for the matchmaker, the maximum rent is zero under μ . \square

Corollary 2. In the one-to-one matching problem, the stable assignment correspondence $SA : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is implemented by the matchmaker game in SNE.

4 Many-to-one Matching

We can extend our domain of the problem to the Kelso-Crawford many-to-one assignment problem without imposing any restrictions on complementarity or substitutability of workers (Kelso and Crawford, 1982).⁹ Each $f \in F$ has a finite **position (quota)** q_f and each position can hold one worker. If firm

⁹In our model, we do not even need gross substitutability (Kelso and Crawford 1982) to ensure the existence of Nash equilibrium since no matching with zero rent would always be Nash equilibrium.

f hires $W_f \subseteq W$ workers, then output would be $Y_f(W_f)$ ¹⁰, and each worker hired by firm f has disutility from working $d_{wf} \geq 0$ independent of position. A **many-to-one matching** $\mu : W \cup F \rightarrow W \cup F$ is a correspondence such that (i) $\forall w \in \mu(f), \mu(w) = f$; (ii) $|\mu(f)| \leq q_f$; (iii) $f = \mu(w)$ if $w \in \mu(f)$; (iii) $\mu(f) \in F$ implies $\mu(f) = f$ and $\mu(w) \in W$ implies $\mu(w) = w$. Let \mathcal{M} be the set of all matchings μ . An efficient matching would then be $\mu^* \in \arg \max_{\mu \in \mathcal{M}} \sum_{f \in F} \left[Y_f(\mu(f)) - \sum_{w \in \mu(f)} d_{fw} \right]$. An allocation is a list $(v, u, \mu) \in \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$ such that (i) $v_f = 0$ for all $f \in F$ with $\mu(f) = f$, (ii) $u_w = 0$ for all $w \in W$ with $\mu(w) = w$ and (iii) $v_f + \sum_{w \in \mu(f)} u_w = Y_f(\mu(f)) - \sum_{w \in \mu(f)} d_{fw}$ for all $f \in F$ with $\mu(f) \subseteq W$. Definition for efficient allocation and individual rational allocation remain the same. An allocation is **stable assignment** if it is individually rational and there is no pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $Y_f(W_f) - \sum_{w \in W_f} d_{wf} > v_f + \sum_{w \in W_f} u_w$.

In a matchmaker game of a many-to-one matching is the same as before: each firm $f \in F$ submits $\sigma_f : W \rightarrow \mathbb{R}_+$ and each worker $w \in W$ submits $s_w : F \rightarrow \mathbb{R}_+$ as strategies.

Proposition 3¹¹ In the many-to-one matching problem, in every Nash equilibrium $((\sigma, s), \mu)$, (1A) for all $f \in F$ with $\mu(f) = \{f\}$, $R^*(N, (\sigma, s)) = R(N, (\sigma, s), \mu) = R^*(N \setminus \{f\}, (\sigma, s))$; (1B) for all $f \in F$ with $\mu(f) \neq \{f\}$, and all $w \in \mu(f)$, there exists μ' such that (i) $|\mu'(f)| < |\mu(f)|$, (ii) $R^*(N, (\sigma, s)) = R(N, (\sigma, s), \mu')$, and (iii) $w \notin \mu'(f)$; and (2) for all $w \in W$, there exists μ'' such that (i) $\mu''(w) = w$, and (ii) $R^*(N, (\sigma, s)) = R(N, (\sigma, s), \mu'') = R^*(N \setminus \{w\}, (\sigma, s))$.

Proof. Since (2) is a special case of (1), we focus on case (1).

Case (1A) is trivial, since we can use the same matching μ in order to achieve the same rent. Thus, we will work on case (1B). Clearly, if $\sigma_f(w) = s_w(f)$ for all $w \in \mu(f)$, then we can find μ' that satisfies all three conditions: the matchmaker makes no money by matching f and workers, so might as

¹⁰If production is additive, $Y_f(W_f) = \sum_{w \in W_f} Y_f(w)$, then we would have Alcalde, Perez-Castrillo, and Romero-Medina (1998).

¹¹It is easy to see Proposition 1 in fact is special case of this proposition.

well cancel matching (let $\mu'(f) = \{f\}$). Thus, let us focus on $\mu(f) \in W$ and $\sigma_f(w) > s_w(f)$ for some $w \in \mu(f)$.

(i) Suppose firm f reduces its salary offers uniformly by $\epsilon > 0$: i.e., $\sigma'_f(w) = \max\{\sigma_f(w) - \epsilon, 0\}$. The matchmaker will switch its choice from μ to $\mu' \in R^*(N, (\sigma'_f, \sigma_{-f}, s))$. Clearly, $\mu'(f) \neq \mu(f)$, since otherwise, firm f would have chosen σ'_f . Since μ' maximizes the total rent under $(\sigma'_f, \sigma_{-f}, s)$, $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(N, (\sigma, s), \mu') - \epsilon |\mu'(f)| \geq R(N, (\sigma, s), \mu) - \epsilon |\mu(f)| = R(N, (\sigma'_f, \sigma_{-f}, s), \mu)$ holds. However, $\mu \in A^*(N, (\sigma, s))$ implies $R(N, (\sigma, s), \mu') \leq R(N, (\sigma, s), \mu)$ so that $|\mu'(f)| \leq |\mu(f)|$ holds.

We further claim that $|\mu'(f)| < |\mu(f)|$. Suppose not. Then we have $R(N, (\sigma, s), \mu') = R(N, (\sigma, s), \mu)$ and $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(N, (\sigma'_f, \sigma_{-f}, s), \mu)$. But then, firm f can improve its payoff by $\epsilon > 0$ by choosing σ''_f such that $\sigma''_f(w) = \sigma'_f(w) = \sigma_f(w) - \epsilon$ for all $w \in \mu(f)$, and $\sigma''_f(w) = 0$ for all $w \notin \mu(f)$ as the matchmaker is forced to choose μ . This is a contradiction.

(ii) Suppose not. Define $\delta \equiv R(N, (\sigma, s), \mu) - R(N, (\sigma, s), \mu') > 0$ and let $\epsilon < \frac{\delta}{|\mu(f)| - |\mu'(f)|}$. Then, we have $R(N, (\sigma, s), \mu) - \epsilon (|\mu(f)| - |\mu'(f)|) > R(N, (\sigma, s), \mu) - \delta = R(N, (\sigma, s), \mu')$, which is a contradiction.

(iii) Suppose not. Then, there exists $w \in \mu(f)$ such that for all μ' with $R(N, (\sigma, s), \mu) = R(N, (\sigma, s), \mu')$, $w \in \mu'(f)$ holds. Let σ'''_f be such that $\sigma'''_f(w) = \sigma_f(w) - \epsilon$ and $\sigma'''_f(w') = \sigma_f(w')$ for all $w' \neq w$. Then, since the matchmaker is forced to match f and w . Thus, firm f can improve its payoff by adopting σ'''_f instead of σ_f . This is a contradiction. \square

Somewhat surprisingly, we can see that the fundamental equations apply to each position instead of each firm. It is easy to observe that Theorem 1 extends to this case by decomposing firms to positions. Let us separate firm f into q_f **positions** $f' = \{f'_1, \dots, f'_{q_f}\}$ where each position offering the same wages. Denote $F' \equiv \bigcup_{f \in F} \{f'_1, \dots, f'_{q_f}\}$ the set of divided firms and $N' \equiv F' \cup W$. Then, we can generate a one-to-one matching of positions and workers. Let **μ -decomposed matching** $\bar{\mu} : W \cup F' \rightarrow W \cup F'$ be a bijection such that (i) $\bar{\mu}(f'_i) = w$ if there exists $f' \ni f'_i$ such that $w \in \mu(f)$; (ii) $\bar{\mu}(w) = f'_i$ if $\mu(w) = f$; (iii) $\bar{\mu}(f) \in F'$ implies $\bar{\mu}(f'_i) = f'_i$ and $\bar{\mu}(w) \in W$ implies $\bar{\mu}(w) = w$. Note

that the proof of Theorem 1 utilizes only Proposition 1 and the matchmaker's profit maximization behavior given the system of net rent on each pair of firms and workers (generated from σ and s). Since Proposition 3 implies that the statement of Proposition 1 applies to μ -decomposed matching in an artificial one-to-one matching problem, Theorem 1 directly implies that the same statement holds for the matchmaker game in the many-to-one matching problem.

Theorem 3. In the many-to-one matching problem, the matchmaker's rent is zero in every Nash equilibrium.

Corollary 3. In every many-to-one matchmaker game, the individually rational correspondence $IR : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is implemented by the matchmaker game in Nash equilibria.

Now, we consider an extension of Theorem 2. Unfortunately, there is no natural way to define a truthful equilibrium in natural way (as we will see in the conclusion). We may consider the additive environment, and we may let each position act as a firm using a truthful strategy (a position-wise truthful strategy). Let's call a Nash equilibrium in position-wise truthful strategy a position-wise TNE. However, even with this device, we can only partially generalize Theorem 2.

Theorem 4. In the many-to-one matching problem, the set of outcomes of SNE is equivalent to the set of stable assignments.

Proof. From Theorem 3, all Nash equilibrium has a zero rent, so does SNE. Let (v, u, μ) be an SNE outcome, and suppose that it is not a stable assignment. Then, there is a pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $Y_f(W_f) - \sum_{w \in W_f} d_{wf} > v_f + \sum_{w \in W_f} u_w$. Consider $\sigma'_f(w) = u_w + d_{wf} - \frac{\epsilon}{2}$ for all $w \in W_f$ and $\sigma'_f(w') = 0$ for all $w' \notin W_f$, and $s'_w(f) = u_w + d_{wf} + \frac{\epsilon}{2}$ for all $w \in W_f$ and $s'_{w'}(f)$ is prohibitively high for all $w' \notin W_f$, where $\epsilon > 0$ satisfies $\epsilon < \frac{1}{|W_f|} \left[Y_f(W_f) - \sum_{w \in W_f} d_{wf} - \left(v_f + \sum_{w \in W_f} u_w \right) \right]$. Since in the SNE, the

matchmaker gets no rent, she is happy to match up f and W_f to make a positive rent. This is a contradiction. Thus, an SNE outcome is a stable assignment.

Now, let (v, u, μ) be a stable assignment. Consider the following strategy. For all $f \in F$ and all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = u_w + d_{wf}$ and $\sigma_f(w') = 0$ and $s_{w'}(f)$ is prohibitively high for $w' \notin \mu(f)$. The matchmaker chooses μ and gets zero rent. Given the strategy (σ, μ) , the matchmaker would create new match only when a pair $(f', W_{f'}) \in F \times 2^W$ with $|W_{f'}| \leq q_{f'}$ provide a nonnegative rent. However, by the definition of stable assignment, there is no pair $(f', W_{f'}) \in F \times 2^W$ with $|W_{f'}| \leq q_{f'}$ such that $Y_{f'}(W_{f'}) - \sum_{w \in W_{f'}} d_{wf'} > v_{f'} + \sum_{w \in W_{f'}} u_w$. Thus, there is no subset of players who agree to offer a nonnegative rent to the matchmaker to create a new matching. Thus, a stable assignment is supportable by an SNE. \square

We cannot get an equivalence result to TNE even in the additive environment, as we will see in the next section. Corollary 2 also partially extends.

Corollary 4. In the many-to-one matching problem, suppose that gross substitutability is assumed. Then, in every many-to-one matchmaker game, the stable assignment correspondence $SA : \mathcal{Y} \times \mathcal{D} \rightarrow \mathbb{R}^{I+J} \times \mathcal{M}$ is implemented by the matchmaker game in SNE.

Without the gross substitute assumption, SA may be empty-valued. This is why we require the assumption. Note that Corollaries 3 and 4 are not affected by the presence of quotas.

5 Concluding Remarks: Relationship with the Common Agency Games

A common agency game is a complete information multi-principal-one-agent game introduced by Bernheim and Whinston (1986). The agent chooses an agenda, but she can take side payments from each principal. Principals can affect the agent's decision by offering a menu of side payments: a side

payment schedule for agendas. The agent maximizes the sum of her own utility and side payments from the principals by choosing an agenda. We can consider our matchmaker's problem as a common agency game by interpreting a matching μ as an agenda, and letting the matchmaker intrinsically indifferent over μ (except for side payments). A **common agency game** Γ is described by $(N + 2)$ tuples:

$$\Gamma \equiv \{A, (V_k)_{k \in N \cup \{0\}}\},$$

where A is the set of actions, $V_k : A \rightarrow \mathbb{R}$ is k 's (quasi-linear) payoff function, 0 denotes an agent, and N is the set of principals. In the extensive form of the game the principals simultaneously offer contingent payments to the agent who subsequently chooses an action that maximizes her total payoff. A strategy for each principal $k \in N$ is a function $T_k : A \rightarrow [b_k, \infty)$ which is a monetary reward (or punishment) of $T_k(a)$ to the agent for selecting a , where b_k is the lower bound for the payment. For each action a , principal k receives a net payoff:

$$U_k(a) = V_k(a) - T_k(a).$$

The agent chooses an action that maximizes her total payoff. That is, given strategy profile $T = (T_k)_{k \in N}$, the matchmaker selects action in the set $M(T)$ with:

$$M(T) \equiv \arg \max_{a \in A} \left[V_0(a) + \sum_{k \in N} T_k(a) \right].$$

The common agency game is merely a game among principals, although, strictly speaking, a tie-breaking rule among $M(T)$ needs to be specified for the matchmaker.

An outcome of a game Γ is (T, a) . An outcome (a^*, T^*) is **Nash equilibrium** if $a^* \in M(T^*)$ and there is no $k \in N$ such that $T_k : A \rightarrow [b_k, \infty)$ and $A \in M(T_k, T_{-k}^*)$ such that $U_k(\mu, T_k, T_{-k}^*) > U_k(\mu^*, T^*)$ for all $k \in N$. Bernheim and Whinston [3] argued that what they called "truthful strategies" are quite crucial in this game. A strategy T_k is said to be **truthful relative to \bar{a}** if and only if for all $a \in A$ either (i) $U_k(a) = U_k(\bar{a})$ or (ii) $U_k(a) < U_k(\bar{a})$

and $T_k(a) = b_k$.¹² An outcome (a^*, T^*) is a **truthful Nash equilibrium (TNE)** if and only if it is a Nash equilibrium, and T_k^* is truthful relative to a^* for all $k \in N$. Our matchmaker game in the one-to-one assignment game can be embedded to the class of common agency game by reinterpreting players' strategies.

Proposition 5. A matchmaker game of a one-to-one assignment game can be embedded in the class of common agency games. A TNE in a matchmaker game is directly translated to a TNE in a common agency game generated from the matchmaker game, and vice versa.

Proof. Let the matchmaker be an agent. Let \mathcal{M} be the set of actions A . Firm f receives monetary payoff $V_f : \mathcal{M} \rightarrow \mathbb{R}$ where $V_f(\mu) \equiv y_{f\mu(f)}$, worker w receives monetary payoff $V_w : \mathcal{M} \rightarrow \mathbb{R}$ with $V_w(\mu) \equiv -d_{w\mu(w)}$, and the matchmaker's (denoted by 0) monetary payoff is $V_0(\mu) = 0$ for all $\mu \in \mathcal{M}$ (The matchmaker do not have any preference over matchings themselves.)

Although in the original common agency problem, principals are able to choose any contribution menu over matchings in \mathcal{M} , we restrict their strategy sets since agents only care about their partners, not about the entire matching. A strategy for firm f that is generated from its menu σ_f is a function $T_f : \mathcal{M} \rightarrow \mathbb{R}_+$, where $T_f(\mu) \equiv \sigma_f(\mu(f))$. A strategy for worker w that is generated from s_w is a function $T_w : \mathcal{M} \rightarrow \mathbb{R}_-$, where $T_w(\mu) \equiv -s_w(\mu(w))$. We can set a lower bound for the value for $T_w(\mu)$ without losing anything, since worker w would not be matched anyway, if $T_w(\mu) < -y_{\mu(w)w}$ holds. Thus, we assume that for each $k \in N = W \cup F$, there is a lower bound b_k : i.e., $T_k(\mu) \geq b_k$ must be satisfied for all $k \in N$. Thus, a game with a matchmaker can be represented as a common agency game. Clearly, a truthful strategy in σ_f or s_w can trivially be extended to a truthful strategy T_k , and vice versa. \square

Laussel and Le Breton (2001) define no-rent property as a property that the agent obtains no rent in any TNE. A common agency game possesses **no-rent property** if and only if all truthful equilibria outcome leaves no rent to

¹²Note that the constraints we imposed on T_k ($T_k(\mu) = T_k(\mu')$ if $\mu(k) = \mu'(k)$) are clearly satisfied under truthful strategies.

the matchmaker. Since TNE is a subset of Nash equilibrium, our Theorem 1 shows that our matchmaker game satisfies the no-rent property. Laussel and Le Breton (2001) prove that if a cooperative game from a common agency game is convex,¹³ then Γ possesses no-rent property. Although convexity is satisfied in interesting class of common agency games such as public good provision game, in our matching problem, convexity is clearly not satisfied.¹⁴ Theorem 1 provides another interesting class of common agency games that possesses the no-rent property.

The readers who are familiar with the literature in common agency game may find odd is that we did not mention **coalition-proof Nash equilibrium** (CPNE: Bernheim and Whinston 1986, and Bernheim, Peleg, and Whinston 1987) which is the central solution concept in common agency games.¹⁵ There is a reason for this. The standard definition of coalition-proof Nash equilibrium requires that all reduced games (the outsiders of a coalition keep their strategies fixed, and the members of the coalition play a game) belong to the same class of games. Unfortunately, however, in our game, this is not true. If outsiders make their salary offers and demand to coalition-members, then the matchmaker will have preferences over matchings she chooses. We have assumed that the matchmaker cares only profit from matching. This is the reason that we did not mention coalition-proof Nash equilibrium in this paper. However, if we allow the matchmaker to have preference over matchings, we can extend Theorem 2 and Corollary 2.¹⁶

Theorem 2’. The sets of outcomes of TNE, SNE and CPNE in our matchmaker game of the one-to-one assignment game, and the set of stable assign-

¹³A system $(V(S))_{S \subseteq N}$ is **convex** if and only if for all $S, T \subseteq N$, $V(S \cup T) + V(S \cap T) \geq V(S) + V(T)$ holds.

¹⁴For example, imagine $N = \{f_1, w_1, w_2\}$ with $y_{11} = y_{12} = 1$. Letting $S = \{f_1, w_1\}$ and $T = \{f_1, w_2\}$, we can see a violation of convexity.

¹⁵Bernheim and Whinston (1986) show that TNE and CPNE are equivalent. Under the no-rent property, Laussel and Le Breton (2001) and Konishi, Le Breton, and Weber (1999) show the equivalences of CPNE and the core of W_Γ , and CPNE and SNE, respectively.

¹⁶The proof of equivalence is very simple. By definition, a SNE is a CPNE. And the outcome of a CPNE is a stable assignment, since otherwise, there is a pair which deviates from a CPNE. But in this domain, such a two-person deviation is credible anyway.

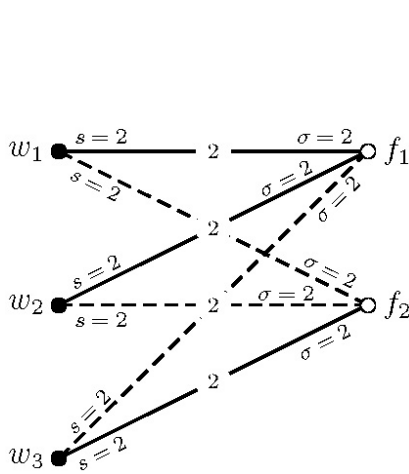
ments (the core) are all equivalent.

Corollary 2’. The set of stable matchings (the core) of the one-to-one assignment problem is implemented by our matchmaker game in CPNE and SNE.

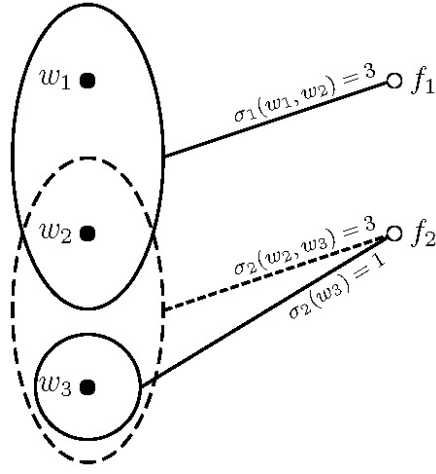
Although the above results indicate a close relationship between our matchmaker game and the common agency game, we can say that the above results are specific to the one-to-one problem. In a many-to-one matching problem, the situation is completely different. What are the differences? In a one-to-one problem, firm cares about who to be matched with: thus, a firm bids on each worker as a potential partner. In contrast, in a many-to-one problem, a firm must bid on each subset of workers. The problem of this is that there is no longer a one-to-one relationship between a salary for a worker and a bid. Moreover, the definition of TNE is no longer clear when we talk about a salary offer to each worker. In fact, as we will see below (Example 4), in the common agency game generated from a matchmaker game may leave a positive rent to a matchmaker in a Nash equilibrium or in a TNE even in an additive environment¹⁷.

Example 4. There are two firms and three workers $N = \{f_1, f_2, w_1, w_2, w_3\}$. All firms and workers are symmetric. Each firm has two positions $q_{f_1} = q_{f_2} = 2$. For all $i = 1, 2$ and all $j = 1, 2, 3$, $d_{w_j f_i} = 0$ and $y_{f_i w_j} = 2$. Unique SNE (up to permutations) in a matchmaker game is (σ, s, μ) such that $\sigma_{f_i}(w_j) = 2$ and $s_{w_j}(f_i) = 2$ for all i and j , and $\mu(f_1) = \{w_1, w_2\}$ and $\mu(f_2) = \{w_3\}$. The salaries are pinned down due to excess demand for workers. Clearly, there is no rent for the matchmaker in this case. However, in a common agency game, we have the following TNE (σ, s, μ) : $\sigma_{f_i}(\{w_j\}) = 1$ for all i and j , and $\sigma_{f_i}(\{w_j, w_{j'}\}) = 3$ (if firm f_i is willing to pay 3 in total if it is matched

¹⁷An **additive environment** (Alcalde et al. 1998) is such that for all $f \in F$ and all $W_f \subseteq W$ with $|W_f| \leq q_f$, $Y_f(W_f) = \sum_{w \in W_f} y_{fw}$ holds. This environment has neither complementarity nor substitutability, and it can be considered as the simplest way to extend the one-to-one matching results in the many-to-one domain.



Example 4a: Unique SNE with zero rent in matchmaker game



Example 4b: Positive rent TNE under common agency game

with subset $\{w_j, w_{j'}\}$ for all i, j , and j' , and $w_j(f_i) = 1$ for all i and j , and $\mu(f_1) = \{w_1, w_2\}$ and $\mu(f_2) = \{w_3\}$. This is a TNE, since firms are indifferent between hiring one or two workers. However, the matchmaker receives a rent 1 from f_1 . Note that firms are better off in this TNE in common agency game: they obtain positive profits.

This example shows that a common agency interpretation of our matchmaker game is valid only in the case of one-to-one assignment game. However, note that the efficient matching is achieved in every coalition-proof nash equilibrium of the common agency game, although the matchmaker earns a positive rent (Bernheim and Whinston 1986).

One may wonder that we may be able to restore the equivalence results under additive environment by applying TNE to a one-to-one problem played by positions and workers instead of firms and workers. The following simple example show that the set of stable assignments and the outcome set of position-wise TNE are different.

Example 5. Consider a **position-wise TNE**, that is, each position announce a truthful strategy relative to its partner worker. Let $F = \{f\}$ and $W =$

$\{w_1, w_2\}$ with $q_f = 2$. Let $y_{fw_1} = 5$ and $y_{fw_2} = 2$. Clearly, the set of stable assignments is $\{(v, u_1, u_2) \in \mathbb{R}_+^3 : v + u_1 + u_2 = 7, u_1 \leq 5, \text{ and } u_2 \leq 2\}$. However, if we consider position-wise TNE, positions compete with each other, and the set of outcomes is $\{(v, u_1, u_2) \in \mathbb{R}_+^3 : v + u_1 + u_2 = 7, 3 \leq u_1 \leq 5, 0 \leq u_2 \leq 2, \text{ and } u_1 = u_2 + 3\}$.

These two examples show that a common agency game generated from many-to-one matching problem has very different properties from the one-to-one matching problem. Thus, we can conclude that our matchmaker game is generally different from common agency game.

References

- [1] Alcalde, J., D. Perez-Castrillo, and A. Romero-Medina, 1998, Hiring procedures to implement stable allocations, *Journal of Economic Theory* 82, 469-480.
- [2] Aumann, R.J., Acceptable points in general cooperative n -person games, in *Contributions to the theory of games (Vol. IV)*, Princeton University Press (Princeton), 1959.
- [3] Bernheim, D., and M. Whinston, 1986, Menu auctions, resource allocation, and economic influence. *Quarterly Journal of Economics* 101, 1–31.
- [4] Bernheim, D., B. Peleg, and M. Whinston, 1987, Coalition-proof Nash equilibria I: concepts, *Journal of Economic Theory* 42, 1-12.
- [5] Hayashi, T., and T. Sakai, 2009, Nash implementation of competitive equilibria in the job-matching market, forthcoming in *International Journal of Game Theory*, doi:10.1007/s00182-009-0163-8.
- [6] Kelso, A.S., and V. Crawford, 1982, Job Matching, Coalition Formation, and Gross Substitutes, *Econometrica* 50, 1483-1504.

- [7] Konishi, H., M. Le Breton, and S. Weber, 1999, On coalition-proof Nash equilibria in common agency games, *Journal of Economic Theory* 85, 122-139.
- [8] Laussel, D., and M. Le Breton, 2001, Conflict and cooperation: the structure of equilibrium payoffs in common agency, *Journal of Economic Theory* 100, 93-128.
- [9] Roth, A., and M. Sotomayor, *Two-sided matching: a study in game-theoretic modeling and analysis* (ESM 18), Cambridge University Press (New York), 1990.
- [10] Shapley, L.S., and M. Shubik, 1971, The assignment gameI: the core. *International Journal of Game Theory* 1, 111-130.