Signaling in Matching Markets

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Abstract

We evaluate the effect of preference signaling in two sided matching markets. Firms and workers have strict preferences over members of the other side of the market. Each firm makes an offer to exactly one worker. Workers select the best offer from those available to them. The short time frame produces congestion and the market fails to reach a stable outcome. But if workers are able to signal their preferences firms may use this information as guidance for their offer choices. We find that in this signaling setting, it is optimal for firms to make use of these signals in the form of cutoff strategies. However, making use of signals imposes a negative externality on other firms. We find that on average, introducing a signaling technology increases the average number of matches, one possible measure of social welfare. We also evaluate the effect of preference signaling for markets with various structures.

1 Introduction

Many entry-level labor markets, as well as many markets for educational positions share the feature that applicants become available at the same time and search for positions in the near future. An outcome in these markets is stable if no applicant-employer pair would prefer a match with each other to their current match, and no participant would prefer to unilaterally dissolve its pairing and remain unmatched. To ensure stability, significant information about preferences needs to be processed. In many settings, market frictions limit the amount of information that can be processed, so that stable outcomes are an unrealistic hope. However, these frictions may be mitigated by allowing applicants to signal preference information to potential employers. By introducing a signaling mechanism, where signals are (by construction) scarce, and hence credible, we may achieve a second best outcome. In this paper, we model such a signaling mechanism, analyzing behavior and welfare implications. Ultimately we hope the model will provide insight and policy recommendations as to when such a mechanism might be useful, and what form a signaling mechanism should take.

1.1 Congestion in Labor Markets

In two-sided matching markets (e.g. with firms matched to workers, or men to women), the amount of information about preferences that must be transmitted to guarantee that a particular matching is stable can be significant. In a market where firms make costless proposals to workers, in a stable matching, each firm must be convinced that no worker better than its current match would prefer to renege on its match and pair with the firm instead. Segal [7] points out that the minimal informational requirements can be achieved via a firm-proposing deferred acceptance algorithm (see e.g. Roth & Sotomayor [5]). This means that every firm must have made an offer, and been rejected by, every worker it prefers to its current match. To get an idea of the number of offers that must be made, observe that with N workers and N firms with uncorrelated preferences over each other, the expected number of offers is approximately $N \cdot H_N$, where H_N is the Nth harmonic number (Knuth, [4]). When $N = 100$, $100 \times H_{100} \approx 500$ offers must be made. Since offers are often made sequentially, the time to reach a stable matching can be significant (Roth and Xing, [6]).

There are many reasons we may not expect markets to solve this problem, that is, to fully extract all the information necessary to achieve a stable matching. When a market fails to clear because market frictions prevent sufficiently many transactions from being considered, the market experiences *congestion.* A congested labor market may not have enough time to clear simply because, for example, classes start in college, or a job start date might be approaching. Without a fixed endpoint, employers may still experience time constraints, as applicants may begin accepting offers elsewhere on which they cannot costlessly renege. Even when a market experiences no time restrictions, an employer may find itself subject to frictions. For example, offers may be costly, or in the extreme case a firm may be restricted in the number of offers it may make.[1]

In markets where congestion is a problem, employers often face a tradeoff between making an offer to their most desirable candidate (who is still available) and a less desirable candidate who is more likely to accept an offer. In such markets, we might expect applicants to convey their willingness to accept an offer. We term such conveyance of information *preference signaling.* In a market with no frictions, and where all worker match values are known to the firms, applicants' indications of preferences are not useful to the firm.[2] Furthermore, if applicants can privately and costlessly convey such preference information to all employers, this information will not be credible and will again be useless.

Signaling in Practice

There are numerous examples of markets that are prone to congestion, and in many of these some form of preference signaling takes place. In certain markets, formal signaling mechanisms have been introduced. In others, we witness informal signaling, where due to reputation or other considerations, signals may still be credible.

An example of a market where preference signals are unambiguously interpreted by employers as a measure of likelihood of acceptance (and do not convey information about the value of a match) is the market for clinical psychologists as described by Roth and Xing [6].

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[1] For example, departments may face deans who argue that after four rejections, making an offer to a fifth choice candidate should be ruled out, because 'we are not a fifth choice candidate institution.'

[2] For the base model in this paper, we assume that the information gathering stage has taken place, so that firm valuations of candidates are known. Hence, applicant signals of preferences are only valuable to firms in that they can help gauge the likelihood of offers being accepted.

From 1973 to 1998 the market operated under very specific rules that essentially emulated a deferred acceptance algorithm with offers and acceptances made over telephone. In this market, program directors for internships in clinical psychology competed for doctoral students. The rules imposed a uniform time regime in which offers could be made, with fixed start and end times.[3] Because the market operated in a decentralized manner, and offers took time, the outcome of the market was not stable. Indeed, some programs found themselves rejected close to the end, and did not have enough time to make backup offers during the operation of the market. The aftermarket was also heavily regulated, and included only applicants who had not accepted an offer elsewhere. A program that found itself rejected close to the end of the market risked being unable to fill its slot with a desirable applicant. This illustrates the market congestion; time could run out before programs had a chance to make offers to all candidates they are interested in. Based on a site visit in 1993, Roth and Xing described the behavior of one program and its program directors on selection day, which that year lasted from 9 am to 4 pm. The program had 5 positions to fill, and a rank order list of 20 acceptable candidates. The co-directors said that their general strategy in the market was to not "tie up offers with people who will hold them the whole day." At the beginning of the market they made offers to candidates 1, 2, 3, 5 and 12, where 3, 5 and 12 had indicated they would accept an offer immediately. Candidates 1 and 2 were deemed attractive enough to be worth taking a chance on. Later in the day, other candidates called to report that the program was now the highest ranked program on their list, and indeed, the co-directors of the program decided to make the offers that were turned down by candidates 1 and 2 to the best of those candidates. The program eventually hired candidates 3, 5, 8, 10 and 12, all of which had indicated they would accept an offer immediately. Only candidates 1 and 2 received offers without having signaled their intent. Candidates 4, 6, 7, 9 and 11 (who were all preferred to some of the applicants eventually hired) were never made an offer. Roth and Xing summarize this episode by remarking on the program directors' concern over making offers which ran the risk of being rejected late in the day, the consequent attention paid to candidates who indicated that they would accept immediately, and the willingness of candidates to convey such information.

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[3] In its later stages, the interval was a seven hour period within a single day.

An important example of preference signaling is the U.S. college admissions market, in which early application to a college can be interpreted as a preference signal. For many colleges this is a market with two application periods, one early and one 'regular.' Although early application rules vary by college, many schools require that early applicants not send early applications to other schools, so that students are often faced with choosing exactly one school to which they can apply early. A distinction across schools is the commitment implications of early application. Some colleges use 'early action,' in which applicants may apply early but without any commitment to accept an offer from the college should they receive one. Other colleges use 'early decision' programs wherein students apply early and commit to matriculation should they be admitted. These programs, although varying by school, represent a formal way of sending a credible preference signal.[4] Avery et. al [1] examine this market in detail, and show that both early action and early decision applications result in a higher chance of admission than do regular applications.

In 2006, the American Economic Association introduced a mechanism to allow candidates in the economics job market to signal their interest to employers. By using an interface on the AEA website, candidates can select up to two employers to whom to send signals. The AEA relays these signals to employers in early December. Most employers weigh these signals in their decisions about whom to interview during the job market interviews in the AEA meetings. Analysis of data from the first three years of operation suggests that signals are helpful in securing job interviews, especially when used strategically. Note that the congestion problem in this job market stems from the very low cost of applying for positions (resulting in several hundred applications per position), combined with employer time and resource constraints that prohibit them from interviewing, flying out, and making offers to all the candidates they find desirable.

In this paper we provide an explicit model of a labor market with frictions, and consider the effects of introducing a signaling mechanism. We examine the strategic behavior of firms and workers in the presence of such a mechanism, as well as welfare effects and comparative statics. We find that when preferences are uncorrelated it is optimal for firms to make use

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[4] We also observe informal signaling in this market during the regular admissions process via college visits, contact with the admissions office, etc.

of these signals in the form of cutoff strategies. However, making use of signals imposes a negative externality on other firms. We find that on average, introducing a signaling technology increases the average number of matches, one possible measure of social welfare. We hope that the model may provide insight into policy recommendations for second best solutions in congested markets.

2 Model

In this paper, a *market* will consist of a set of firms, a set of workers, and a distribution over firm and workers preferences. We will examine behavior in the market in two congested settings. In the first setting, which we term the *offer game with no signals*, firms will make offers to workers based on a limited knowledge of worker preferences. In the second setting, *the offer game with signals*, before offers are made, workers will have the opportunity to send a signal of their preferences to the firms, who in turn may use these signals to inform their offers.

Let $\mathcal{F} = \{f_1, \ldots, f_F\}$ be the set of firms, and $\mathcal{W} = \{w_1, \ldots, w_W\}$ be the set of workers, with $|\mathcal{F}| = F$ and $|\mathcal{W}| = W$. Firms and workers, indexed by i and j respectively, have preferences over each other as follows. Let Θ_f be the set of all possible firm preference lists over the workers (or rank order lists or ROLs), and let Θ_w be the set of all possible worker preference lists over the firms. Lists $\theta_f \in \Theta_f$ and $\theta_w \in \Theta_w$ are vectors of length W and F respectively. We sometimes refer to an agent that has rank r in agent a's preference list as θ_a^r. Define $\Theta_F = (\Theta_f)^F$ and $\Theta_W = (\Theta_w)^W$, and let $\Theta \equiv (\Theta_f)^F \times (\Theta_w)^W$ indicate preference list profiles. Let $t(\cdot)$ give the distribution over preference list profiles.

Each firm has the capacity to hire at most one worker, and each worker can fill at most one position.[5] Firm f with preferences θ_f values a match with worker w as $g(\theta_f, w)$, where $g(\theta_f, \cdot)$ is a von-Neumann Morgenstern utility function. We assume that firm utility of a match depends only on a worker's rank. That is, for any permutation σ, we have $g(\sigma(\theta_f), \sigma(w)) = g(\theta_f, w)$.[6] We will sometimes abuse notation and write $g(\theta_f, \theta_f^j)$ as $g(j)$,

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[5] Analysis of the case where firms may hire more than one worker and when workers can fill more than one position (e.g. if positions correspond to interviews of workers by firms) can be found in the Appendix.

[6] Let $\sigma : \{1, \ldots, N\} \rightarrow \{1, \ldots, N\}$ be a permutation. Abusing notation, we apply σ to preference lists,

firm utility from matching with its jth ranked worker. Utility to worker j from matching with firm i is given by $h(\theta_w, f)$, where match utility again depends on rank. Though not essential for the results, we will assume that workers and firms derive zero utility from being unmatched, and that any match is preferable to remaining unmatched for all participants.

Definition 1 *A* market *is given by the 5-tuple* $\langle \mathcal{F}, \mathcal{W}, t, g, h \rangle$.

We will consider a special type of market: the *block correlated market.*

Definition 2 *A* block correlated market *is a market* $\langle \mathcal{F}, \mathcal{W}, t, g, h \rangle$ *such that for a partition* $\mathcal{F}_1, \ldots, \mathcal{F}_B$ *of the firms into* blocks, *ordinal preferences (as encompassed in* $t(\cdot)$*) are such that*

1. *For any* $b < b'$, *where* $b, b' \in \{1, \ldots, B\}$, *each worker prefers every firm in* $\mathcal{F}_b$ *to any firm in* $\mathcal{F}_{b'}$.
2. *Each worker's preferences within block* F_b *are uniform and uncorrelated, for all* b.
3. *Firm preferences over workers are uniform and uncorrelated.*

We call distributions $t(\cdot)$ that satisfy the criteria in definition 3 *block uniform.* Block correlated markets are meant to capture the notion that in many two-sided markets, agents may largely agree on the ranking of agents on the other side, but have idiosyncratic preferences within segments of the market. For example, workers might agree on the set of firms that constitute the 'top tier' of the market, but within that tier, preferences are influenced by factors specific to each worker.

2.1 The Offer Game with No Signals

We first examine behavior in a congested market in the absence of a signaling mechanism. Play proceeds as follows: First, preferences are realized for firms and workers. Next, each firm

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workers, and sets of workers such that the permutation applies to the worker indices. For example, suppose $N = 3, \sigma(1) = 2, \sigma(2) = 3$ and $\sigma(3) = 1$. Then we have $\theta_f = (w_1, w_2, w_3) \Rightarrow \sigma(\theta_f) = (w_2, w_3, w_1)$ and $\sigma(w_1) = w_2$.

makes an offer to at most one worker, and offers are made simultaneously. Finally, workers choose at most one offer from those available to them. Sequential rationality ensures that workers will always select the best offer from those available to them. Hence, we assume this behavior for the workers and focus on the reduced game with only firms as players.

The congestion in this market is a direct consequence of our limitation to a single round of offers. Were firms able to make as many offers as they wished, and workers able to collect and compare all offers, firms could simply run down their preference lists, making a new offer each time an offer is rejected. In this setting, however, multiple firms may make offers to the same worker. The worker will reject the less-preferred of these two firms, leaving that firm unmatched. On the other hand, some otherwise available workers might be interested in getting offers, but firms, unable to coordinate, overlap in their offers to other workers instead.

To analyze behavior and outcomes, we model this setting as a Bayesian game. Firm types are simply their preferences θ_F, drawn from a block uniform distribution. Upon realization of its preference list, a firm must select a worker, if any, to make an offer. A mixed strategy for a firm f is a map from the set of preferences lists to the set of distributions over the union of workers with the no-offer option (denoted by $\mathcal{N}$), $s_f : \Theta_f \rightarrow \Delta(\mathcal{W} \cup \mathcal{N})$.[7] We denote a profile of all firms' strategies as $s_F = (s_{f_1}, ...s_{f_F})$.

We focus on strategies that depend on workers' rank within a firm's preference list (ruling out strategies that rely on worker index):

Definition 3 *Firm f's strategy s_f is* anonymous *if for any permutation $\sigma \in \Sigma$, and for any preference profile $\theta_f \in \Theta_f$, we have $s_f(\sigma(\theta_f)) = \sigma(s_f(\theta_f))$.*

As an example, 'always make an offer to my second ranked worker' is an anonymous strategy, whereas 'always make an offer to w_2 is not.

Let us denote function $\pi_f : \left(\Delta^W\right)^F \times \Theta \rightarrow \mathbf{R}$, the payoff to firm f as a function of firm strategies and realized agents' types. We are now ready to define a Bayesian Nash equilibrium of the offer game with no signals.

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[7] In other words, firms select elements of a W-dimensional simplex; $s_f(\theta_f) \in \Delta^W$, where $\Delta^W = \{x \in R^{W+1} : \sum_{i=1}^{W+1} x_i \leq 1$, and $x_i \geq 0$ for each $i\}$.

Definition 4 *Strategy profile $\hat{s}_F$ is a Bayesian Nash equilibrium if for all $f \in \mathcal{F}$ and $\bar{\theta}_f \in \Theta_f$ we have*

$$\hat{s}_f(\bar{\theta}_f) \in \arg \max_{s_f \in \Delta^{W-1}} \mathbb{E}_{\theta_{-f}}(\pi_f(s_f, \hat{s}_{-f}, \theta) \mid \theta_f = \bar{\theta}_f)$$

Due to the uniform distribution of firm preferences, each firm optimally makes an offer to the highest ranked worker on its preference list. The following theorem formally states this result.

Theorem 1 *If the distribution of agent preferences is block uniform, the unique equilibrium of the offer game with no signals is $s_f(\theta_f) = \theta_f^1$ for all $f \in \mathcal{F}$ and $\theta_f \in \Theta_f$.*

2.2 The Offer Game with Signals

We now modify the game so that each worker may send a '*signal*' to exactly one firm. The game now proceeds in three stages.

1. Agents' preferences are realized. Each worker sends a signal to at most one firm; signals are sent simultaneously. Signals are observed only by firms who have received them.
2. Each firm makes an offer to at most one worker; offers are made simultaneously.
3. Each worker may accept at most one offer from the set of offers she receives.

Once again, sequential rationality ensures workers will always select the best offer from those available to them. Hence, we assume this behavior for workers and focus on the reduced game consisting of the first two stages.

In the first stage, a worker sends a signal to a firm, or else chooses not to send a signal. A mixed strategy for worker w is then a map from the set of all possible preference lists to the set of distributions over the union of firms and the no-signal option, denoted $\mathcal{N}$, $s_w : \Theta_w \to \Delta(\mathcal{F} \cup \mathcal{N})$.

In the second stage, each firm decides which worker, if any, to make an offer based on the firm's preference profile and the set of signals it receives in the first stage. A mixed strategy of firm f is then a map from the set of all possible preference lists and the set of all possible profiles of signals to the set of distributions over the union of workers and the no-offer option, again denoted $\mathcal{N}$. That is, $s_f : \Theta_f \times H_W \rightarrow \Delta(\mathcal{W} \cup \mathcal{N})$, where $H_W = \{h : h \subset \mathcal{W}\}$ is the set of all subsets of workers. We denote a profile of all worker and firm strategies as $s_F = (s_{w_1}, ...s_{w_W})$ and $s_F = (s_{f_1}, ...s_{f_F})$ correspondingly.

We again require that all agents have anonymous strategies:

Definition 5 *Firm f's strategy s_f is* anonymous *if for any permutation $\sigma \in \Sigma$, preference profile $\theta_f \in \Theta_f$, and subset of workers $h \subset \mathcal{W}$, we have $s(\sigma(\theta_f), \sigma(h)) = \sigma(s(\theta_f, h))$. Worker w's strategy s_w is* anonymous *if for any permutation $\sigma \in \Sigma$, preference profile $\theta_w \in \Theta_w$ we have $s_w(\sigma(\theta_w)) = \sigma(s_w(\theta_w))$.*

As the offer game with signals is a multi-stage game of incomplete information, we consider sequential equilibrium as solution concept.

Definition 6 *Strategy profile $(\hat{s}_W, \hat{s}_F)$ and posterior beliefs $\hat{\mu}_f(\cdot|h)$ for each firm f and each subset of workers $h \subset \mathcal{W}$ are a sequential equilibrium if:*

- *for any $w \in \mathcal{W}$, $\bar{\theta}_w \in \Theta_W : \hat{s}_w(\bar{\theta}_w) \in \arg\max_{s_w \in \Delta^F} E_{\theta_{-w}}(\pi_w(s_w, \hat{s}_{-w}, \theta) \mid \theta_w = \bar{\theta}_w)$*
- *for any $f \in \mathcal{F}$, $\bar{\theta}_f \in \Theta_f$, $\bar{h} \subset \mathcal{W} : \hat{s}_f(\bar{\theta}_f, \bar{h}) \in \arg\max_{s_f \in \Delta^W} E_{\theta_{-w}}(\pi_f(s_f, \hat{s}_{-f}, \theta) \mid \theta_f = \bar{\theta}_f,\ h_f = \bar{h}, \mu_f = \hat{\mu}_f)$*

where $\hat{s}_{-a}$ denotes the strategies of all agents except a. For any $f \in \mathcal{F}$ and $h \subset \mathcal{W}$, $\hat{\mu}_f(\theta|h) = \prod_{w \in W} \hat{\mu}_f(\theta_w|h)$, and beliefs are defined using Bayes' rule.[8]

3 Equilibrium Analysis

Theorem 1 established the unique equilibrium of the offer game with no signals, in which firms straightforwardly make offers to their most preferred workers. From this point on, we consider the offer game with signals, where firm and worker strategies are more subtle.

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[8] As usual in a sequential equilibrium, permissible off-equilibrium beliefs are defined by considering the limits of completely mixed strategies.

Each firm that receives signals faces a fundamental tradeoff: it can make an offer to a higher ranked worker who has not signaled it, or else to a lower ranked worker who has sent a signal, and who is potentially more likely to accept the offer.

In this section we study the nature of this tradeoff, focusing on how it dictates the structure of firm strategies, namely that firms play intuitive strategies we call *cutoff strategies*. We further examine how this tradeoff is impacted by opponents' strategies, finding that strategies of firms are strategic complements: a firm that moves to favor one of the choices mentioned above induces other firms to favor the same choice. Finally, we demonstrate the existence of an equilibrium, and describe both firm and worker strategies in equilibrium.

We are interested in equilibria when firms that are within each block use the same strategy and all workers use the same strategy. Similar, firms that are within each block have the same beliefs. We call such firms' strategies and firms' beliefs *block symmetric*. We call the equilibria where firms' strategies and beliefs are block symmetric and worker strategies are symmetric *block-symmetric equilibria*.

Let us consider some block-symmetric sequential equilibrium. Since, firms within some block $b \in \{1, ..., B\}$ have the same anonymous strategy, we can denote the exante probability of receiving an offer by some worker from a firm within block b, conditional on the worker sends a signal to it, as q^b. Similarly, we can denote the exante probability of receiving an offer by some worker from a firm within block b, conditional on the worker does not send a signal to it, as p^b. We also denote the probability that some worker sends her signal to a firm within block b in the considering block-symmetric sequential equilibrium as α_b, such that $\alpha_b \in [0, 1]$ and $\sum_{b=1}^{B} \alpha_b \leq 1$. The following proposition characterizes all block-symmetric sequential equilibria that satisfies criterion $D1$ of Cho and Kreps [3][9].

Proposition 1 *Let us consider some block-symmetric sequential equilibrium that satisfies criterion D1 than either*

1. *for any* $b = 1, ..., B$, $q^b = p^b$ *or*

2. *there exists* $b_0 \in \{1, ..., B\}$ *such that* $q^{b_0} > p^{b_0}$ *and*

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[9] See proof of proposition 2 in the Appendix for the definition of criterion D1 or [3]. The statement of the proposition is also correct if we use "universal divinity" of Banks and Sobel [2] instead of criterion D1.

- *for any* $b \in \{1, ..., B\}$ *such that* $\alpha_b > 0$*, we have that* $q^b > p^b$ *and if worker sends her signal to block* b *she sends her signal to her top firm within block* b*.*
- *for any* $b' \in \{1, ..., B\}$ *such that* $\alpha_{b'} = 0$*, the off-equilibrium beliefs of each firm* $f \in F^{b'}$ *are such that* $\mu_f(\Gamma | w \subset h_f) = 1$*, where* $\Gamma = (\theta_w \in \Theta_w : f = \underset{\theta_w}{max}(f' \in F^b))$*.*

Proposition 4 proves that there are two types of block-symmetric equilibria that satisfy criterion D1. Equilibria of the first type are babbling, where firms ignore signals. The outcomes of these equilibria coincide with the outcome in the offer games with no signals. Consequently, the signaling mechanism adds no value in this case. In equilibria of the second type, workers send signals only to their most preferred firm in each block, possibly mixing across these top firms. We call such worker strategies *top-firm strategies.* Moreover, if in equilibrium workers are not prescribed to signal to some block b', the off-equilibrium beliefs of each firm f in b' upon receiving a signal from worker w are such that firm f believes it is nevertheless w's most preferred firm within block b'. We call such off-equilibrium beliefs *top-worker beliefs.*

Consider a firm f that has received signals from the set of workers $h \subset \mathcal{W}$. Call f's most preferred worker overall TRW_f (top-ranked worker) and f's most preferred worker among firms h that have signaled it TSW_f (top-signaled worker). Since these depend on f's preferences, we sometimes write $TRW_f(\theta_f)$ and $TSW_f(\theta_f)$.

The next proposition states that firm f will optimally make an offer to TRW_f or to TSW_f, provided firms $-f$ use anonymous strategies and workers use symmetric, top-firm strategies.

Proposition 2 *Suppose firms* $-f$ *use anonymous strategies and workers use symmetric, top-firm strategies. Consider firm* f *that receives signals from workers* $h \subset W$*. Then the expected payoff to* f *from making an offer to* TSW_f *is strictly greater than the payoff from making an offer to any other worker in* h*. The expected payoff to firm* f *from making an offer to* TRW_f *is strictly greater than the payoff from making an offer to any other worker from set* $W \backslash h$*.*

The symmetry of worker strategies and anonymity of firm strategies dictates that for any two workers $w, w' \in h$, f's expectation that these workers will accept an offer is identical. Hence, if f makes an offer to a worker in h, it should make that offer to its most preferred worker. The same logic holds for workers $\mathcal{W} \backslash h$.

Proposition 2 shows that the tradeoff made by firms is reduced to a binary one between a firm that it ranks highly and a lower-ranked firm, that as we will see, it may find more likely to accept an offer.

In order to understand the behavior of firms, we define a special type of strategy, the *cutoff strategy.*

Definition 7 (Cutoff Strategies) *Strategy s_f is a* cutoff strategy *for firm f if $\exists j_1, \ldots, j_W \in \{0, \ldots, W\}$ such that for any $\theta_f \in \Theta_f$ and any $h \subset W$,* $s(\theta_f, h) = \begin{cases} TSW_f(\theta_f) & \textit{if } \textit{rank}_{\theta_f}(TSW_f(\theta_f)) \leq j \\ TRW_f(\theta_f) & \textit{otherwise.} \end{cases}$

We call $(j_1, \ldots, j_W)$ f's *cutoff vector*, which has as its components *cutoffs* for each possible number of signals received.

For a firm f that employs a cutoff strategy, conditional on the number of signals it has received, the firm need only look at the rank of the highest ranked worker that signaled. If the rank of this worker is above a certain cutoff, then the firm makes an offer to TSW_f. Otherwise the worker makes an offer to TRW_f.

While we have defined cutoffs as integers, we can extend the definition to include all real numbers in the range $(1, W)$ by letting cutoff $j + \frac{a}{b}$ correspond to mixing between cutoff j and cutoff $j+1$ with probabilities $(1 - \frac{a}{b})$ and $\frac{a}{b}$ respectively. Note that this is equivalent to f making offers to TSW_f with rank higher than j, randomizing between TRW_f and TSW_f when TSW_f has rank exactly j, and making offers to TRW_f otherwise.

Cutoff strategies seem reasonable; if it is optimal for firm i to make an offer to his TSW, it seems that in a scenario where i has a better TSW, that worker too should receive an offer. The caveat is that worker signals provide information about the signals other firms receive, affecting their behavior, and hence the optimal decision for firm i. In fact, when we relax the anonymity requirement for firm strategies, we can generate examples of where optimal strategies are not cutoff strategies.

The above example depends heavily on the non-anonymous nature of firm strategies. The next proposition states that provided firms $-f$ use anonymous strategies and workers use symmetric, top-firm strategies, it is optimal for a firm to play cutoff strategies.

Proposition 3 (Optimality of Cutoff Strategies) *Suppose workers use symmetric, top-firm strategies. Then for any strategy s_f for firm f, there exists a cutoff strategy which provides f weakly higher expected payoff than s_f for any anonymous strategies s_{-f} of opponent firms $-f$.*

Under the assumptions of the proposition, when K workers have signaled f, f expects an offer to any of these workers to be accepted with equal probability, and this probability does not depend on the identity of the workers. Furthermore, f's expectation that an offer to TRW_f will be accepted depends only on K, and not on the identity of the workers. Hence, if f finds it optimal to make an offer to TSW_f, it will certainly make an offer to TSW_f with higher rank, provided the number of signals received is the same.

Propositions 2 and 3 together put structure on optimal firm strategies. For any set of signals a firm f receives, it will always make an offer to one of two workers: TSW_f or TRW_f. Which of these two workers to select is dictated by the firm's cutoff.

We now examine how a firm should adjust its behavior in response to changes in the behavior of opponents. We find that responding to signals is a case of strategic complements. One consequence of this result is that strategic complements can very well allow for multiple equilibria. We will see in section 5 that in the case of a single block, where firms are the only meaningful strategic players, multiple equilibria can exist and, in fact, can be Pareto ranked.

Proposition 4 (Strategic complements) *Suppose workers play symmetric, top-firm strategies, and firms $-f$ use cutoff strategies. If firm $f' \in -f$ increases its cutoffs (responds more to signals), firm f will also optimally weakly increase its cutoffs.*

When opponent firms make offers to workers who have signaled to them, making an offer to a worker who has not signaled you is particularly risky. This worker has signaled another firm (in your or in another block) that is very much inclined to make her an offer. The greater this inclination on the part of your opponents, the riskier it is for you to make offers to your TRW, and hence, the more inclined you are to make an offer to your TSW as well.

The last result of the section establishes the existence of equilibria in the offer game with signals. The theorem relies on proposition 3, the optimality of cutoffs result, in that we first demonstrate equilibrium existence when requiring firms to use cutoff strategies. We then use proposition 3 to show that this step is not restrictive.

Theorem 2 *There exists a block-symmetric equilibrium where*

1. *workers play a symmetric, top-firm strategy*
2. *firms play block-symmetric, anonymous, cutoff strategies.*

4 Signals and agents' welfare

In this section we focus on theoretical results about welfare of the agents. We first analyze the welfare impact of a single firm adjusting its strategy to respond more to signals. Building on this result, we draw conclusions about both the number of matches and the welfare of agents in equilibrium.

Denote function $\mu : \left(\Delta^{W-1}\right)^F \times \left(\Delta^{F-1}\right)^W \times \Theta \to \mathbf{R}$ that yields the expected total number of matches in the market as a function of agents' strategies and agents' types.

Proposition 5 (Number of Matches) *Assume agents employ block-symmetric strategies. Fix strategies of firms* $-f$ *as* $s_{-f}(\cdot)$*. Let firm* f*'s strategy* $s_f(\cdot)$ *differ from* $s'_f(\cdot)$ *only in that for some number of signals* l *and for any preference profile* $\theta_f \in \Theta_f$*,* $s_f(\theta_f, |h| = l) = TRW_f(\theta_f) = \theta_f^1$ *while* $s'_f(\theta_f, |h| = l) = TSW_f(\theta_f, h)$*. Then*

$$\mathbb{E}_\theta[\ |\mu(s_f(\theta), s_{-f}(\theta), \theta)|\ ] \leq \mathbb{E}_\theta[\ |\mu(s'_f(\theta), s_{-f}(\theta), \theta)|\ ].$$

To summarize the intuition in the proof, observe that when a firm switches his offer his TRW to his TSW, it is the outside offers of these workers that determine the impact on the total number of matches. If both firms have outside offers, or if neither has an outside offer, the number of matches is unaffected. When exactly one firm has an outside offer, it is more likely to be the TRW, as this worker has signaled to another firm, while the TSW has not. Hence, by making an offer to its TSW, the firm maximizes the expected total number of matches. In addition to increasing the expected number of matches, response to signals also unambiguously increases workers' welfare.

Proposition 6 *Assume that agents employ some block-symmetric strategies. Let us fix strategies of all firm except firm* f*,* $s_{-f}(\cdot)$*. Let firm* f*'s strategy* $s_f(\cdot)$ *differs from* $s'_f(\cdot)$ *only in that for some number of signals* l *and for any preference profile* $\theta_f \in \Theta_f$ $s_f(\theta_f, |h| = l) = TRW_f(\theta_f) = \theta_f^1$ *while* $s'_f(\theta_f, |h| = l) = TSW_f(\theta_f, h)$*.For each worker* $w \in W$ *and each profile* $\theta_w = \bar{\theta}_w$ *the expected worker* w*'s payoff is greater when firm* f *response more to signals*

$$\mathbb{E}_\theta[\pi_w(s_i(\theta), s_{-i}(\theta), \theta)|\theta_w = \bar{\theta}_w] \leq \mathbb{E}_\theta[\pi_w(s'_i(\theta), s_{-i}(\theta), \theta)|\theta_w = \bar{\theta}_w]$$

As a corollary of the above propositions we can compare the welfare of workers and the expected number of matches in the unique equilibrium of the offer game with no signals and a block-symmetric equilibrium of the offer game with signals. Similar to the case of uniform distribution of preferences, we have an ambiguous effect of the introduction of signals on the welfare of firms. Basically, the signals in the offer game with signals are cheap talk and there always exist an equilibrium when signals are ignored. What is interesting is to compare the welfare properties of new equilibria. The theorem stated below summarizes these results.

Theorem 3 *The expected number of matches and the expected welfare of workers in a block-symmetric equilibrium in the offer game with signals are greater than the corresponding parameters in the unique equilibrium of the offer game with no signals. The welfare of firms in these equilibria are incomparable, i.e. the welfare of firms in a block-symmetric equilibrium the offer game with signals could be greater or smaller than in the unique equilibrium of the offer game with no signals, depending on cardinal utility of firms.*

5 Equilibrium Ranking in a Single Block

We analyze the properties of the markets with one block of firms in this section. Now, workers' preferences are uniformly distributed. As previously, firms' preferences are also uniformly distributed. As a corollary of proposition 2, we get that either all workers are indifferent which firm to send a signal or they send their signals to their top K firms in any symmetric equilibrium. As a consequence of proposition 3, we get that if all workers send their signals to their top K firms and we fix the strategies of all firms except firm f, then there exists a cutoff strategy for firm f that is optimal in the set of all possible strategies. The strategic complements result of proposition 4 also hold. However, we are able to obtain a stronger existence result for single block markets. The reason for this is that its an optimal for workers to send their signals to top K firms for any symmetric cutoff strategy of firms. The optimal strategies of workers can change in many block markets and do not change in single block markets.

Theorem 4 *There exists a symmetric equilibrium in pure cutoff strategies when 1) workers send their signals to top K firms and 2) firms play a symmetric cutoff strategy. The set of symmetric equilibria in pure cutoff strategies constitutes a complete lattice*

Note that the above theorem states the existence a symmetric equilibrium in pure strategies, compare to the statement of theorem *4* that establishes the existence of mixed strategy equilibria.

The lattice structure of symmetric cutoff strategies equilibria is a consequence of proposition 4 and Tarski theorem (see Appendix for more details). The existence of lattice structure identifies the presence of a partial order of symmetric cutoff equilibria as well as a symmetric equilibrium with the highest and lowest cutoff strategy.

As a corollary of propositions 5 and 6 and the fact workers send their signals to top K firms in any symmetric equilibria (except the babbling equilibrium when firms do not respond to signals), we get the following comparison among symmetric cutoff equilibria.

Proposition 1 *In a symmetric equilibrium with greater cutoffs:*

- *the expected number of matches is higher*

- *workers have higher expected payoffs*

- *firms have lower expected payoffs*

The result that the expected number of matches is higher in the equilibrium with the largest cutoff seems very intuitive. More firms respond to signals, more likely firms offers are accepted, because they make offers to workers where they have lower rank.

In the same direction, if firms respond more to signals, workers get better matches as they send their signals to top K firms. Hence, workers get greater and better matches in equilibria with the greater cutoffs. Therefore, workers prefer equilibria when firms respond more to signals.

The propositions states that in the case of multiple cutoff equilibria, firms prefer the equilibrium with the smallest cutoff. This result may seem unusual, in that it appears to indicate that firms prefer less signaling to more. However, note that this comparison only holds for equilibria. The equilibria when firms respond more to signals are more competitive equilibria, and firms have lower payoffs.

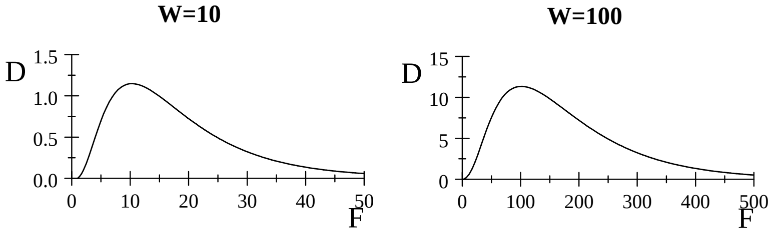
6 Market Structure and The Value of a Signaling Mechanism

In this section we analyze how the expected increase in the number of matches from the introductions of the signaling mechanism - the value of the signaling mechanism - differs across the markets. Our main findings are the following. The signaling mechanism adds the most value for the balanced markets, markets where the number of firms and the number of worker of the same magnitude. We also show that the optimal number of signals - that maximize the expected increase in the number of matches - is greater than one, $K > 1$, for the markets, where each worker could have many positions to feel (interviews). We show, by way of simulations, that the optimal number of signals increases when workers have more interviews. Finally, we show that the expected increase in the number of matches from the introductions of the signaling mechanism decreases for the markets with many periods of interactions.

To isolate the impact of a signaling mechanism on the number of matches in the market, we consider the *pure coordination model*, wherein firms and workers want to match, but are *almost* indifferent over the identity of the match. Specifically, we consider cardinal utilities from being matched to a partner being almost the same across partners. If agent a has preference profile θ_a, it prefers to be matched with partner θ_a^k rather than with partner $\theta_a^{k'}$ $k' > k$, though the difference between utility intensities is negligible. Under this assumption, there is the unique equilibrium in the offer game without signals - each firm makes an offer to its top ranked worker (TRW) and each worker accepts the best offer among available ones. As well as there is the unique symmetric equilibrium in the offer game with signals when firms respond to signals in the offer game with signals - each worker sends signals to her top K firms, each firm makes an offer to TSW if it receives at least one signal; otherwise, it makes an offer to its top ranked worker (TRW).

Define $S_K(F, W)$ and $U(F, W)$ be the expected number of matches in the pure coordination model with F firms and W workers with and without a signaling mechanism, respectively. Define $D_K(F, W) \equiv S_K(F, W) - U(F, W)$, the expected increase in matches from the introduction of the signaling mechanism where each worker can send up to K signals - the value of the signaling mechanism.

We first consider markets with one signal, $K = 1$, and analyze the question when the signaling mechanism creates the most additional matches. For illustrative purposes, let us consider the value of signaling mechanism for the markets of different sizes. Figure I graphs $D_1(F, W)$ as a function of F for fixed $W = 10$ and $W = 100$ and $D_1(F, W)$ as a function of W for fixed $F = 10$ and $F = 100$.



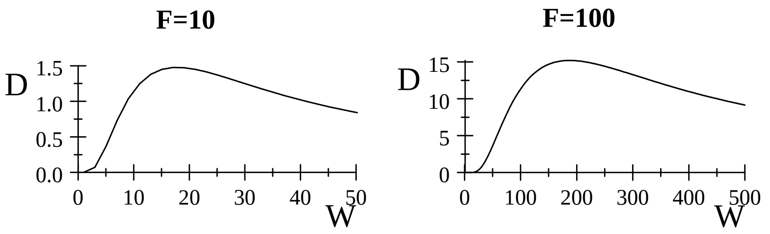


Figure I. The value of the signaling mechanism for different market sizes

The graphs allow to conjecture that the value of the signaling mechanism in terms of the percentage increase in the expected number of matches is similar for the markets of different size, but with the same ratio of the number of workers over the number of firms. The introduction of the signaling mechanism is the most beneficial for balanced markets when the number of firms and the number of workers are of the same magnitude. The signals are not very helpful in other cases. To understand this logic of balance, it is helpful to think about the endpoints. With W workers and very few firms, those firms will almost certainly match with or without a signaling mechanism. On the other hand, when there are very many firms, then most workers will get offers with or without a signaling mechanism, though there is still a chance that some workers will have no offers. Hence, a signaling mechanism offers little benefit at the extremes. More precisely, we are able to get the following result about the maximum value of the signaling mechanism.

Proposition 2 (Balanced markets) *For fixed W, $D(F,W)$ attains its maximum value at $F = x_0W + O_W(1)$, where $x_0 \approx 1.01211$ is a constant and $O_W(1)$ is a function that is smaller than a constant for large W. For fixed F, $D(F,W)$ attains its maximum value at $W = y_0F + O_F(1)$, where $y_0 \approx 1.884\,2$ is a constant and $O_F(1)$ is a function that is smaller than a constant for large F.*

We obtain the results of the proposition by the calculation of an explicit formula for $D(F,W)$. We show that the expected increase in the number of matches can be represented as

$$D(F,W) = Fg(\frac{W}{F}) + O_F(1)$$

or as

$$D(F,W) = Wt(\frac{F}{W}) + O_W(1)$$

where $g(\cdot)$ and $t(\cdot)$ are some functions.

Therefore, $D(F,W)$ is homogeneous of order one for large markets. This means that we can evaluate the properties of some sample market, and the properties of larger markets, but with the same ratio of firms over workers will be the same. This means that we can also use Figure I to investigate quantitative gains from the introduction of the signaling mechanism in markets of . In the most balanced markets, the increase in the number of matches is substantial. For fixed number of workers the maximum increase is 11% of the total number of firms, whereas for a fixed number of workers the maximum increase 15% of the total number of workers. Furthermore, the returns to a signaling mechanism are substantial over a wide range of market conditions. For example, only when the number of firms outweigh the number of worker by more than fivefold do the gains to a signaling mechanism drop to below 2%.

6.1 The optimal number of signals and the number of interviews

We consider the pure coordination model where each worker can have several positions to fill, called interviews, in this subsection. Each worker utility function is additive with respect to the number of interviews. Moreover, in order to stay in the frame of pure coordination model we assume that the value of each interview is the same for each worker.

Similar to previous analysis, one could show that if each worker has I positions to fill there is the unique equilibrium in both the offer game without signals - each firm makes an offer to its top ranked worker (TRW) and each worker accepts the best I offers among available ones. As well as there is the unique symmetric equilibrium in the offer game with signals when firms respond to signals in the offer game with signals - each worker sends signals to her top K firms, each firm makes an offer to TSW if it receives at least one signal; otherwise, it makes an offer to its top ranked worker (TRW). Again, each worker accepts the best I offers among available ones.

In the line with the previous notation, we denote the expected increase in the number of matches from the introduction of a signaling mechanism when each worker has I interview positions and she can send up to K signals as $D_{I,K}(F, W)$. We use simulations in order to analyze markets with F firms and W workers, and I interviews.

Simulation result. Let us consider some market with F firms, W workers, and I interviews. $D_{I,K}(F, W)$ is a single picked function of K.

The figure below shows the results of the simulations for the market with $W = 50$ workers and $F = 50$, $F = 100$, and $F = 150$ firms with $I = 1$, $I = 2$, and $I = 3$ interviews correspondingly (solid, dashed, and dot-dashed lines).

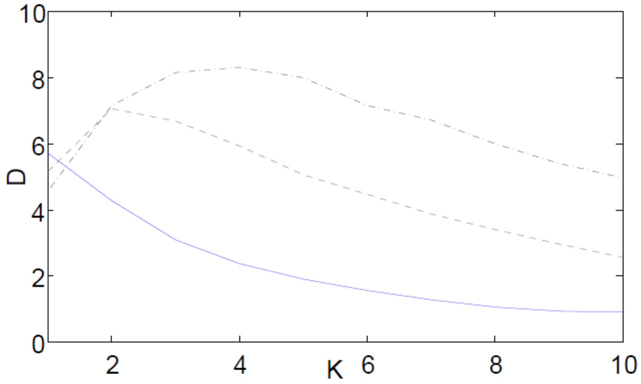


Figure II. Optimal number of signals and the number of interviews.

Simulations show that if each worker have only one interview that $D_{1,K}(F, W)$ is decreasing function of K (the pick is $K^* = 1$). However, the pick is greater than one if workers can have many interviews, $I > 1$. In other words, if we consider the optimal number of signals for the market (F, W, I) as a function of I, $K^*(I)$, simulations suggests that it should be an increasing function.

6.2 Markets with many periods of interactions

We consider the matching markets where agents can interact up to L periods in this subsection. For the purpose of clarity of the exposition we consider the markets with $K = 1$ signal and $I = 1$ interview. If we consider the offer game with signals, the market consists of $L + 1$

periods, where period 0 is the period when workers send their signals to firms and periods $1, ..., L$ are periods of agents' interaction.

Period 0. Workers send their signals.

1. *Each worker sends one signal to some firm. Workers send signals simultaneously. Each signal is observed only by the firm that receives it.*

Periods 1, ..., L. Agents' interaction. Each period consists of two stages:

1. *Each firm makes one offer to some worker. Firms make offers simultaneously.*

2. *Each worker can accept one offer from the set of offers she receives.*

The offer game without signals and L periods is similar to the one described above except it does not have period 0. As previously, the offers are binding and agents that form a pair leave the market. The other agents can participate in the remaining periods. We assume that after the end of each period all agents observe the agents that have left the market in previous periods.

Under the assumption that agents preferences are uniformly distributed and that the utility from being matched is *almost* the same across agents, there is the unique symmetric sequential equilibrium in the offer game without signals when each firm makes an offer to its top worker and each worker accepts its best offer in each period (if agents are not matched in previous periods). The reason for this is that there are no incentives to delay offers or offer acceptance as agents care only about being matched[10].

Similarly, there is the unique symmetric sequential equilibrium when firms respond to signals. Each worker sends her signal to her top firm at period 0. Each worker accepts the best available offer at each period. Each firm makes an offer to its TSW if some workers, which have not exited the market in previous periods, send their signals to it; otherwise the firm makes an offer to its TRW (among workers that are in the market). Again, there are no incentives to delay offers and offer acceptances as agents care only about being matched.

The next proposition shows that the effect of the signaling mechanism decreases with the introductions of additional periods of interactions.

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[10] We could presume that there is a discount factor δ for being matched in the latter periods.

Proposition 3 *The expected increase in the number of matches from the introduction of the signaling mechanism,* $D^L_{1,1}(F,W)$, *is a decreasing function of the number of periods of interactions,* L.

Proof.

Let us first, compare the expected number of matches in the markets with one and two periods of interactions. The signaling mechanism increases the expected number of matches in the first period. All matched pairs leave the market. Therefore, the number of market participants in the offer game without signals is greater then the number of participants in the offer game with signals in the second period. However, if workers can send only one signal, the firms that receive at least one signal left the market at period 1. Therefore, the second period of the offer game with signals and (F_2, W_2) agents is essentially an offer game without signals with (F_2, W_2). As proposition 2 shows the number of matches in a market with one period is proportional to the size of the market. Therefore, the expected number of matches in the second period in the market with signals is greater than in the market without signals. Therefore, the difference between the expected number of matches in the offer game with signals and the offer game without signals decreases if we consider the second period of interaction. The similar logic applies if we consider markets with L periods of interactions. □

Using simulations we are also able to predict quantitatively the change in the expected number of matches with the introduction additional periods of interactions. Proposition 4 provides the results. A typical change in the expected number of matches for markets with many periods of interactions is shown on Figure VI for the market with F=100 firms, W=100 workers and $L \in \{1, ..., 5\}$ periods of interactions.

Proposition 4 *For the markets with the same number of agents on the both sided of the market* $F = W$ *and* $K = 1$

- *the increase of the expected number of matches due the introduction of signaling mechanism decreases by* 55% *if* $L = 2$ *periods of interactions is allowed (compare to the market of the same size with* $L = 1$) .

- *the increase of the expected number of matches due the introduction of signaling mechanism decreases by* 85% *if* $L = 3$ *periods of interactions is allowed (compare to the market of the same size with* $L = 1$) .
- *there is almost no difference in the expected number of matches (less than* 0.5% *of the total number of agents) in the markets with signals and without signals if* $L = 4$ *periods of interaction is allowed.*

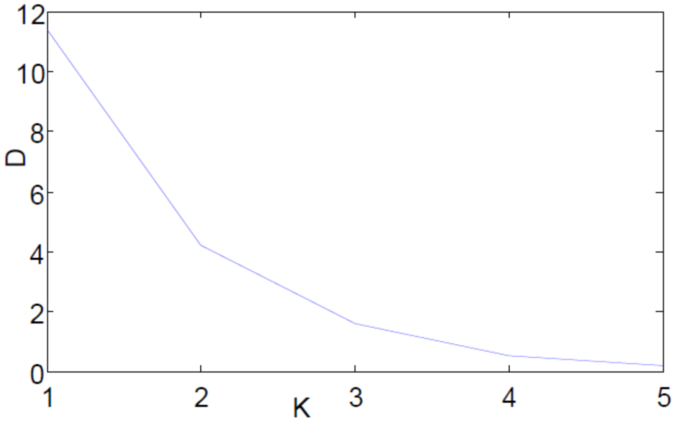

Figure III. The value of the signaling mechanism with $F = 100$ firms and $W = 100$ workers and many periods of interactions.

7 Conclusion

Market congestion is a reality, and the abundance of signaling in congested markets suggests that signaling deserves consideration as a means of achieving second-best outcomes.

We have examined a natural signaling mechanism; allowing workers to signal to several firms. In a setting of uncorrelated preferences, workers will send this signal to their top choice firms, and the firms use this information as guidance for their offers. We find that in this signaling setting, it is optimal for firms to make use of these signals in the form of cutoff strategies. However, making use of signals imposes a negative externality on other firms, so that while beneficial for workers, the overall welfare effect of a signaling technology on firms is ambiguous. We find that on average, introducing a signaling technology increases the average number of matches, one possible measure of social welfare.

We have also analyzed the value of the signaling mechanism for the markets with different structures. The signaling mechanism adds the most value for the balanced markets, markets where the number of firms and the number of worker of the same magnitude. If we allow workers to have several positions to fill, interviews, the optimal number of signals increases with the number of interviews workers have. Finally, we show that the expected increase in the number of matches from the introduction of the signaling mechanism decreases for the markets with many periods of interactions.

Ultimately the goal is to provide policy advice for a broad set of environments as to when an organized signaling mechanism might improve outcomes, and should this be the case, what forms the signaling mechanism might take. Our hope is that the approach in this paper will serve as a tool and as a benchmark; a framework for examining settings with alternative preference/market assumptions, and a point of comparison for alternative signaling mechanisms.

A Appendix. Block-correlated preferences.

Proof of theorem 1.

The logic of the proof is the following. If distribution $t(\cdot)$ is block uniform, firms use anonymity strategies, and workers chooses the best offer among available at the last stage, the probability of being accepted by each worker is the same. Hence, each firm prefers to make an offer to its top worker.

Let us consider some firm f with some profile of preferences $\bar{\theta}$. We compare two strategies of firm f: make an offer to its top worker $s_f(\bar{\theta}) = \bar{\theta}_f^1$ and make an offer to some other worker $s'_f(\bar{\theta}) = w' = \theta_f^k$, $k > 1$. We consider the following sets of agents' preferences:

$$\begin{aligned}
&\bar{\Theta} \equiv \{\theta \in \Theta | \theta_f = \bar{\theta}\} \\
&\bar{\Theta}_+ \equiv \{\theta \in \bar{\Theta} | \pi_f(s_f(\theta), s_{-f}(\theta), \theta) > 0\} \\
&\bar{\Theta}_- \equiv \{\theta \in \bar{\Theta} | \pi_f(s'_f(\theta), s_{-f}(\theta), \theta) > 0\}
\end{aligned}$$

If set $\bar{\Theta}_-$ is empty, $\bar{\Theta}_- = \varnothing$, we are done because any matched worker gives positive utility to firm f. If it is not empty, let us take some $\theta \in \bar{\Theta}_-$. Let us construct one-to-one correspondence from $\bar{\Theta}_-$ to $\bar{\Theta}_+$. Namely, we find $\theta' \in \bar{\Theta}_+$ which uniquely corresponds to $\theta \in \bar{\Theta}_-$ and for any $\theta' \in \bar{\Theta}_+$ there is only one $\theta \in \bar{\Theta}_-$. In order to accomplish this task we perform the following permutation of preference profiles:

$$\begin{aligned}
\text{for any } f' \neq f \quad & f' : (..., w, ...w', ...) \to (..., w', ...w, ...) \\
& w : (f_{l_1}, ..., f_{l_W}) \to (f_{d_1}, ..., f_{d_W}) \\
& w' : (f_{d_1}, ..., f_{d_W}) \to (f_{l_1}, ..., f_{l_W})
\end{aligned}$$

For new profile θ' we have not changed firm f's preferences, $\theta'_f = \bar{\theta}$ and $\theta' \in \bar{\Theta}$. Moreover, both profiles

$$(\theta_w = (f_{l_1}, ..., f_{l_W}), \theta_{w'} = (f_{d_1}, ..., f_{d_W}))$$

and

$$(\theta_w = (f_{d_1}, ..., f_{d_W}),\ \theta_{w'} = (f_{l_1}, ..., f_{l_W}))$$

are in the support of block uniform distribution of agents preferences $t(\cdot)$ and equally likely.

According to the anonymity assumption

$$\text{for any } \theta_{f'} \in \Theta_f \Rightarrow \sigma\left(s_{f'}(\theta_{f'})\right) = s_{f'}\left(\sigma(\theta_{f'})\right)$$

Therefore, firm f' chooses w for profile θ if and only if it chooses w' for profile θ'. The payoff from an offer to worker w' more than 0, $\pi_f(s'_f(\theta), s_{-f}(\theta), \theta) > 0$, only if firm f is the best firm who make an offer to worker w'. Hence firm f is the best firm who makes an offer to worker w for profile θ'. Therefore, $\pi_f(s_f(\theta'), s_{-f}(\theta'), \theta') > 0$ and $\theta' \in \bar{\Theta}_+$. It means that there exists one-to-one correspondence such that for any $\theta \in \bar{\Theta}_-$ there exists $\theta' \in \bar{\Theta}_+$ which is different for different θ by construction. Moreover each θ and θ' are in the support of block uniform function $t(\cdot)$. We obtain that $\left|\bar{\Theta}_+\right| \geq \left|\bar{\Theta}_-\right|$. Using the reverse construction one may show that actually $\left|\bar{\Theta}_+\right| = \left|\bar{\Theta}_-\right|$. Hence, the probabilities of being accepted by worker w and worker w' are the same.

We know that whenever firm f's offer is accepted

$$\pi_f(s'_f(\theta), s_{-f}(\theta), \theta) = g(\bar{\theta}, w') > 0, \; \pi_f(s_f(\theta), s_{-f}(\theta), \theta) = g(\bar{\theta}, w) > 0$$

and that $g(\bar{\theta}, w) > g(\bar{\theta}, w')$ (w is strictly better that w' according to $\bar{\theta}$). Hence,

$$\frac{\left|\bar{\Theta}_+\right|}{\left|\bar{\Theta}\right|} g(\bar{\theta}, w) > \frac{\left|\bar{\Theta}_-\right|}{\left|\bar{\Theta}\right|} g(\bar{\theta}, w')$$

And overall we get that

$$E_\theta(\pi_f(s_f(\theta), s_{-f}(\theta), \theta) | \theta \in \bar{\Theta}) > E_\theta(\pi_f(s'_f(\theta), s_{-f}(\theta), \theta) | \theta \in \Theta^w)$$

Hence, firm f's strategy of making an offer to the top worker dominates any other strategy and there is a unique equilibrium of the offer game without signals: $s_f(\theta_f) = \theta_f^1$ for all $f \in F$ and $\theta_f \in \Theta_f$. □

Proof of proposition 2.

We first define criterion D1 in our environment[11]. Let us consider some block-symmetric

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[11] See Cho and Kreps [3] for more detailed discussion.

sequential equilibrium and some worker w and firm f. Worker w sends some off-equilibrium message m to firm f, if it sends a signal when the equilibrium strategy prescribes zero probability of sending a signal to firm f or it does not send a signal when the equilibrium strategy prescribes sending a signal to firm f with probability equal 1. According to the definition of anonymous strategies the latter one can happen only firm f is the only one firm in its block. However, the symmetry of workers strategies then ensures that all workers send their signals to firm f with probability 1, which can happen only in the case of babbling equilibrium. Therefore, we only analyze the off-equilibrium messages of the first type mentioned above.

Let us denote the expected equilibrium payoff of firm f as u_f^* and the expected equilibrium payoff of worker w as u_w^*. We consider worker w that sends some off-equilibrium signal to firm f. For each firm f's type $\bar{\theta} \in \Theta_f$, each set of signals that firm f could receive, $\bar{h} \subset W$, we denote the mixed best response of firm f that has beliefs $\bar{\mu}$ and the other agents use strategies s_{-f} as

$$MBR(\bar{\theta}, \bar{h} \cup w, \bar{\mu}) = \arg\max_{s_f \in \Delta^W} E_{\theta_{-w}}(\pi_f(s_f, s_{-f}, \theta) \,|\theta_f = \bar{\theta},\ h_f = \bar{h} \cup w,\ \mu_f = \bar{\mu}_f).$$

Then, we denote the mixed best response of firm f for all types of firm f and any possible profiles of signals it may receive, conditional on the fact that worker w sends her signal to it as

$$MBR(w, \bar{\mu}) = \{MBR(\bar{\theta}, \bar{h} \cup w, \bar{\mu}) \text{ for all } \bar{\theta} \in \Theta_f,\ \bar{h} \subset W\}$$

And for any $\Omega \subset \Theta_w$, we denote the set of best responses by firm f to probability assessments concentrated on the set Ω.

$$MBR(w, \Omega) = \bigcup_{\{\mu_f : \mu_f(\Omega) = 1\}} MBR(w, \mu_f)$$

We also denote for any type $t \in \Theta_w$

$$\begin{aligned} D_t &= \{\phi \in MBR(w, \Theta_w),\ u_w^*(t) < E_{\theta_{-w}}(\pi_w(s_w, \phi, s_{-w,f}, \theta) \mid \theta_w = t)\} \\ D_t^0 &= \{\phi \in MBR(w, \Theta_w),\ u_w^*(t) = E_{\theta_{-w}}(\pi_w(s_w, \phi, s_{-w,f}, \theta) \mid \theta_w = t)\} \end{aligned}$$

We say that the pair (t, m) may be pruned from the game if firm f's off-equilibrium beliefs should put zero probability on worker w's type t upon receiving message m. Using the above notations criterion D1 can be stated as follows.

Criterion D1. If for some worker w's type $t \in \Theta_w$ there exists a second worker w's type $t' \in \Theta_w$ with $D_t \cup D_t^0 \subseteq D_{t'}$, then (t, m) may be pruned from the game.

The intuition behind this criterion is that whenever type t either wishes to defect and send m or is indifferent, some other type t' strictly wishes to defect. When we prune the pair (t, m), we believe that it is infinitely more likely that m has come from type t'.

We first show that there cannot be a block-symmetric sequential equilibrium such that $q^b < p^b$ for some $b \in \{1, ..., B\}$ that satisfies criterion $D1$. Let us consider such equilibrium. Then, no worker sends a signal to any firm from block F^b on the equilibrium path. Then, firm $f \in F^b$ can receive a signal from some worker w only on off-equilibrium path. If it is beneficial for some type $\theta_w \in \Theta_w$ to deviate from the equilibrium path and send a signal to firm f (in this case firm f responds to worker w signal), then it is beneficial for any type $\theta'_w \in \Theta_w$, such that firm f is the top firm within block b, to deviate. Therefore, the only types of worker w that are not pruned in firms believes according to criterion $D1$ are such where firm f is the top firm within block b. Therefore, if it is optimal for firm f to make an offer to worker w when it does not receive her signal, it is optimal for firm f to make an offer to worker w when it receives its signal. This means that $q^b < p^b$ cannot be part of an equilibrium.

We now analyze the case when for any $b = 1, ..., B$ $q^b \geq p^b$. It is easy to observe, that there is exist a block-symmetric sequential equilibrium that satisfies criterion $D1$, when for any $b = 1, ..., B$ $q^b = p^b$. For example, each worker uses the strategy that prescribes sending its signal to firms with equal probability independently on their preferences and firms, play the equilibrium strategy of the offer game with no signals. The equilibrium beliefs are block-uniform, i.e. if firm receives a signal from worker w its beliefs coincides with its priors. As one could see there are no off-equilibrium paths needs to be specified. Therefore, this equilibrium satisfies criterion $D1$.

Let us now consider the case when there exists $b_0 \in \{1, ..., B\}$, such that $q^{b_0} > p^{b_0}$ in some block-symmetric sequential equilibrium. We denote the probability that a worker sends her signal to firms within block b as α_b, $\alpha_b \in [0,1]$ and $\sum_{b=1}^{B} \alpha_b \leq 1$. Let us consider some b, such that $\alpha_b > 0$. If there are at least two workers $F \geq 2$, agents use anonymous block-symmetric strategies, and agents' types are uncorrelated, each worker is unmatched with positive probability in any equilibrium. As immediate consequence of this is that $\alpha_b > 0$ and $q^b \leq p^b$ are incompatible in an equilibrium (otherwise worker w can benefit by shifting the probability mass α_b to block b_0, when signal has positive effect). Then, if $q^b > p^b$ worker w should send her signal to her top firm within block F^b, as it delivers her the greatest expected payoff.

Let us now consider $b' \in \{1, ..., B\}$ such that $\alpha_{b'} = 0$. Consider some firm f from block $F^{b'}$ that receives a signal from some worker w. Similarly to the discussion in the first part of the proposition, the only beliefs that survive applying criterion D1 are such that worker w has preferences where firm f is the top firm in block b'.

$$\mu_f(\Gamma | w \subset h_f) = 1, \text{where } \Gamma = (\theta_w \in \Theta_w : f = \underset{\theta_w}{max}(f' \in F^b))$$

□

Proof of proposition 2.

Consider some firm f that belongs to block F_b, $b \in \{1, ..., B\}$. Firm f receives signals from the set of workers $\bar{h} \subset W$. Let us first prove that the strategy of making an offer to TSW_f strictly dominates the strategy of making an offer to any other worker. We denote worker TSW_f as w and take some other worker w' that has sent a signal to firm f, $w' \in \bar{h}$. Firm f believes that it is worker w's top firm within block F_b.

Let us fix strategies of all agents and consider the set of preference profiles such that firm f's preference profile and the set of signals firm f receives are unchanged.

$$\bar{\Theta} \equiv \{\theta \in \Theta | \theta_f = \bar{\theta}_f, h_f = \bar{h}\}$$

For each $\theta \in \bar{\Theta}$ we construct the permutation of profile θ and denote it as profile θ' as follows.

We exchange ranks of w and w' in the preference profiles of all firms except firm f :

$$\text{for any } f' \in F,\ f' \neq f,\ f' : (..., w, ...w', ...) \longrightarrow (..., w', ...w, ...)$$

We also exchange preference profiles of worker w and worker w' :

$$w : (f_{l_1}, ..., f_{l_W}) \rightarrow (f_{d_1}, ..., f_{d_W})$$
$$w' : (f_{d_1}, ..., f_{d_W}) \rightarrow (p_{l_1}, ..., p_{l_W})$$

We assumed that workers w and w' use the same strategies. Hence, the preference profile of firm f, and the set of workers that have sent their signals to firm f are unchanged, $\theta_f = \bar{\theta}_f$ and $h_f = \bar{h}$.

Note that formally workers play mixed strategies. Hence, there is no one-to-one correspondence between preference profiles and signaling behavior. However, our assumptions of anonymity and symmetry of strategies guarantee that when we exchange workers' profiles, workers play the same strategies and the fraction of signals sent to each group is unchanged (independent on the exact worker's profile). Therefore, the fraction of signals that worker w and worker w' send to each block of firms is unchanged.

Hence $\theta' \in \Theta^w$. Let us consider two strategies of firm f: $s_f(\bar{\theta}_f, \bar{h}) = w$ and $s'_f(\bar{\theta}_f, \bar{h}) = w'$. From the assumption of that strategies of firms are anonymous whenever making an offer to worker w' is successful for profile θ', making an offer to worker w is also successful for profile θ. Therefore, the probability that worker w accepts firm f's offer equals to the probability that worker w' accepts firm f's offer because the distribution of types is block-uniform. However, the payoff of firm f from being matched with worker w is greater than its payoff from being matched with worker w', i.e. $g(\bar{\theta}_f, w) > g(\bar{\theta}, w')$. Therefore,

$$E_\theta(\pi_f(s(\theta), s_{-f}(\theta))|\theta \in \Theta^w) > E_\theta(\pi_f(s'(\theta), s_{-f}(\theta))|\theta \in \Theta^w)$$

Hence, the strategy of making an offer to TSW_f strictly dominates the strategy of making an offer to any other worker that has sent a signal to firm f.

The logic for set $W \backslash h$ is the same. The probability of being accepted by any worker among $W \backslash h$ is the same. Hence, firms prefer its most valuable worker, TRW_f, to others workers in $W \backslash f$. One could also consult the proof of theorem 1 for more details. $\square$

Proof of proposition 3.

Workers play some symmetric anonymous strategy that prescribes sending their signals to top firms. We break the proof into two parts. First we prove that the identities of workers that have sent a signal to some firm f do not influence the decision of firm f provided that the total number of signals firm f receives is constant and TSW_f does not change. Second we prove that if firm f chooses TSW_f when it receives signals from set $h \subset W$ then it still chooses TSW_f if the number of received signals does not change and TSW_f has a smaller rank (TSW_f becomes more valuable).

Let us consider some firm f with preference list $\theta_f = \bar{\theta}$ that receives signals from the set of workers $\bar{h} \subset W$. Assume that firm f makes an offer to TSW_f if it receives the set of signals $\bar{h}$.Then we want to show that if set $\bar{h}'$ has the same TSW_f and $\left|\bar{h}\right| = \left|\bar{h}'\right|$, it is still optimal for firm f to make an offer to TSW_f. For simplicity, we consider the case when $\bar{h}$ and $\bar{h}'$ differ only in one signal. Hence, there exist worker w and worker w', such that the former one belongs to set $\bar{h}$, but it does not belong to set $\bar{h}'$; while the latter one belongs to $\bar{h}'$, but it does not belong to $\bar{h}$. More general case directly follows.

Proposition 2 establishes that the optimal choice of firm f is either TSW_f, or TRW_f, or some lottery between them. We consider two strategies of firm f.

$$s_f(\bar{\theta}, \cdot) = TSW_f$$
$$s'_f(\bar{\theta}, \cdot) = TRW_f$$

We consider two sets:

$$\Theta^{\bar{h}} \equiv \{\theta \in \Theta | \theta_f = \bar{\theta}, h_f = \bar{h}\}$$
$$\Theta^{\bar{h}'} \equiv \{\theta' \in \Theta | \theta_f = \bar{\theta}, h_f = \bar{h}'\}$$

and two subsets of these sets

$$\Theta^{\bar{h}}_{+} \equiv \{\theta \in \Theta^{h} |\ \pi_f(s_f(\theta), s_{-f}(\theta), \theta) < \pi_f(s'_f(\theta), s_{-f}(\theta), \theta)\}$$
$$\Theta^{\bar{h}}_{+} \equiv \{\theta' \in \Theta^{h'} |\ \pi_f(s_f(\theta'), s_{-f}(\theta'), \theta') < \pi_f(s'_f(\theta'_f), s_{-f}(\theta'), \theta')\}$$

We construct a bijection between $\Theta_+^{\bar{h}}$ and $\Theta_+^{\bar{h}'}$. For any profile $\theta \in \Theta^{\bar{h}}$ we construct profile θ' in the following way

$$\begin{aligned}
&\text{for any } f' \neq f, \quad f : (..., w, ...w', ...) \rightarrow (..., w', ...w, ...) \\
&\; w : (f_{l_1}, ..., f_{l_W}) \rightarrow (f_{d_1}, ..., f_{d_W}) \\
&\; w' : (f_{d_1}, ..., f_{d_W}) \rightarrow (p_{l_1}, ..., f_{l_W})
\end{aligned}$$

Therefore, firm f preference profile and and the set of workers that have sent a signal to firm f has not changed, i.e. $\theta'_f = \bar{\theta}$ and $h'_i = \bar{h}'$.

Note that formally workers play mixed strategies. Hence, there is no one-to-one correspondence between preference profiles and signaling behavior. However, our assumptions of anonymity and symmetry of strategies guarantee that when we exchange workers' profiles, workers play the same strategies and the fraction of signals sent to each group is unchanged (independent on the exact worker's profile). Therefore, the fraction of signals that worker w and worker w' send to each block of firms is unchanged.

Let us denote $TRW_f = w''$. In accordance with the anonymity assumption

$$s_f(\bar{\theta}, \bar{h}) = w'' \Rightarrow s_f(\sigma(\bar{\theta}), \sigma(\bar{h})) = \sigma(w'') = w''$$

Firm f makes an offer to TRW_f for profile θ if and only if it makes an offer to TRW_f for profile θ'. Therefore, if it is optimal to make an offer to TRW_f for profile $\theta' \in \Theta_+^{\bar{h}'}$

$$\pi_f(s_f(\theta'), s_{-f}(\theta'), \theta') < \pi_f(s'_f(\theta'), s_{-f}(\theta'), \theta')$$

i.e. $\theta' \in \Theta_+^{\bar{h}'}$, it is optimal for firm f to make an offer to TRW_f for profile $\theta \in \Theta^{\bar{h}}$

$$\pi_f(s_f(\theta), s_{-f}(\theta), \theta) < \pi_f(s'_f(\theta), s_{-f}(\theta), \theta)$$

and vice versa. We should also mention that by construction $\theta \in \Theta^{\bar{h}}$ is different for different $\theta' \in \Theta^{\bar{h}'}$. A similar construction is valid for the reverse direction. Hence, there is a one-to-one correspondence between $\Theta_+^{\bar{h}'}$ and $\Theta_+^{\bar{h}}$. In addition,

$$\pi_f(s_f(\theta), s_{-f}(\theta), \theta) < \pi_f(s'_f(\theta), s_{-f}(\theta), \theta) \Leftrightarrow \pi_f(s'_f(\theta), s_{-f}(\theta), \theta) = g(\bar{\theta}, \bar{\theta}^1) > 0$$

because $g(\bar{\theta}, TRW_f) > g(\bar{\theta}, TSW_f)$. As a result

$$\begin{gathered} E_\theta(\pi_f(s(\theta'), s_{-f}(\theta'), \theta')|\theta' \in \Theta^{\bar{h}\prime}) < E_\theta(\pi_f(s'(\theta'), s_{-f}(\theta'), \theta')|\theta' \in \Theta^{\bar{h}\prime}) \\ \Leftrightarrow E_\theta(\pi_f(s(\theta), s_{-f}(\theta), \theta)|\theta \in \Theta^{\bar{h}}) < E_\theta(\pi_f(s'(\theta), s_{-f}(\theta), \theta)|\theta \in \Theta^{\bar{h}}) \end{gathered}$$

Therefore, if firm f strictly prefers to make an offer to TSW_f for profile θ, it also strictly prefers to make an offer to TSW_f for profile θ'. Note that there can be the case when firm f is indifferent between making an offer to TSW_f and making an offer to TRW_f for some $\bar{h}$. Therefore, firm f could change its behavior when it receives the sets of signals with the same TSW_f and the same number of signals, $|\bar{h}| = |\bar{h}'|$. However, the strategy of making an offer to TSW_f for the set of signals $\bar{h}'$ is weakly dominated by the strategy of making an offer to TRW_f for the set of signals $\bar{h}$.

Now we prove that if firm f chooses TSW_f then it still chooses TSW_f if TSW_f has a smaller rank (TSW_f becomes more valuable). Assume that firm f's preference profile is $\theta_f = \bar{\theta}$. Then, it is optimal for firm f to make an offer to TSW_f when it receives the set of signals $\bar{h}$, $s_f(\bar{\theta}, \bar{h}) = TSW_f$. We consider the set of signals $\bar{h}'$ that differs from $\bar{h}$ only in the best (for firm f) worker. Moreover, we consider the case when the difference in rank between two top ranked workers equals to one.

$$\begin{gathered} \text{for any } w \in \bar{h}/\{TSW_f\} \Leftrightarrow w \in \bar{h}'/\{TSW'_f\} \\ rank_f\left(TSW'_f\right) = rank_f\left(TSW_f\right) - 1 \end{gathered}$$

More general case directly follows. Our goal is to show that it is still optimal for firm f to make an offer to TSW'_f when it receives the set of signals $\bar{h}'$, $s(\bar{\theta}, \bar{h}') = TSW'_f$.

Note that sets $\bar{h}$ and $\bar{h}'$ differs only in one signal. We construct the same permutation as constructed above. However, now if firm f makes an offer to TRW_f for two different profiles it gets the same payoff in case the offer is accepted (the payoff equals to $g(\bar{\theta}, \bar{\theta}^1)$). Whereas the payoff from making an offer to TSW_f' for the second profile is strictly greater than payoff from making an offer to TSW_f for the first one. Hence, if it is optimal for firm to make an offer to TSW_f when it receives the set of signals $\bar{h}$, it is optimal to make an offer to TSW_f' when it receives the set of signals $\bar{h}'$. $\square$

Proof of proposition 4.

Let us consider firm f with preference profile $\theta_f = \bar{\theta}$ that receives some set of signals $\bar{h}$. We denote two strategies of firm f as

$$\begin{aligned} s_f(\bar{\theta}, \bar{h}) &= TRW_f \\ s_f'(\bar{\theta}, \bar{h}) &= TSW_f \end{aligned}$$

We also consider two sets of strategies of all firms except firm f, s_{-f} and s_{-f}', that vary only in that firm $f' \neq f$ makes an offer to different workers: $s_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'}) = TRW_{f'}$ while $s_{f'}'(\bar{\theta}_{f'}, \bar{h}_{f'}) = TSW_{f'}$ for some pair $(\bar{\theta}_{f'}, \bar{h}_{f'})$. We want to show that firm f's payoff from making an offer to TRW_f decreases whereas firm f's payoff from making an offer to TSW_f increases when firm f' responds more to signals, i.e. plays strategy $s_{f'}'(\bar{\theta}_{f'}, \bar{h}_{f'})$ instead of $s_{f'}(\bar{\theta}_{f'}, \bar{h}_{f'})$.

$$\begin{aligned} I)\ TRW_f &: E_\theta(\pi_f(s_f(\theta), s_{-f}(\theta), \theta) | \theta_f = \bar{\theta}, h_f = \bar{h}) \geq E_\theta(\pi_f(s_f(\theta), s_{-f}'(\theta), \theta) | \theta_f = \bar{\theta}, h_f = \bar{h}) \\ II)\ TSW_f &: E_\theta(\pi_f(s_f'(\theta), s_{-f}(\theta), \theta) | \theta_f = \bar{\theta}, h_f = \bar{h}) \leq E_\theta(\pi_f(s_f'(\theta), s_{-f}'(\theta), \theta) | \theta_f = \bar{\theta}, h_f = \bar{h}) \end{aligned}$$

I. We first prove the first statement. We define:

$$\begin{aligned} \bar{\Theta} &\equiv \{\theta \in \Theta | \theta_f = \bar{\theta}, h_f = \bar{h}\} \\ \bar{\Theta}_+ &\equiv \{\theta \in \bar{\Theta} | \pi_f(s_f(\theta), s_{-f}(\theta), \theta) < \pi_f(s_f(\theta), s_{-f}'(\theta), \theta)\} \\ \bar{\Theta}_- &\equiv \{\theta \in \bar{\Theta} | \pi_f(s_f(\theta), s_{-f}(\theta), \theta) > \pi_f(s_f(\theta), s_{-f}'(\theta), \theta)\} \end{aligned}$$

If set $\bar{\Theta}_+$ is empty the statement has been proved. Otherwise, we take some $\theta \in \bar{\Theta}_+$ and denote $s_f(\bar{\theta}, \bar{h}) = TRW_f = w$.Then, we should have

$$\begin{cases} s_{f'}(\theta_{f'}, h_{f'}) = TRW_{f'} = w \\ s'_{f'}(\theta_{f'}, h_{f'}) = w' \neq w \end{cases}.$$

Note that the situation when firm $f \in F_b$ and f'$\in F_l$ $l > b$ cannot happen. In this case the offer of firm f' offer is always worse than the offer of firm f. Hence, firm f is from worse or the same group as firm f', i.e. $l \leq b$.

Note that worker w has signaled neither to firm f nor to firm f'. This allows us to construct $\theta' \in \Theta^{w}_{-}$. We switch places of workers w and w' in the preference profile of firm f' as well as we switch the preferences of workers w and w'[12]:

$$\begin{aligned} f' &: (w, ..., w', ...) \to (w, ...w', ...) \\ w &: (f_{l_1}, ..., f_{l_W}) \to (f_{d_1}, ..., f_{d_W}) \\ w' &: (f_{d_1}, ..., f_{d_W}) \to (p_{l_1}, ..., f_{l_W}) \end{aligned}$$

Note that workers could play mixed strategies. Hence, there is no one-to-one correspondence between preference profiles and signaling behavior. However, our assumption of anonymity and strategies symmetry guarantees that when we exchange workers' profiles, workers play the same strategies and the fraction of signals sent to each group is unchanged (independent on the exact worker's profile).

We get preference profile θ' such that $\theta_f = \bar{\theta}$, $h_f = \bar{h}$. Hence, $\theta' \in \bar{\Theta}$. Also $s_{f'}(\theta'_{f'}, h'_{f'}) = s_{f'}(\sigma(\theta_{f'}), \sigma(h_{f'})) = \sigma(w) = w'$. From the assumption that only firm f' prevents firm f from being matched with worker w for profile θ, the offer of firm f to worker w is successful for profile θ'. If firm f' uses strategy $s'_{f'}(\theta_{f'}, h_{f'}) = w$, the offer of firm f is unsuccessful for preference profile θ'. If firm f is from the group F_b, $b > l$, worker w always prefers firm f' to firm f. If firm f and firm f' are form the same group, $b = l$, worker w has signaled to firm f' for profile of preferences θ'. This means that worker w prefers firm f' to firm f for profile θ'.

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[12] For the case of one signal we may exchange the whole preference profiles of w and w'. For the case of many signals that might cause a problem since worker w' may also signal to firm f.

We should also investigate the behavior of a firm that receives worker w's signal for profile θ, say firm f_y. If firm f_y makes an offer to worker w for profile θ, since worker w's changes her behavior when firm f' changes its strategy, firm f_y should be worse than both firm f and firm f'. Hence, firm f_y's offer cannot change the incentives of worker w. If worker w' sends her signal to firm f_y then firm f_y either makes an offer to worker w' or to worker TRW_y not to worker w. Hence, firm f_y does not change incentives of the other agents. Hence $\theta' \in \bar{\Theta}_-$.

We have constructed an injective function $\bar{\Theta}_+ \to \bar{\Theta}_-$. Hence, $\left|\bar{\Theta}_-\right| \geqslant \left|\bar{\Theta}_+\right|$[13]. We know that the payoff from making an offer to worker $w = TRW$ can take only two values: $g(\bar{\theta}, \bar{\theta}^1) > 0$ in case of success and 0 in case of failure. Therefore

$$E_\theta(\pi_f(s_f(\theta), s_{-f}(\theta), \theta)|\theta_f = \bar{\theta}, h_f = \bar{h}) \geq E_\theta(\pi_f(s_f(\theta), s'_{-f}(\theta), \theta)|\theta_f = \bar{\theta}, h_f = \bar{h})$$

II. Let us now show that if firm f' responds more to signals, firm f's payoff from making an offer to TSW_f increases. If firm f receives a signal from worker w it means that this firm is the best firm in its group F_b. Therefore, worker w prefers the offer of firm f to an offer from any other firm f' from group F_l $l \geq b$. Therefore, the change of the behavior of any firm f' from group F_l, $l \geq b$, does not influence firm f's payoff.

If we consider any firm f' from group F_l, $l < b$, it can draw away worker w's offer from firm f only if it makes an offer to worker w. However, it can make an offer to worker w only when she is its $TRW_{f'}$. Hence, if firm f' responds more to signals, it makes an offer to its TRW_f more rarely. This means it is less often draw worker w away from firm f. Therefore, the payoff of firm f from making an offer to TSW_f increases.

$$E_\theta(\pi_f(s'_f(\theta), s_{-f}(\theta), \theta)|\theta_f = \bar{\theta}, h_f = \bar{h}) \geq E_\theta(\pi_f(s'_f(\theta), s'_{-f}(\theta), \theta)|\theta_f = \bar{\theta}, h_f = \bar{h})$$

As a corollary of I) and II), we have that if firm f' increases its cutoff point for some structure of signals (responds more to signals), firm f will also optimally (weakly) increase its cutoff point for each structure of signals. □

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[13] One may show that generally it is impossible to have a correspondence in the other direction.

Proof of theorem 2.

Let us denote the set of strategies of all agents as S. A typical element of S, $s \in S$, consists of firms' and workers' strategies $s = (s_F, s_W)$ where $s_F = (s_{f_1}, ..., s_{f_F})$ and $s_W = (s_{w_1}, ..., s_{w_W})$.

The strategy of firm f, s_f, is a vector of real numbers of size $W+1$ that specifies cutoff points for each structure of signals firm f could receive, $s_f = (j_f^0, ..., j_f^W)$, where for each $l = 0, ..., W$, $j_f^l \in [0, W]$. We denote the set of possible firms strategies as $T_f = [0, W]^F$.

The strategy of worker w, s_w, is a vector of size B that specifies the probability that she sends her signal to the top firm of specific block $s_w = (\alpha_w^1, ..., \alpha_w^B)$, where for each $b = 1, ..., B$, $\alpha_w^b \geq 0$ and $\sum_{b=1}^{B} \alpha_w^b \leq 1$. We denote the set of possible workers' strategies as $T_w = \{(\alpha^1, ..., \alpha^B) : \alpha^b \geq 0 \text{ and } \sum_{b=1}^{B} \alpha^b \leq 1\}$.

From definition follows that

T_f and T_w are *non-empty*, *convex*, and *compact.*

We introduce function $g : S \to 2^S$ such that

$$\begin{aligned} g_f(s) &= \arg\max\nolimits_{\alpha \in T_f} U_f(\alpha, s_{-f}) \\ g_w(s) &= \arg\max\nolimits_{\beta \in T_w} U_w(\beta, s_{-w}) \end{aligned}$$

where $U_f(\alpha, s_{-f})$ is expected utility of firm f that employs strategy α when the other agents employ strategies s_{-f}. Similar $U_w(\beta, s_{-w})$ is expected utility of worker w that employs strategy β when the other agents employ strategies s_{-w}.

More specifically, if firm f receives $|h_f| = l$ signals the corresponding component of $g(\cdot)$:

$$g_f(|h_f| = l, s) = \arg \max_{\alpha \in [0,W]} E_{\theta_{-f}}(\pi_f(\alpha, s_{-f}(\theta_{-f}), \theta) | \, |h_f| = l)$$

Similar for $w \in W$

$$g_w(s) = \arg \max_{\beta \in T_W} E_{\theta_{-w}}(\pi_w(\beta, s_{-w}(\theta_{-w}), \theta))$$

Note that $g_f(s)$ and $g_w(s)$ do not depend on preference profiles as we consider only anonymous strategies.

As one could see function $\pi_f(\alpha, s_{-f}(\theta_{-f}), \theta)$ and $\pi_w(\beta, s_{-w}(\theta_{-w}), \theta)$ are continuous in agents' strategies. Function π_a is also concave function of its first argument. Hence, $g(s)$ is a continuous correspondence with closed graph. Hence, $g(s)$ has a fixed point by Kakutani theorem.

Note that we should also specify beliefs when the above set of strategies is a sequential equilibrium. The corresponding beliefs are determined by Bayesian rule. If a firm receives a signal from some it believes that it is the worker's top firm within its block. If a firm does not receive a signal it believes that worker's preferences are block-uniformly distributed. The specification of beliefs finishes the proof of the existence a block-symmetric equilibrium in mixed strategies. □

Proof of proposition 5.

We define

$$\begin{aligned}\Theta^+ &= \{\ \theta \in \Theta\ :\ |\mu(s_i(\theta), s_{-i}(\theta), \theta)| < |\mu(s_i(\theta), s'_{-i}(\theta), \theta)|\ \} \quad \text{and} \\ \Theta^- &= \{\ \theta \in \Theta\ :\ |\mu(s_i(\theta), s_{-i}(\theta), \theta)| > |\mu(s_i(\theta), s'_{-i}(\theta), \theta)|\ \}.\end{aligned}$$

Sets Θ^+ and Θ^- are the profiles for which firm f's strategy swap affects the number of matches. In particular, this means that firm f receives l number of signals

$$\theta \in \Theta^+ \cup \Theta^- \Rightarrow |h_f| = l.$$

Let us take some specific profile $\theta \in \Theta^+ \cup \Theta^-$ and denote $TRW_f = w'$ and $TSW_f = w$. If $\theta \in \Theta^+$, it must be that

$$|\mu(s_i(\theta), s'_{-i}(\theta), \theta)| - |\mu(s_i(\theta), s_{-i}(\theta), \theta)| = 1. \tag{1}$$

It must be the case that without firm f's offer, w' has an offer from another firm, and w does not. Similarly, if $\theta \in \Theta^-$, it is w who has an outside offer, and hence

$$|\mu(s_i(\theta), s_{-i}(\theta), \theta)| - |\mu(s_i(\theta), s'_{-i}(\theta), \theta)| = 1. \tag{2}$$

We will now show that $|\Theta^{+}| \geq |\Theta^{-}|$. Equations (1) and (2) along with the fact that each $\theta \in \Theta^{+} \cup \Theta^{-}$ happens with the same probability will then be enough to prove the result.

Construct function $\psi : \Theta \to \Theta$ as follows: Let $\psi(\theta)$ be the profile in which workers have preferences as in θ, but firms $-i$ all swap the positions of workers w' and w in their preference lists. If $\theta \in \Theta^{-}$, without firm f's offer, w has an offer from another firm, and w' does not. But then when preferences are $\psi(\theta)$, without firm i's offer

i. Worker w' ***must*** have another offer.

ii. Worker w ***cannot*** have another offer.

To see i), note that under θ, worker w sends a signal to firm f, so his outside offer must come from a firm f' who has ranked him first. Under $\psi(\theta)$, this firm ranks w' first. If w' has not sent a signal to firm f', then by anonymity, w' gets the offer of firm f'. If worker w' has signaled to firm f', worker w' again gets the offer of firm f'

To see ii), suppose to the contrary that under $\psi(\theta)$, worker w does in fact receive an offer from some firm $f' \neq f$. Since w sends a signal to firm f', w must be $\text{TRW}_{f'}$ under $\psi(\theta)$, so that w' is $TRW_{f'}$ under θ. But then by anonymity w' receives the offer of firm f' under θ, a contradiction.

From i) and ii), we have

$$\theta \in \Theta^{-} \Rightarrow \psi(\theta) \in \Theta^{+}.$$

Since $\psi(\cdot)$ is injective, we have $|\Theta^{+}| \geq |\Theta^{-}|$, which is enough to prove the result. □

Proof of Proposition 6.

For every profile θ such that the switch $(s \to s')$ causes a net worker welfare decrease, we can find a profile such that the change causes a net worker welfare increase of greater magnitude as follow. Let us consider som profile $\theta \in \Theta$ and denote $TRW_f = w'$ and $TSW_f = w$.

Now swap the positions of w' and w in the lists of all firms $-f$. Swap the lists of worker w' and worker w, except for firm f, which maintains its ranking in the two lists.

This switch has the effect of swapping the outside (non f) offers of worker w' and worker w, except possibly adding some offers for w' (who may not sent a signal to firm f) and possibly removing some offers to worker w (who has sent a signal to firm f). Under this new profile θ', net worker welfare is greater from f's offer to w, and because of the extra outside offers for worker w' (and fewer for worker w), the welfare difference is greater than under θ.

Finally, since under anonymous strategies, workers all receive ex ante identical utility, this change must improve welfare for all workers.□

Proof of theorem 3.

Let us consider some block-symmetric equilibrium of the offer game with signals (s_F, s_W). Let us denote the strategy of firms when they ignore all the signals as s_F^0. One could notice that though this is possibly not an equilibrium set of strategies, the expected number of matches and the welfare of workers in the offer game with signals when agents employ strategies (s_F^0, s_W) equals the expected number of matches and the welfare of workers in the offer game with no signals. Then, we can apply the results of proposition 5 and proposition 6 to obtain the results about the number of matches and the welfare of workers.

Similar to the case of uniform distribution of preferences the signals ambiguously influence the welfare of firms. The next example illustrates the point. Though this example considers only the case of the uniform distribution of preferences (one block of firms) its result could be easily generalized for the case of block-uniform preferences.

Example 1 *Let us consider that case when there are two firms* $\{f_1, f_2\}$ *and two workers* $\{w_1, w_2\}$. *The utility of a firm from being matched with its best and second best worker is* δ_1 *and* δ_2 *correspondingly,* $\delta_1 > \delta_2$. *We compare two games: the offer game with no signals and the offer game with signals.*

i) If signals are not allowed each firms makes its offer to its top worker. Hence, the expected welfare of each firm equals $U_f = \frac{3}{4}\delta_1$.

ii) When signals are allowed a symmetric equilibrium can be characterized by a single cutoff point (firm f *receive one signal). We have the following characterization of equilibria* *existence and welfare properties of firms for the offer game with signals.*

	$\delta_2 \leq \frac{1}{2}\delta_1$	$\delta_2 > \frac{1}{2}\delta_1$
$j_{\|h\|=1} = 1$	$\frac{3}{4}\delta_1$, *eqilibrium*	-, *not equilibrium*
$j_{\|h\|=1} = 2$	$\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$, *equililibrium*	$\frac{1}{2}\delta_1 + \frac{1}{2}\delta_2$, *equilibrium*

If $\delta_2 \leq \frac{1}{2}\delta_1$ there are two equilibria in the model when signals are allowed. One of the equilibrium coincides with the equilibrium in the game without signals. Hence, the general result for one block case applies. The welfare of firms is weakly smaller in the offer game with signals compare to the offer game without signals. However, if $\delta_2 > \frac{1}{2}\delta_1$, there is only one equilibrium in the offer game with signals when firms always respond to signals. In this case firm's welfare in the unique equilibrium in the offer game with signals is greater then in the unique equilibrium in the game without signals. $\square$

B Appendix. Equilibrium Ranking in a Single Block

Proof of theorem 4.

We use the result of proposition 4 and Tarski fixed-point theorem [see 8] in order to prove the existence of equilibria. The strategy of firm f is a vector of cutoff points, $X^f = (x_0^f, ..., x_F^f)$. Each cutoff point corresponds to a different number of signals firm f could receive. Let us denote the set of all cutoff strategies of F firms as L with $\Gamma = \left(X^1, ..., X^F\right) \in L$ being its typical element. We impose the following partial order on L, $\Gamma \geq_L \Gamma' \Leftrightarrow X^j \geq X'^j \Leftrightarrow x_k^j \geq x_k'^j$ $\forall j = 1, ..., F$, $\forall k = 0, ..., F$ which is reflexive, antisymmetric and transitive[14]. The set of all strategies and the imposed partial order is complete lattice $(L, \geq_L)$ because L is a finite grid. We consider function $f = (f_1, ..., f_F)$, $f : L \to L$ such that $f_j(\gamma) = \underset{X'}{argmax}(U^j(X', X^{-j})$.Proposition 4 shows that if some firm increases its cutoff strategy for some number of received signals then it is profitable for the other firms to (weakly) increase their cutoff strategies. Hence, function f is weakly increasing on L.

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[14] See .... for the definitions of reflexive, antisymmetrica, and transiive order.

Therefore, there the set of fixed points is not empty and it comprises of a complete lattice with respect to $\geq_L$ according to Tarski fixed-point theorem. $\square$

C Appendix. The value of a signaling mechanism.

Proof of proposition 2.

We first calculate the explicit formula for the increase in the expected number of matches from the introduction of the signaling mechanism with one signal.

Lemma C1 *Let us consider the markets with $W \geq 2$ and $F \geq 2$. The expected number of matches in the offer game with W workers and F firms with no signals equals*

$$U(F, W) = W\left(1 - \left(1 - \frac{1}{W}\right)^F\right) \tag{3}$$

The expected number of matches in the offer game with W workers and W firms with one signal equals

$$S(F, W) = F\left(\begin{array}{c} 1 - (\frac{F-1}{F})^W + \frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}}\left(1 - \frac{F-1}{W}\left(1 - (\frac{F-2}{F-1})^W\right)\right) * \\ * \left(1 - \left(1 - \frac{(\frac{F-2}{F-1})^{W-1}}{W}\right)^{F-1}\right) \end{array}\right) \tag{4}$$

The increase of the expected number of matches from the introduction of signaling mechanism

$$D_1(F, W) = S(F, W) - U(F, W) \tag{5}$$

Proof of lemma C1

Let us first calculate the expected number of matches in the offer game without signals. The unique equilibrium of the game with no signals is that each firm makes an offer to its

TRW and workers accept the best offer among available ones. Let us denote the probability that firm f makes an offer to a particular worker as $p = \frac{1}{W}$. Therefore, the probability that firm f is matched equals:

$$U_f(F,W) = (1-p)^{F-1} + C^1_{F-1}p^1(1-p)^{F-2}\frac{1}{2} + ... C^j_{F-1}p^j\,(1-p)^{F-1-j}\,\frac{1}{j+1} + ... + C^{F-1}_{F-1}p^{F-1}\,(1-p)^0\,\frac{1}{F}$$

Intuitively, if firm f makes an offer to worker w, j firms among the other $F-1$ firms simultaneously make an offer to worker w with probability $C^j_{F-1}p^j(1-p)^{F-1-j}$. Hence, firm f is matched with worker w only with probability $\frac{1}{j+1}$. The sum over all possible j from 0 to $F-1$ gives us the overall probability of firm f being matched.

$$\begin{aligned}
U^f(F,W) &= \sum\nolimits_{j=0}^{F-1} C^j_{F-1}p^j\,(1-p)^{F-1-j}\,\frac{1}{j+1} && (6)\\
&= \sum\nolimits_{j=0}^{F-1} \frac{(F-1)!}{j!\,(F-1-j)!}p^j\,(1-p)^{F-1-j}\,\frac{1}{j+1} && (7)\\
&= \sum\nolimits_{j=0}^{F-1} \frac{1}{Fp}\frac{F!}{(j+1)!\,(F-(1+j))!}p^{j+1}\,(1-p)^{F-(1+j)} && (8)\\
&= \frac{1}{Fp}\sum\nolimits_{j=0}^{F-1} \frac{F!}{t!\,(F-t)!}p^t\,(1-p)\,\backslash && (9)\\
&= \frac{1}{Fp}\left(\sum\nolimits_{j=0}^{F-1} \frac{F!}{t!\,(F-t)!}p^t\,(1-p)^{F-t} - (1-p)^F\right) && (10)\\
&= \frac{1}{Fp}\left(1-(1-p)^F\right) && (11)\\
&= \frac{W}{F}\left(1-\left(1-\frac{1}{W}\right)^F\right) && (12)
\end{aligned}$$

Therefore, the expected number of matches in the game with no signals equals

$$U(F,W) = W\left(1-\left(1-\frac{1}{W}\right)^F\right) \qquad (13)$$

Let us now calculate the expected number of matches in the offer game with signals, when each worker can send only one signal. As described above there is the unique equilibrium in the offer game with signals. Each worker sends her signal to her top firm, each firm makes its offer to TSW (top signaled worker) if it receives at least one signal, otherwise it makes an offer to its TRW (top ranked worker).

We first calculate the probability of being matched by some firm f. We denote the set of workers that signal to firm f as $h_f \subset W$. If firm receives at least one signal, $|h_f| > 0$, it guarantees itself a match, because each worker send a signal to her top firm. If firm receives no signals, firm makes an offer to its top worker. This worker accepts firms' offer only if firm f is the best firm in the worker's preferences among the firms she receives an offer from.

$$S^f(F,W) = P(|h_f| > 0) * 1 + P(|h_f| = 0) * P(TRW_f \text{ accepts firm } f'\text{s offer} \,|\, |h_f| = 0) \quad (14)$$

Firm f receives no signals with probability equal

$$P(|h_f| = 0) = (1 - \frac{1}{F})^W \quad (15)$$

For further notation, we denote $q = (1 - \frac{1}{F})^W$.

If firm f receives no signals, $P(|h_f| = 0)$, it makes an offer to TRW_f which we denote as worker $w = TRW_f$. Firm can be matched with worker w only if she does not receive an offer from its top firm. Worker w receives an offer from its top firm, say firm f_0, conditional on $|h_f| = 0$ with probability equal

$$\begin{aligned} G &= P(|h_{f_0}| = 1 \,|\, |h_f| = 0) * 1 + P(|h_{f_0}| = 2 \,|\, |h_f| = 0) * \frac{1}{2} + ... + P(|h_{f_0}| = W \,|\, |h_f| = 0) * \frac{1}{W} \quad (16) \\ &= \textstyle\sum_{j=0}^{W-1} C_{W-1}^{j} \left(\frac{1}{F-1}\right)^j (1 - \frac{1}{F-1})^{W-j-1} \frac{1}{j+1} \end{aligned}$$

Intuitively, firm f_0 receives a signal from a particular worker with probability $\frac{1}{F-1}$ because firm f receives no signals. Then, if firm f_0 receives signals from j other workers, worker w receives an offer from firm f_0 with probability $\frac{1}{j+1}$. Similarly to equation (6) the expression for G could be simplified

$$\begin{aligned} G &= \textstyle\sum_{j=0}^{W-1} C_{W-1}^{j} \left(\frac{1}{F-1}\right)^j (1 - \frac{1}{F-1})^{W-j-1} \frac{1}{j+1} \quad (17) \\ &= \frac{F-1}{W} \left(1 - (1 - \frac{1}{F-1})^W\right) \end{aligned}$$

In addition, worker w does not receive an offer from firm f_0 with probability $1 - G$. If worker w does not receive an offer from her top firm $-$ firm f_0 - firm f competes with firms (excluding itself and firm f_0) that have received no signals from workers. The probability that some firm f' among firms $F \backslash \{f, f_0\}$ receives no signals conditional on the fact that worker w sends her signal to firm f_0 and $|h_f| = 0$ equals $t = (1 - \frac{1}{F-1})^{W-1}$. Note that the probability that firm f' does not receive a signal from a worker equals $1 - \frac{1}{F-1}$, because firm f receives no signals, $|h_f| = 0$. There are also only $W - 1$ workers that can send a signal to firm f', because worker w sends her signal to firm f_0.

Therefore, the probability that some firm f' among firms $F \backslash \{f, f_0\}$ receives no signals and makes an offer to worker w equals $\frac{t}{W}$ conditional on the fact that worker w sends her signal to firm f_0. Therefore, the probability that worker w prefers offer of firm f to other offers conditional on the fact that firm f receives no signals and worker w sends her signal to firm f_0, equals[15]

$$\sum_{j=0}^{F-2} C_{F-2}^{j} \left(\frac{t}{W}\right)^{j} (1 - \frac{t}{W})^{F-2-j} \frac{1}{j+1} = \frac{W}{(F-1)\,t} \left(1 - \left(1 - \frac{t}{W}\right)^{F-1}\right) \tag{18}$$

Then the probability that worker w accepts firm f's offer equals

$$\begin{aligned} P(\text{firm } f'\text{s offer to } TRW_f \text{ is accepted}|\,|h_f| &= 0) = (1-G)\left(\frac{W}{(F-1)\,t}\left(1 - \left(1 - \frac{t}{W}\right)^{F-1}\right)\right) \quad (19) \\ &= \left(1 - \frac{F-1}{W}\left(1 - (1 - \frac{1}{F-1})^{W}\right)\right) * \\ &\quad * \frac{W}{(F-1)\,t}\left(1 - \left(1 - \frac{t}{W}\right)^{F-1}\right) \end{aligned}$$

---

[15] Note, that the maximum number of offers worker w could get equals to $M - 1$ as it does not receive an offer from its top firm f_0.

Overall, the probability of firm f being matched in the offer game with signals equals

$$\begin{aligned} S^f(F,W) &= 1-q+q*P(\text{firm } f'\text{s offer to } TRW_f \text{ is accepted}|\,|h_f|=0) \qquad (20)\\ &= 1-q+q*\frac{W}{(F-1)\,t}\left(1-\frac{F-1}{W}\left(1-(1-\frac{1}{F-1})^W\right)\right)*\\ &\quad *\left(1-\left(1-\frac{t}{W}\right)^{F-1}\right)\\ &= 1-(\frac{F-1}{F})^W+\frac{W\,(F-1)^{2W-2}}{F^W\,(F-2)^{W-1}}\left(1-\frac{F-1}{W}\left(1-(\frac{F-2}{F-1})^W\right)\right)*\\ &\quad *\left(1-\left(1-\frac{(\frac{F-2}{F-1})^{W-1}}{W}\right)^{F-1}\right) \end{aligned}$$

The expected number of signals in the offer game with signals equals $S(F,W)=FS^f(F,W)$.

□

Lemma C1 establishes that the increase in the number of matches due to the introduction of the signaling mechanism equals

$$D_1(F,W)=F\left(\begin{array}{c} 1-(\frac{F-1}{F})^W+\frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}}\left(1-\frac{F-1}{W}\left(1-(\frac{F-2}{F-1})^W\right)\right)*\\ *\left(1-\left(1-\frac{(\frac{F-2}{F-1})^{W-1}}{W}\right)^{F-1}\right) \end{array}\right)-W\left(1-\left(1-\frac{1}{W}\right)^F\right)$$

Let us first fix W and calculate where $D(F,W)$ attains its maximum. We consider large markets, where F and W are large. Let us denote $x=\frac{F}{W}$. Then using Taylor expansion formula

$$(1-a)^b=\exp(-ab+O(a^2b)) \qquad (21)$$

we have

$$U(F,W)=W\left(1-\left(1-\frac{1}{W}\right)^F\right)=W(1-e^{-\frac{F}{W}+O(\frac{F}{W^2})})=W(1-e^{-x+O(x/W)})$$

Let us consider the number of matches in the offer game with signals

$$S(F,W)=Wx\left(1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}+A*B\right)$$

where

$$\begin{aligned}
A &= \left(1-\frac{F-1}{W}\left(1-(\frac{F-2}{F-1})^{W}\right)\right)\\
B &= \frac{W\left(F-1\right)^{2W-2}}{F^{W}\left(F-2\right)^{W-1}}*\left(1-\left(1-\frac{(\frac{F-2}{F-1})^{W-1}}{W}\right)^{F-1}\right)
\end{aligned}$$

We first calculate an approximation of A for large markets. Using (21) we get

$$1-(1-\frac{1}{F-1})^{W}=1-e^{-\frac{W}{F}+O(\frac{W}{F^2})}=1-e^{-x+O(\frac{1}{x^2W})}$$

Then

$$\begin{aligned}
A &= 1-\frac{F-1}{W}\left(1-(\frac{F-2}{F-1})^{W}\right)\\
&= 1-\frac{F-1}{W}\left(1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}\right)\\
&= 1-x\left(1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}\right)+O(\frac{1}{xW})
\end{aligned}$$

We now calculate an approximation of B for large markets.

$$\begin{aligned}
\frac{W\left(F-1\right)^{2W-2}}{F^{W}\left(F-2\right)^{W-1}} &= \frac{W}{F}\left(\frac{F-1}{F}\right)^{W-1}\left(\frac{F-1}{F-2}\right)^{W-1}\\
&= \frac{1}{x}e^{-\frac{W-1}{F}+O(\frac{1}{x^2W})}e^{\frac{W-1}{F-1}+O(\frac{1}{x^2W})}\\
&= \frac{1}{x}e^{O(\frac{1}{x^2W})}
\end{aligned}$$

Also, we have that

$$\begin{aligned}\left(1-\left(1-\frac{Z^{W-1}}{W}\right)^{F-1}\right) &= 1-e^{-\frac{Z(F-1)}{W}+O(\frac{x}{W})}\\ &= 1-e^{-Zx+O(\frac{x}{W})}\end{aligned}$$

where $Z = (\frac{F-2}{F-1})^{W-1} = e^{-\frac{W}{F}+O(\frac{W}{F^2})} = e^{-\frac{1}{x}+O(\frac{1}{x^2W})}$. Then, we have that

$$\begin{aligned}B &= \frac{W\,(F-1)^{2W-2}}{F^W\,(F-2)^{W-1}} * \left(1-\left(1-\frac{(\frac{F-2}{F-1})^{W-1}}{W}\right)^{F-1}\right)\\ &= \frac{1}{x}e^{O(\frac{1}{x^2W})}(1-e^{-xe^{-\frac{1}{x}}+O(x/W)})\end{aligned}$$

Overall, we have

$$\begin{aligned}D(F,W) &= F\left(\begin{array}{c}1-(\frac{F-1}{F})^W+\frac{W(F-1)^{2W-2}}{F^W(F-2)^{W-1}}\left(1-\frac{F-1}{W}\left(1-(\frac{F-2}{F-1})^W\right)\right)*\\ *\left(1-\left(1-\frac{(\frac{F-2}{F-1})^{W-1}}{W}\right)^{F-1}\right)\end{array}\right)-W\left(1-\left(1-\frac{1}{W}\right)^F\right)\\ &= Wx\left(\begin{array}{c}1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}+\left(1-x\left(1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}\right)+O(\frac{1}{xW})\right)*\\ *\frac{1}{x}e^{O(\frac{1}{x^2W})}(1-e^{-xe^{-\frac{1}{x}}+O(x/W)})\end{array}\right)-W(1-e^{-x+O(x/W)})\\ &= W\left(x-xe^{-\frac{1}{x}}+\left(1-x\left(1-e^{-\frac{1}{x}}\right)\right)(1-e^{-xe^{-\frac{1}{x}}})-1+e^{-x}\right)+O(1)\\ &= Wg(x)+O(1)\end{aligned}$$

where the graph of $g(x)$ looks like

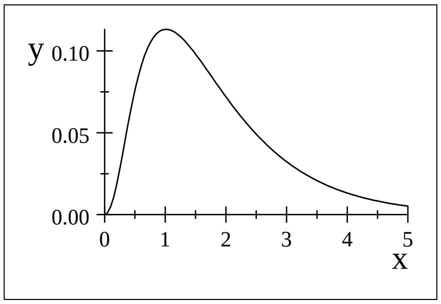

Figure IV

The maximum is attained at $x_0 = 1.012113$. Therefore we know that for fixed W, D

attains its maximum value at $F = x_0 W + O(1)$.

Let us first fix F and calculate the value of W where $D(F, W)$ attains its maximum. We consider large markets, where F and W are large. Let us again denote $x = \frac{F}{W}$. Using the same approximations we get

$$\begin{aligned} U(F,W) &= W\left(1-\left(1-\frac{1}{W}\right)^{F}\right) \\ &= \frac{F}{x}(1-e^{-\frac{F}{W}+O(\frac{F}{W^2})}) \\ &= \frac{F}{x}(1-e^{-x+O(\frac{x^2}{F})}) \end{aligned}$$

Let us consider the number of matches in the offer game with signals

Overall, we have

$$\begin{aligned} D_1(F,W) &= S(F,W) - U(F,W) \\ &= F\left(\begin{array}{c} 1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}+\left(1-x\left(1-e^{-\frac{1}{x}+O(\frac{1}{x^2W})}\right)+O(\frac{1}{xW})\right) \\ *\frac{1}{x}e^{O(\frac{1}{x^2W})}(1-e^{-xe^{-\frac{1}{x}}+O(x/W)}) \end{array}\right) - \frac{F}{x}(1-e^{-x+O(\frac{x^2}{F})}) \\ &= F\left(1-e^{-\frac{1}{x}}+\left(1-x\left(1-e^{-\frac{1}{x}}\right)\right)\frac{1}{x}(1-e^{-xe^{-\frac{1}{x}}})-\frac{1}{x}\left(1-e^{-x}\right)\right)+O(1) \\ &= Wf(x)+O(1) \end{aligned}$$

where the graph of $f(x)$ looks like

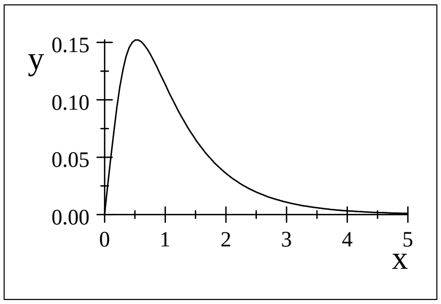

Figure V

The maximum is attained at $x_{00} = 0.53074$. Therefore we know that for fixed F, D attains its maximum value at $W = y_0 F + O(1)$, where $y_0 = 1/x_{00} = 1.8842$.

D Appendix. Simulations.

In order to support (check) our theoretical results presented above we calculate the increase of the expected number of matches from the introduction of signaling mechanism using completely different approach - computer simulations. There is the unique equilibrium in both games with and without signaling mechanism. Therefore, we do not need to optimize agents' behavior, rather we need to calculate the expected number of matches for different realizations of agents' preferences and estimate the average effect of the introduction of signals for markets of different size. We use MATLAB[16] for these purposes.

D.1 Simulations. Balanced markets.

Figures presented below illustrate our computer simulation results for balanced markets. These figures correspond to Figure I.1 presented earlier and show that the theoretical and simulation results coincide (within the accuracy of the estimations). The fact that the curve is not smooth is due to simulation approach. The size of deviations can be easily decreased by the increase of the number of repetitions we used to calculate the expected number of matches.

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[16] MATLAB. Copyright 1984-2008 The MathWorks, Inc. Version 7.7.0.471 (R2008b).

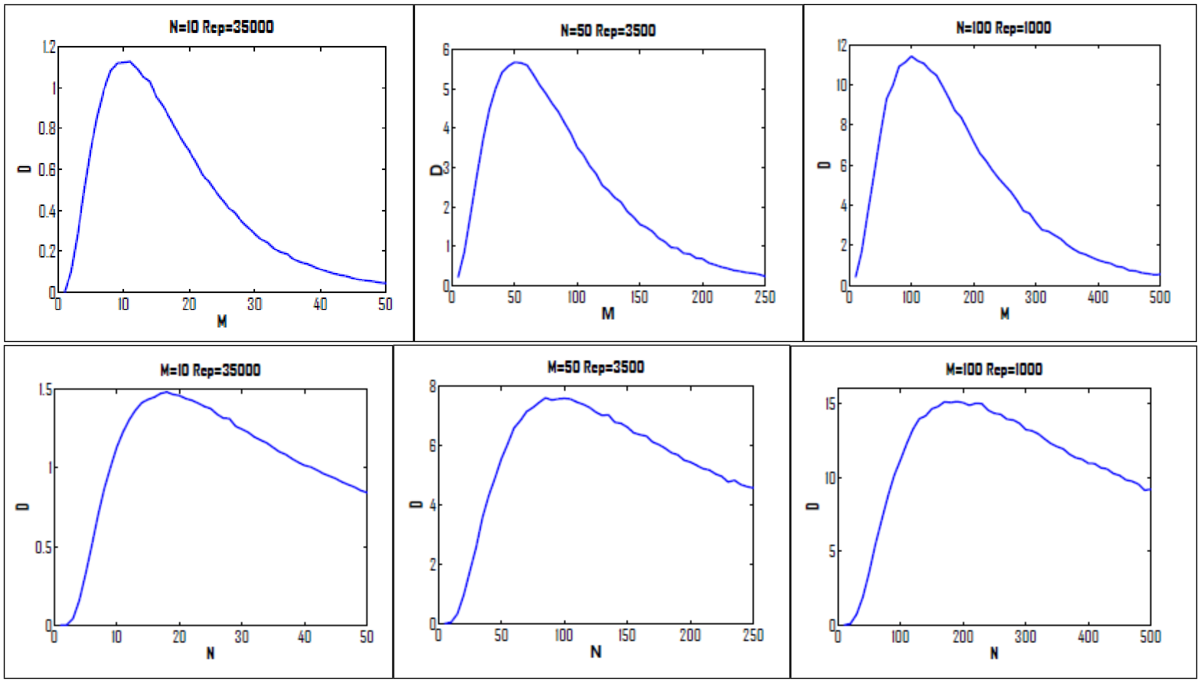
N=10 Rep=35000
D
0
0.2
0.4
0.6
0.8
1
1.2
M
0
10
20
30
40
50
N=50 Rep=3500
D
0
1
2
3
4
5
6
M
0
50
100
150
200
250
N=100 Rep=1000
D
0
2
4
6
8
10
12
M
0
100
200
300
400
500
M=10 Rep=35000
D
0
0.5
1
1.5
N
0
10
20
30
40
50
M=50 Rep=3500
D
0
2
4
6
8
N
0
50
100
150
200
250
M=100 Rep=1000
D
0
5
10
15
N
0
100
200
300
400
500

Figure I.1

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