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On the robustness of herds

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In a series of interesting papers Banerjee (1992) and Bikhchandani, Hirshleifer and Welch (1992) develop stylized models of herd-like behavior in abstract setup. In the model each agent receives a private signal about the state, observes the past decisions of other agents and takes some action. These early models are quite stark in their assumptions: agents move in an exogenously specified sequence, agents make simple zero-one decisions, and agents are not allowed to share information even though they may have an incentive to do so. This literature recognizes that these assumptions are stark. For example, Banerjee argues that it would be more natural to let the order of choice be endogenous, but that this modification is rather complicated. The starkness of these assumptions have led economists to both question the robustness of the results and the relevance of herd behavior in applied contexts (Lee 1995). In this paper we relax the stark assumptions and find that there are still herds.

The applied context we are most interested in is one of investment in a foreign country by domestic lenders. We are motivated by the recent crises in Latin America and Asia. [1]We think the herding story is an attractive explanation of these crises but in this applied context as well as others the stark assumptions of the simple model seem strained: investors cannot make investments either before their turn in the order or after their turn, they must invest all or none of their money, and they cannot share the information. If herds are to be relevant they should be robust to a relaxation of these assumptions.

We begin by developing a model with endogenous timing of moves, but we retain the zero-one decisions and we don't allow communication. In the model agents choose when, if at all, to invest one unit in a risky project. In each period a single investor receives a private signal about the profitability of the risky project. Investors observe the aggregate amount of investment in each period and optimally decide whether to invest or to wait for more information. Waiting is costly because of discounting. In the model if agents become either sufficiently optimistic or pessimistic about profitability they take the same action regardless of their private signal and information can get trapped. We say that an outcome is a herd if investors decisions do not vary with their signals and the aggregate outcome differs from that with public signals. We show that in the stationary equilibrium there are both herds of investment and herds of no investment. We also show that the equilibrium outcomes are identical to those in a version of the model in which the order of moves is exogenously specified. In this sense herds are robust to endogenous timing.

Next, we extend the model to allow for continuous investment decisions but do not allow communication. In the model investors must choose when, if at all, to invest any nonegative amount in a risky project. We show that there are both herds of investment and herds of no investment. Interestingly, in the corresponding version with exogenous timing it turns out that are no herds of investment, because whenever agents invest, the size of their investment reveals their signal perfectly. (This result is shown in Lee 1995). In contrast, with endogenous timing, if investors become sufficiently optimistic they all invest thereby forgoing to opportunity to receive and act on a later signal. In this sense, when investment decisions are continuous, herds are not merely robust to endogenous timing but become more likely than with exogenous timing.

Finally, we consider the incentives to communicate in a version of the endogenous timing model with zero-one investment decisions. We allow investors to send messages about their signals. We assume there is an early-mover advantage to investing in that the return is decreasing in the aggregate number of investors. We show that there is a unique stationary equilibrium which coincides with that in the version without communication. In this sense, herds are robust to information-sharing.

Our paper is most closely related to Caplin and Leahy (1993) and Chamley and Gale (1994). Both of these papers allow for private signals and endogenous timing of decisions. In Caplin and Leahy's model a continuum of agents at each date agents must decide whether to continue with their projects or suspend them. The profitablity of projects is perfectly correlated across agents and in each period each agents gets a private signal about this profitability. Caplin and Leahy show that there is some deterministic date at which agents who have received all negative signals suspend their projects. The fraction of agents who suspend projects perfectly reveals the underlying state and all agents either continue their projects to completion or abandon them forever. Thus, the aggregate number of completed projects coincides with that under public information. In this sense, while Caplin and Leahy's model generates collapses of investment but not herds.

Chamley and Gale consider a model in which all signals are received in the first period. In equilibrium agents follow mixed strategies over investing and waiting in each period. With a large number of agents the time profile of investment has an extreme form in that in the first period there is a negligible amount of investment followed in the second period by either everyone investing or no one investing. Chamley and Gale show that the equilibrium outcomes differ from those with public information. In this sense, Chamley and Gale's model generates herds but at most one period of low investment. In contrast, in our model there can be many periods of low investment followed by a burst of investment.

1. Endogenous timing

Consider an economy populated with N risk neutral agents each of whom has 1 unit of resources to invest. There are two assets, a safe domestic asset with gross return that is normalized to 1 and a risky foreign asset. The payoff on the risky foreign asset depends on the state denoted $y \in \{G, B\}$, where G is referred to as the good state and B is referred to as the bad state. The state of the foreign country is unknown, although the distribution is common knowledge to the agents. The common prior probability of the good state at date 0 is p_0.

Each agent starts at period 0 with one unit invested in the domestic asset and in each period t, for $t = 0, \ldots, T_1$ chooses whether or not to switch his investment from the safe domestic asset into the risky foreign asset. Once an agent has switched his money he must leave it there until period T, which satisfies $T > T_1$. In period T, the foreign asset pays a gross return of R per period if the state is good or a gross return of 0 if the state is bad. This return is continuously compounded and consumption occurs at T. Hence, an investor that switches at time t, gets a total expected return of

$$e^{R(T-t)}\text{Prob}_t(\text{state is } G).$$

where $\text{Prob}_t(\text{state is } G)$ is the conditional probability that the lender assigns to the state being G while an investor who never switches gets a total return of 1. Notice that after an agent invests in the risky project there are no actions to be taken. Hence, we do not need to define payoffs or strategies for such agents.

Agents receives signals $s \in \{G, B\}$ about the state as follows. In each period $t = 0, \ldots, T_1$ one signal arrives to the economy and is randomly distributed to one and only one agent among the set of agents who have not already received a signal.[2] The signals are informative and symmetric in the sense that

$$\text{(1)} \qquad \Pr(s = G \mid y = G) = \Pr(s = B \mid y = B) = q > 1/2.$$

The only publicly observable events are the number of investments in each period. Let n_t denote the number of new investments at t. The public history $h_t = (n_0, n_1, , \ldots, n_{t-1})$ records the aggregate number of positive investments in each period up through the beginning of period t. Agents also record the signal they receive, if any, and the date they receive it. Thus, the history of an agent i at t who received a signal at r is $h_{it} = (h_t, s_r, r)$ and we let $(h_t, \emptyset, \emptyset)$ denote the history of an agent who has not received a signal. Notice that at each date t, given their histories, agents can be described as belonging to one of several groups. Any agent that has already invested is *inactive*. The active agents consist of a *newly informed* agent who received the signal at the beginning of period t, *previously informed* agents who received a signal at some date r before t and *uninformed* agents who have not yet received a signal. Notice that given the structure of signals, the probability that an uninformed agent receives a signal at t is $\pi_t = 1/(N - t)$.

An agent's strategy and beliefs are sequences of functions $x_t(h_{it})$ and $p_t(h_{it})$ that map their histories into actions and priors over the state. The payoffs are defined as follows. Let $V_t(h_{it})$ denote the payoff for an agent that switches his assets from the safe to the risky project at t conditional on the history h_{it}, then

$$V_t(h_{it}) = e^{R(T-t)} p_t(h_{it}),$$

Let $W_t(h_{it})$ denote the payoff for an agent that waits at time t. The payoff to waiting is given by

$$W_t(h_{it}) = \sum_{h_{it+1}} \mu_t(h_{it+1}|h_{it}) \max\{V_{t+1}(h_{it+1}), W_{t+1}(h_{it+1})\})$$

where $\mu_t(h_{it+1}|h_{it})$ is the conditional distribution over history at $t+1$ given the history at t. Notice that the conditional distributions $\mu_t(h_{it+1}|h_{it})$ are induced from the strategies and the structure of exogenous uncertainty of the game in the obvious way. Notice also that we have imposed symmetry by supposing that all agents who have the same histories take the same actions and have the same beliefs. Here, a *perfect Bayesian equilibrium* is a set of strategies $x_t(h_{it})$, a set of conditional distributions $\mu_t(h_{it+1}|h_{it})$ and a set of beliefs $p(h_{it})$ such that i) for every history h_{it}, such that the agent has not switched to the risky project before t, $x_t(h_{it})$ is optimal, ii) the conditional distributions $\mu_{it}(h_{it+1}|h_{it})$ and the beliefs $p_{it}(h_{it})$ are consistent with Bayes' rule wherever possible and arbitrary otherwise.

In constructing an equilibrium we will find it useful to let $P_G(p)$ and $P_B(p)$ be the posteriors associated with a good and a bad signal respectively, when the prior is given by p. Thus,

$$P_G(p) = \frac{pq}{pq + (1-p)(1-q)} \tag{2}$$

$$P_B(p) = \frac{p(1-q)}{p(1-q) + (1-p)q}. \tag{3}$$

where q is given in (1). Let $P(0) = p_0$, $P(1) = P_G(P(0)), P(2) = P_G(P(1))$ and so on, and let $P(-1) = P_B(P(0))$, $P(-2) = P_B(P(-1))$ and so on. Thus, $P(k)$ for $k > 0$, is the prior probability that the state is good if k good signals have been received and $P(k)$ for $k < 0$, is the prior probability that the state is good if k bad signals have been received. Notice from the symmetry in (1) that

$$P_G(P_B(p)) = P_B(P_G(p)) = p. \tag{4}$$

It follows from (4) that the effect on the prior of a given set of signals is summarized by the number of good minus the number of bad signals in the set. Thus, for example, receiving two good signals and one bad signal yields that same prior as receiving one good signal.

We begin by analyzing the region of the parameter space that satisfies the following:

$$(5) \qquad 1 < e^{R(T-T_1)}P(0)$$

$$(6) \qquad 1 > e^{RT}P(-1)$$

$$(7) \qquad e^{RT}P(0) < \nu_G(P(0))e^{R(T-1)}P(1) + \nu_B(P(0)).$$

Here $\nu_G(p) = P(s = G|p) = pq + (1-p))(1-q)$ and $\nu_B(p) = P(s = B|p)) = p(1-q) + (1-p)q$ denote the probabilities that the signal is good and bad, respectively, given a prior of p. Note that because $R > 1$ these assumptions imply the following. Assumption (5) implies that at any date t, between the two options of investing in the risky asset given belief $P(0)$ or never investing in the risky asset, it is better to invest. Assumption (6) implies that at any date t, between the two options of investing in the risky asset given belief $P(-1)$ or never investing in the risky asset, it is better to not invest. Assumption (7) implies

$$(8) \qquad e^{R(T-t)}P(0) < \nu_G(P(0))e^{R(T-t-1)}P(1) + \nu_B(P(0))$$

so that at any date t, investing with beliefs $P(0)$ is dominated by waiting one period and investing if and only if a good signal is realized.

To understand the role of (7), note that in this model, waiting and receiving information is beneficial because agents having the option of not investing if the signals are sufficiently bad. We call this benefit the *no investment option value.* The cost of waiting comes from a kind of discounting in that agents forgo the flow return from investing. Assumption (7) requires that the no investment option value be large relative to discounting.

We begin with an informal description of the equilibrium outcomes. At the beginning of date 0 one agent receives a signal and is the newly informed agent. That agent invests if the signal is good and does not otherwise. All uninformed agents wait.

The decisions at date 1 depends on the history from date 0. If there was positive investment at date 1 all agents invest at date 1. We say that this history starts *a cascade with investment*. If there was zero investment at date 0 then the uninformed agents wait while the newly informed agent at date 1 invests if his signal is good and waits otherwise.

At the beginning of date 2 if there has been no investment at either dates 0 or 1 then no agent invests at date 2 or any subsequent date. We will say that this history starts a *cascade with no investment*. If there has been no investment at date 0 but an investment at date 1 then both the uninformed agents and the previously informed agent wait and the newly informed agents invests if his signal is good and does not otherwise.

More generally, histories of the form (1), $(0, 1, 1)$, $(0, 1, 0, 1)$, $\ldots$, $(0, 1, 0, 1, \ldots, 0, 1, 1)$ start cascades with investment. Histories of the form $(0, 0)$, $(0, 1, 0, 0)$, $\ldots$, $(0, 1, 0, 1, \ldots, 0, 1, 0, 0)$, start cascades with no investment.

More formally we proceed as follows. The strategy for all uninformed and previously informed agents is

$$(9) \qquad x_t(h_{it}) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_{it}) \geq P(1) \\ \\ 0 \text{ otherwise} \end{array} \right\}$$

for $t \leq T_1 - 1$ and $x_{T_1}(h_{iT_1}) = 1$ if and only if $p_{T_1}(h_{iT_1}) \geq P(0)$. The strategy for newly informed agents is

$$(10)\quad x_t(h_{it}) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_{it}) \geq P(0) \\ \\ 0 \text{ otherwise} \end{array} \right\}$$

for $t \leq T_1$. Notice that the uninformed and previously informed agents need to be more optimistic than newly informed agents in order to invest before T_1.

The beliefs of uniformed agents at history $h_{it} = (h_t, \emptyset, \emptyset)$ are recursively defined. Given $p_{t-1}(h_{it-1})$ and a total investment of n_{t-1} at $t-1$, the beliefs at t are given as follows. For $p_{t-1}(h_{it-1})$ equal to either $P(-1)$ or $P(0)$

$$(11)\quad p_t(h_{it}) = \left\{ \begin{array}{c} P_B(p_{t-1}(h_{it-1})) \text{ if } n_{t-1} = 0, \\ \\ P_G(p_{t-1}(h_{it-1})) \text{ if } n_{t-1} = 1 \\ \\ P(2) \text{ if } n_{t-1} \geq 2 \end{array} \right\}.$$

where $p_0(h_{i0}) = p_0$. For $p_{t-1}(h_{it-1})$ either greater than or equal to $P(1)$ or less than or equal to $P(-2)$, $p_t(h_{it}) = p_{t-1}(h_{it-1})$.

The beliefs of the newly informed agents at history $h_{it} = (h_t, S, t)$ are given as follows. These beliefs are simply those of the uninformed agent, updated by the newly informed agent's signal, namely $p_t(h_t, S, t) = P_S(p_t(h_t, \emptyset, \emptyset))$ for $S = G, B$.

The beliefs of the previously informed agent at t who received his signal in $t-1$ with history $h_{it} = (h_t, S, t-1)$ are defined as follows. If no other agent invested at $t-1$, this agent's beliefs are the same as they were in period $t-1$, namely $p_t(h_t, S, t) = P_S(p_{t-1}(h_{t-1}, \emptyset, \emptyset))$ for $S = G, B$. If some other agent invested at $t-1$, $p_t(h_t, S, t) = P(2)$. The beliefs of previously informed agents who received their signals before period $t-1$ are recursively defined using (11) except that the recursion starts at r, with the beliefs of the newly informed agent at r, namely $p_r(h_{ir}, S, r)$.

Built into these beliefs is the idea that agents look at previous agents' actions and try to infer their signals. On the equilibrium path and for deviations that they cannot detect agents infer the following. Consider the uninformed agents. If they see one unit of investment at t and the strategies specify that a newly informed agent receiving a good signal should invest while a newly informed agent receiving a bad signal should not invest, they infer the newly informed agent received a positive signal. They update beliefs in a similar fashion when they see zero investment at t. Notice that when the newly informed agent acts differently based on their signal, their deviations cannot be detected by uninformed agents so that uninformed agents beliefs are updated as if there were no deviation by the informed agent.

On the equilibrium path and for undetectable deviations the newly informed agents simply update the beliefs of the uninformed agents with their private signal. The previously informed agent that was newly informed at $t-1$, simply updates the beliefs of the newly informed agent at $t-1$ appropriately. The previously informed agent that was newly informed at $r < t-1$, simply updates the beliefs of the previously informed agent at $t-1$ appropriately.

For detectable deviations agents infer the following. If an uninformed agent sees more than one unit of investment, beliefs are updated to an optimistic level, namely, $P(2)$. If an uninformed agent sees other deviations, beliefs are left unchanged. Previously informed agents behave similarly. A newly informed agent at t who is active at $t+1$ and sees investments by others also updates beliefs to the optimistic level $P(2)$.

These strategies and beliefs induce the conditional distributions $\mu_t(h_{it+1}|h_{it})$ in the obvious manner. We will show that these strategies and beliefs are an equilibrium. An important feature of the strategies is that cutoff level for investment for the uninformed agents is higher than that for the newly informed agents. To understand why this is necessary suppose first that both types of agents invest if their beliefs are greater than or equal to $P(0)$. To see why this cannot be an equilibrium consider a deviation by an uninformed agent at $t = 0$ with beliefs $P(0)$ to waiting. Since the newly informed agent invests if and only if his signal is good, the deviating agent learns the value of the signal. By (7) this deviation increases payoffs.

Suppose next that the cutoff level for both types of agents is $P(1)$. Suppose the first signal is B. The newly informed at 0 does not invest and the other agents infer he got a bad signal and their priors are $P(-1)$. The newly informed agent at date 1 is supposed to wait regardless of his signal. Thus the prior of the uninformed agent stays at $P(-1)$ and thus all newly informed agents at all future dates also wait. After a history of signals B, G, the newly informed at date 1 has a prior of $P(0)$. A deviation to investing, by (5), raises the payoffs.

These arguments help explain why the cutoff levels of the informed and uninformed agents must be different. We now show that when these cutoffs have the form in (9) and (10) the strategies and beliefs are an equilibrium.

Proposition 1. Under assumptions $(5) - (7)$, the strategies and beliefs in (9)-(11) constitute a perfect Bayesian equilibrium.

Proof. By construction the beliefs in (11) satisfy Bayes' rule. We repeatedly use the observation that by construction for any history h_{it}, $p_t(h_{it}) = P(k)$ for some integer k .

Consider first optimality for histories with no detectable deviations. Consider the strategies of the uninformed agents. For a history of the uninformed agent with $p_t(h_{it}) = P(1)$ the uninformed agent invests and receives

$$e^{R(T-t)}P(1)$$

Suppose the agent deviates and waits. If for all future histories the agent ends up investing then waiting merely reduces the length of time of investment in the high return project and he loses e^R in each period. Thus, the only way that this deviation can be profitable is that there are some future histories in which this agent never invests. Consider the most pessimistic information the agent could receive. Recall that for such a history all other active agents invest at t. Thus, by waiting the uninformed agent receives no new information from others. By waiting the uninformed agent could receive a signal in the future. But even if the future signal is $s = B$, this agent's belief will be $P(0)$ and by (5) he will invest. Thus, even under the most pessimistic information it is optimal to invest, and hence waiting is not profitable. Clearly, for histories with $p_t(h_{it}) \geq P(1)$ it is also optimal to invest.

Consider next the strategies of the informed agents at some history h_{it}. From (6) it follows that deviating to investing when the strategies specify waiting is not optimal, namely for histories in which the newly informed agent's beliefs are $P(k)$ for $k \leq -1$. It also easy to check that deviating in a cascade is not optimal. A cascade is associated with a history in which the uninformed agents' beliefs are either $P(1)$ or $P(-2)$. After such histories either all agents have already invested or they will never invest. In either case deviations are not profitable.

The interesting histories are those in which the uninformed agents' beliefs are $P(0)$ or $P(-1)$ and the newly informed agent has just received a good signal. The strategy for the newly informed agent specifies invest and suppose this agent deviates and waits, presumably to garner information about the signals of subsequent informed agents. If for all future histories the agent ends up investing then waiting merely reduces the length of time of investment in the high return project. Thus, the only way that this deviation can be profitable is that there are some future histories in which this agent never invests. After such a deviation the beliefs of the newly informed agent are always 2 higher than that of the uninformed agents. The reason is that the newly informed agent′s private signal raised his beliefs by 1 and the deviation by the newly informed agent did not affect his own beliefs while it lowered the uninformed agents' beliefs by 1.

Consider first a history h_{it} in which the uniformed agents' beliefs are $P(-1)$ and suppose that the newly informed agent receives a good signal and hence has beliefs $P(0)$. If the newly informed agent deviates and waits then this deviation triggers a cascade with no investment. To see this note that the deviation causes the uninformed agents' beliefs to be $P(-2)$ permanently. Given these beliefs uninformed agents never invest. Future newly informed agents update their beliefs to at most $P(-1)$ and do not invest either. Thus this deviation garners no new information. The beliefs of the deviating agent remain at $P(0)$. Assumption (5) then implies that given these beliefs it is optimal for the newly informed agent to invest at t.

Next, consider a history h_{it} in which the uniformed agents' beliefs are $P(0)$ and the newly informed agent receives a good signal and hence has beliefs $P(1)$. Suppose the newly informed agent deviates and waits. In the subsequent period the uninformed agents' beliefs fall to $P(-1)$ while the informed agent's beliefs stay at $P(1)$. Recall that the deviating agent's prior is always two above that of the uninformed agents and that a cascade of no investment starts when the priors of the uniformed agents reach $P(-2)$. Thus, the most pessimistic information that the deviating agent can garner leads to a prior of $P(0)$. At this prior it is optimal to invest and thus the deviation is not profitable.

Consider next histories at t with deviations detectable by uninformed agents and previously informed agents. Suppose that the first detectable deviation occurs at $s < t$. There are several types of such histories at t.. First, there can be deviations at s to investing during a cascade without investment. Under this type of history all agents' beliefs are at $P(-2)$ at $s+1$. The only agents whose beliefs can change from $s+1$ through t are those that become newly informed and their beliefs are either $P(-1)$ or $P(-3)$. Second, at the onset of a cascade with investment an agent can deviate and not invest. His beliefs are at $P(2)$, and regardless of the future signal he might receive, he should invest. Finally, there are number of histories which involve by simultaneous deviations by two or more agents that need not be considered.

Consider next histories at t with deviations detectable only by newly informed agents. The only interesting deviation that only a newly informed agent can detect is one in which the prior was at $P(0)$,the newly informed agent received a bad signal and some other agent invested. In this case uninformed agents' beliefs are at $P(1)$, the newly informed agents' beliefs are at $P(2)$ and all agents should invest. It is easy to check that this is indeed optimal. *Q.E.D.*

Next we show that within a certain class the equilibrium we have constructed is unique. We will say that a collection of strategies and beliefs is a *symmetric, stationary equilibrium* if it is a perfect Bayesian equilibrium and the strategies have the cutoff rule form. Namely, there are sets of integers I and I' such that for $t \leq T_1 - 1$ the strategies of the uninformed and previously informed agents are of the form

$$(12) \quad x_t(h_{it}) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_{it}) \in I \\ \\ 0 \text{ otherwise} \end{array} \right\}$$

while the strategy for newly informed agents is of the form

$$(13) \quad x_t(h_{it}) = \left\{ \begin{array}{c} 1 \text{ if } p_t(h_{it}) \in I' \\ \\ 0 \text{ otherwise} \end{array} \right\}.$$

(Notice that we allow the strategies at T_1 to differ from those for $t \leq T_1 - 1$.)

Proposition 2. The strategies and beliefs described in (9)-(11) is the unique symmetric stationary equilibrium.

Proof. Consider period T_1. By (5) and (6) it follows that all agents invest if and only if their beliefs are greater than or equal to $P(0)$. Consider next period $T_1 - 1$. The newly informed agent can learn nothing by waiting and hence invests if and only if his beliefs are greater than or equal to $P(0)$. Thus, the set I' is the set of integers with $k \geq 0$. An uninformed agent who enters the period with beliefs $P(0)$, by waiting learns the signal of the newly informed agent. By (7) it is optimal to wait. An uninformed agent who enters the period with beliefs greater or equal to $P(1)$ will invest in period T_1 regardless of the newly informed agent's action. Hence, waiting only postpones investment and it is optimal to invest immediately. Thus, the set I is the set of integers with $k \geq 1$. Q.E.D.

Next, we show that this model has the same type as cascades as a model with exogenous timing of investments. To reduce our model to one with exogenous timing suppose that only the newly informed agent can invest at t and $N = T_1 + 1$. Let the strategies of the newly informed agents be the same as those in the endogenous timing model and let the beliefs be appropriately modified. It is obvious that these strategies and beliefs are the unique equilibrium of the exogenous timing model. In particular, histories of the form (1), $(0, 1, 1)$, $(0, 1, 0, 1)$, $\ldots, (0, 1, 0, 1, \ldots, 0, 1, 1)$ start cascades with investment. Histories of the form $(0, 0)$, $(0, 1, 0, 0), \ldots,$ $(0, 1, 0, 1, \ldots, 0, 1, 0, 0)$, start cascades with no investment. Thus, we have shown the following.

Proposition 3. The strategies described in (10) and the associated beliefs is the unique equilibrium of the model with exogenous timing. Furthermore, for any history of signals the aggregate investment at T_1 is the same as in the model with endogenous timing.

Next, we are interested in exploring if there is any sense in which these cascades are mistakes relative to some benchmark. One benchmark is the public information version of the above game. In this version, the agent at t receives his signal at t and chooses whether or not to invest at t, but now suppose that at each t the signal at t is publicly observable at the beginning of period t. It is easy to show that even if actions always fully reveal the private information, under (5)-(7) the equilibrium of the private information game will not coincide with those in the public information game. The reason is that in the private information game the uninformed agents can only react to the revealed information with a one period lag, while in the public information game they can react immediately.

A more interesting benchmark is a public information game which captures the informational lags built into the private information game. Consider a game with public information lags in which the uninformed agents learn the realization of the date t signal after they have made their period t investment decisions. Thus, at t the relevant history of agents is the history of past investments $(m_0, m_1, \ldots, m_{t-1})$ together with the history of the signals. For the newly informed agent the history of signals $s^t = (s_0, s_1, \ldots, s_t)$, while that of all other agents is s^{t-1}. An equilibrium is defined as before. For any equilibrium the outcome path can be defined recursively from the strategies, and depends only on the history of signals. The equilibrium strategies induce outcomes at each date. Let $m_t(s^t)$ denote the aggregate number of positive investments at t for history s^t. Similarly, in the private information game, let $n_t(s^t)$ denote the aggregate number of positive investments along the equilibrium path at t for history s^t.

We say that the private information game has a *herd at* s^t if i) for all future histories s^r containing $s^t, n_r(s^r)$ is that same for all s^r and ii) for some future history s^{T_1} containing s^t, $\sum_{k=0}^{T_1} n_k(s^k) \neq \sum_{k=0}^{T_1} m_k(s^k)$ where $s^k \in s^{T_1}$ for all k. We say that the game has a *herd of investment* if it has a herd for some s^t and

$$\sum_{k=0}^{t} n_k(s^k) = N$$

where the summations are over $s^k \in s^t$. We say that the game has a *herd of no investment* if it has a herd for some s^t and

$$\sum_{k=0}^{t} n_k(s^k) < N \text{ and } n_r(s^r) = 0 \text{ for all } s^r,\ r > t \text{ and } s^t \in s^r.$$

The first clause in our definition of a herd requires that individuals make the same decisions regardless of their signals. The second clause requires that these decisions are mistakes relative to the benchmark game of public information with lags. Several authors (including Banerjee 1992) have defined notions of herd-like behavior which only require the first clause. With only the first clause, if we started agents with prior below $P(-T_1)$ they would never invest in either game regardless of the signals. In a sense everyone is doing the same thing because they are all doing the right thing. The second clause ensures that when everyone is doing the same thing, relative to the public information game, they are doing the wrong thing.

To pin down the equilibrium of the game of public information with lags we need to strengthen our assumptions. We assume

(14) $\quad e^{RT} P(1) <$

$$\nu_G(P(1))e^{R(T-1)}P(2) + \nu_B(P(1))\left[\nu_G(P(0))e^{R(T-2)}P(1) + \nu_B(P(0))\right]$$

We think of (14) as strengthened version of (7), since one can show that, given (6), (14) implies (7). This assumption implies that at any t investing at $P(1)$ dominated by the following strategy. Wait until $t+1$, if the signal is good invest, if the signal is bad, wait until $t+2$ and invest if and only if the signal is good.

Proposition 4. Under (5), (6), (14), the equilibrium has both herds of investment and herds of no investment.

Proof. To see that there is a herd of investment consider the history of signals $s^{T_1} = (G, B, B, \ldots, B)$. In the private information game, $n_0 = 1$ and $n_1 = N - 1$. In the public information game, using (5), (6), (14), $m_t = 0$ for all t. Likewise there are clearly herds of no investment. Consider the history of signals $s^{T_1} = (B, B, G, \ldots, G)$. In the private information game $n_t = 0$ for all t, while in the public information game for $T_1 \geq 4$ the prior rises above $P(0)$ and there is positive investment. Q.E.D..

In (14) we guarantee that in the public information game it is optimal to wait at $P(1)$. Suppose instead that we replaced (14) with an assumption that guarantees that in the public information game it is optimal to invest at $P(1)$. Obviously, the equilibrium outcomes of the private information game would be unaffected, but the positive cascades would not be herds because they would coincide with outcomes in the public information game.

So far, we have focused on what we consider to be the most interesting region of the parameter space. We briefly discuss the characteristics of symmetric stationary equilibria for other regions. If the rate of return R is so low that $e^{RT}P(1) < 1$ then agents never invest regardless of the signals. If the rate of return R is so high that $e^{R(T-T_1)}P(-1) > 1$, then all agents invest at time 0. If R is lower than what we have but satisfies, $e^{RT}P(0) < 1$, but $e^{R(T-T_1)}P(1) > 1$ then the equilibrium is very similar to the equilibrium described above. Specifically, in the equilibrium described above, it takes one good signal to set a positive herd and two bad signals to set off a negative herd, while here it will take two good signals to set off a positive herd and one bad signal to set off a negative herd. Next, if parameters are such that the value of waiting at $P(0)$ is less than the value of immediately investing (the direction of the inequality is reversed in the analog of (7) , where the right side is now the payoff to the optimal strategy following waiting) then all agents invest at 0. Finally, we focussed on a region of the parameter space such that, except for the last period, the equilibria are stationary in that they depend only on the prior, and not on time. If, for example, we assumed that (6) was replaced by $e^{RT}P(-1) > 1 > e^{R(T-T_1)}P(-1)$, then the equilibria are necessarily nonstationary and would depend on time as well as the prior.

We have also focused attention on stationary equilibria. Under assumptions (5) – (14) there are also nonstationary equilibria. For example, consider the strategies we have proposed with the single change that the newly informed agent at date 0 waits regardless of the signal, and the beliefs in period 1 of the uninformed agents stay at $P(0)$. To show this is an equilibrium we need only show that the newly informed agent prefers to wait when he gets a G. By (14) waiting is optimal.

2. Continuous investment

In the basic model agents either invested all of their money in the foreign project or none. Here we let agents make a once and for all decision to invest any nonnegative amount and assume the foreign project has decreasing returns. These changes will imply that, conditional on investing, agents will adjust their investments smoothly as a function of their beliefs. Nevertheless, we find that there are herds similar to those with zero/one investment.

We consider a variant of the basic model in which the investment project has decreasing returns and there is a nonnegativity constraint on investment. For an investment of x at t in the foreign project the foreign project's payoff is $e^{R(T-t)}f(x)$ in the good state and 0 in the bad state, where $f(x)$ is a strictly increasing concave function with $f'(0)$ finite. The rest of the model is unchanged.

For some arbitrary belief p that the state is G, an agent who invests at t chooses the size of the investment x to solve

$$V(p,t) = \max_x e^{R(T-t)} f(x)p + (1-x)$$

subject to

$$0 \leq x \leq 1.$$

The first order condition at an interior point is

$$(15) \quad f'(x) = e^{R(t-T)}/p.$$

Let $\underline{p}(t) = e^{R(t-T)}/f'(0)$. Clearly, the agent's optimal investment is given by

$$(16) \quad x(p,t) = \left\{ \begin{array}{c} (f')^{-1}(e^{R(t-T)}/p) \text{ if } p \geq \underline{p}(t) \\ 0 \text{ otherwise} \end{array} \right\}.$$

We make the following assumptions.

$$(17) \quad V(P(0), T_1) > 1$$

(18) $V(P(-1), 0) = 1$

(19) $V(P(0), 0) < \nu_G(P(0))V(P(1), 1) + \nu_B(P(0))$

and

(20) $V(P(1), t) > v_G(P(1))V(P(2), t+1) + v_B(P(1))V(P(0), t+1)$

(21) $V(P(2), t) > v_G(P(2))V(P(3), t+1) + v_B(P(2))V(P(1), t+1)$

Assumptions (17)-(19) are the analogs of (5)-(7). To understand the roles of (19), (20), and (21) note that in this model, waiting and receiving information has two benefits. The first, is the no investment option value discussed earlier. The second, which we call the *fine-tuning option value,* comes from better information allowing agents who have already decided to invest to adjust the size of their projects. The cost of waiting comes from the same kind of discounting as before. Assumption (19) requires that no investment option value be large relative to discounting. Assumptions (20) and (21) require that fine-tuning option value is small relative to discounting in the following sense. Assumption (20) requires that investing at a prior of $P(1)$ dominates waiting one period, receiving a signal and then investing the optimal higher amount if the signal is good and the optimal lower amount if the signal is bad. Assumption (21) imposes a similar requirement at a prior $P(2)$. Note that if the size of the risky project is fixed, as in the previous section, the fine-tuning option value is zero, and hence (20) and (21) is automatically satisfied.

We use assumption (20) and (21) to show that starting at history in which uninformed agents' priors are $P(0)$, a newly informed agent who has received a good signal will invest immediately rather than attempting to learn from the investments of future newly informed agents and then optimally adjusting the size of his investment.

The histories, strategies, and beliefs are defined as follows. Let $X_t = (x_{1t}, \ldots, x_{Nt})$ denote the amount of the date t investments of the N agents. We let n_t denote the number of agents that invested positive amounts at t. The public history $h_t = (X_0, X_1, , \ldots, X_{t-1})$. The strategies are that in all periods, except at T_1, uniformed and previously informed agents invest a positive amount $x(p, t)$ if and only if $p \geq P(1)$. At T_1, they invest a positive amount $x(p, T_1)$ if and only if $p \geq P(0)$. Newly informed agents invest a positive amount $x(p, t)$ if and only if $p \geq P(0)$. The beliefs of agents are defined analogously to those in the previous section.

The proof that these strategies and beliefs are an equilibrium is essentially the same as the proof of Proposition 1. The only part of the proof that needs elaboration is showing that at a history in which uninformed agents' priors are $P(0)$, a newly informed agent who has received a good signal will invest immediately. We use assumption (20) and (21) to show that investing immediately dominates attempting to learn from the investments of future newly informed agents and then optimally adjusting the size of his investment.

More formally, consider a history h_{it} in which the uniformed agents' beliefs are $P(0)$ and the newly informed agent receives a good signal and hence has beliefs $P(1)$. Suppose the newly informed agent deviates and waits. Recall that in all subsequent periods, the beliefs of this deviating agent are 2 higher than that of the uninformed agents (1 for the private signal and 1 for the deviation). Recall that a herd with investment starts when uninformed agents' beliefs are $P(1)$ and a herd of no investment starts when uninformed agents' beliefs are $P(-2)$. Hence, in any period after the deviation, the deviating agent's beliefs can be one of four values, $P(0)$, $P(1)$, $P(2)$, or $P(3)$. Moreover, if this agent's beliefs reach $P(0)$

or $P(3)$ they stay there because the uninformed agents are then in either a cascade without investment or a cascade with investment.

We recursively solve for the strategy the yields the highest payoff following the deviation at t as follows. Suppose that the agent has not invested until T_1. For all four beliefs (17) implies that it is optimal to invest. Suppose that the agent has not invested until $T_1 - 1$. If the agent's beliefs are at either $P(0)$ or $P(3)$ they stay there at T_1 and discounting makes it optimal to invest at $T_1 - 1$. If the agent's beliefs are at $P(1)$ (20) makes it optimal to invest immediately while if the agent's beliefs are at $P(2)$ (21) makes it optimal to invest immediately. Repeating this argument, conditional on waiting at t, the best deviation is for the agent to invest at $t+2$. Assumptions (20) and (21) then imply that investing at t, dominates waiting and then investing at $t+2$.

We have proved the following.

Proposition 5. Under (17)-(21), the above strategies and beliefs constitute a perfect Bayesian equilibrium.

To show that there are herds we need another assumption to pin down the equilibrium of the game of public information with lags. We assume the analog of (14),

$$(22) \quad V(P(1),0) < \nu_G(P(1))V(P(2),1) + \nu_B(P(1)) \left[\nu_G(P(0))V(P(1),2) + \nu_B(P(0))\right].$$

The following proposition is immediate.

Proposition 6. Under $(17) - (22)$, there are both herds of investment and herds of no investment.

Next, we contrast the type of herd behavior in the endogenous timing model with that in the exogenous timing model. To reduce our model to one with exogenous timing suppose that only the newly informed agent can invest at t and $N = T_1$. Let the strategies of the newly informed agents be the same as those in the endogenous timing model. Let the beliefs be adapted in the obvious fashion.

In the model the exogenous timing a newly informed agent that does not invest at t can never invest in the future, while when there is endogenous timing such an agent can invest in the future. Because of this difference we need to modify our definition of a herd. Let equilibrium investment outcomes at t for a given sequence of signals s^t in the private and the public information games be denoted $x_t(s^t)$ and $y_t(s^t)$ respectively. We will say that the private information game with exogenous timing has a *herd of investment* if for some s^t, i) for all future histories s^r containing s^t, $x_r(s^r) > 0$ for all s^r and ii) for some future history s^r containing s^t, $x_r(s^r) \neq y_r(s^r)$. We will say that the private information game with exogenous timing has a *herd of no investment* if for all s^t, i) for all future histories s^r containing s^t, $x_r(s^r) = 0$ for all s^r and ii) for some future history s^r containing s^t, $x_r(s^r) \neq y_r(s^r)$.

We will show that, in contrast to model with endogenous timing, with exogenous timing there can be no herds of investment. Intuitively, with exogenous timing each agent has a take it or leave it option to invest. Thus as long as the prior is above the cutoff level $\underline{p}(t)$,the agent invests some positive amount, say x_t, given by (16). Using (15), the belief of the agent can be inferred to be

$$p_t = e^{R(t-T)}/f'(x_t)$$

and, given some public prior $p_{t-1,}$ the action x_t reveals the agent's signal. More formally we have the following

Proposition 7. In the equilibrium of the model with exogenous timing, there are herds of no investment but there are no herds of investment.

Proof. By the above construction $(0, \ldots, 0)$ is a herd of no investment. To see that there are no herds of investment note first that for any outcome in which investment is always positive, say $(x_0, \ldots, x_N)$ the investment decisions can be inverted to give unique beliefs $(e^{R(0-T)}/f'(x_0), \ldots, e^{R(T_1-T)}/f'(x_N))$ and since Bayes' rule is monotone, a unique path of signals. A moment's reflection makes it clear that the only way information can be trapped is either in outcome paths that start with two consecutive zeros, or in outcome paths which have $x(P(0))$ and then two consecutive zeros. In each of these however, the beliefs after the two zeros are $P(-2)$ and there can be no more investment. Hence, for such outcome paths $x_N = 0$ and such outcome paths cannot be herds of investment. Q.E.D.

3. Incentives to share information

One question that arises in the above formulation is that it seems that agents have an incentive to share information. Here we consider a version of the model in which agents can share information by sending messages and show that they do not have such an incentive. We consider a variant of the model in which there is a flow benefit to being one of the early-investors in the risky project. This confers a type of early-mover advantage on the informed investor which provides incentives to mislead other investors. We show that there does not exist a truth-telling equilibrium of the communication game.

The model is the same as before except for the following two changes. First, we let the rate of return earned in the risky project be $\bar{R}$ if there are fewer than T_1 investors and R otherwise, where $\bar{R} > R$. Second, we assume that at the beginning of each period t each informed agent sends a message $m_t \in \{G, B\}$ about his signal in that period. All other agents receive the message and then decide whether or not to invest in period t.

The publicly observable events are the number of investments in each period and the messages. The public history $h_t = (n_0, n_1, \ldots, n_{t-1}, m_0, m_1, \ldots, m_t)$ records the aggregate investment in each period up through the end of period $t-1$ and the messages in each period up through and including the message sent at the beginning of period t. The histories, strategies and beliefs of newly informed agents, previously informed agents and uninformed agents are defined analogously to those in Section 1.

We impose the following assumptions.

(23) $1 < e^{R(T-T_1)}P(0)$

(24) $1 > e^{\bar{R}T}P(-1)$

(25) $e^{\bar{R}T}P(0) < \nu_G(P(0))e^{R(T-1)}P(1) + \nu_B(P(0))$

Assumptions (23)-(25) play the same role as before. Under these assumptions truth-telling for all histories is not an equilibrium. To see this consider a newly informed agent at T_1 that inherits a prior of $P(-1)$ from period $T_1 - 1$. If truth-telling were an equilibrium then if the informed agents sends a message G, the priors of all other investors rise to $P(0)$ and they invest, while if the informed agents sends a message B then the priors of all other investors fall to $P(-2)$ and they do not invest. Clearly, if the informed agent gets a good signal and lies by sending a message B he gains the early-mover advantage. Thus, truth-telling cannot be an equilibrium for all histories.

It is useful to define $H_t = (h_{t-1}, n_{t-1})$ to be the public history at the beginning of period t, before the period t message is sent, and to define the inherited prior at t to be the prior based on such a history. The beginning of period t history for any agent is $H_{it} = (H_t, s_r, r)$.

We define a symmetric stationary equilibrium of the communication game to be one with strategies for investment of the following form. There are functions x_U, x_I, and m such that for $t \leq T_1 - 1$ the investment strategies of the uninformed and previously informed agents are of the form

(26) $\quad x_t(H_{it}, m_t) = x_U(p_t(H_{it}), m_t),$

the investment strategy for newly informed agents is of the form

(27) $\quad x_t(H_{it}, s_t) = x_I(p_t(H_{it}, s_t))$

while the message strategy is of the form

(28) $\quad m_t(H_{it}, s_t) = m(p_t(H_{it}, s_t), s_t).$

(Notice that we allow the strategies at T_1 to differ from those for $t \leq T_1 - 1$.) We will say that a message is uninformative if it is the same for both signals. We then have

Proposition 8. Under (23)-(25), the symmetric stationary equilibrium of the communication game is essentially unique, in that there is a unique investment outcome. This outcome equals that of the symmetric stationary equilibrium of the no-communication game.

Proof. We will show that $x_U = 1$ if and only if $p_t(H_{it}) \geq P(1)$ for all m_t, $x_I = 1$ if and only if $p_t(H_{it}, s_t) \geq P(0)$, and the message strategy is uninformative when the inherited prior is $P(0), P(-1)$, or $P(-2)$. (Notice that if these strategies are followed, the only inherited priors at which there will be active agents at the end of the period are $P(0), P(-1)$ and $P(-2)$. Off the equilibrium path, the message strategies and beliefs can be filled in an analogous manner to that in the basic model.)

Consider period T_1. By (23) and (24) it follows that all agents invest if and only if their beliefs are greater than or equal to $P(0)$. Furthermore, the uninformed and the previously informed agents' decisions are independent of the newly informed agent's message. To see this suppose that the inherited prior is $P(0)$ or $P(-1)$. If the uninformed agents' decisions depend on the newly informed agent's message then the newly informed agent has an incentive to mislead.

Consider next period $T_1 - 1$. Consider first the newly informed agent at $T_1 - 1$. Since this agent's decision at T_1, if he waits, is independent of the messages at T_1, he can learn nothing by waiting and hence invests if and only if his beliefs are greater than or equal to $P(0)$. Thus, $x_I = 1$ if and only if $p_t(H_{it}, s_t) \geq P(0)$. If the inherited prior is either $P(0)$ or $P(-1)$ the uninformed and the previously informed agents' decisions must be independent of the newly informed agent's message, otherwise the newly informed agent has an incentive to mislead. Hence, in any symmetric stationary equilibrium at any date the message must be uninformative at an inherited prior of $P(0)$ or $P(-1)$. It remains to be shown that at an inherited prior of $P(-2)$ the message is uninformative. We consider this case below. If the inherited prior is $P(1)$ or higher, everyone invests and thus the message is irrelevant.

Consider next an uninformed agent at $T_1 - 1$. Suppose this agent has an inherited prior $P(0)$ or $P(-1)$. If this agent waits, he can observe the newly informed agent's investment decision at $T_1 - 1$ and hence can infer the signal and take the optimal decision in period T_1. At inherited prior $P(0)$, (25) implies that it is optimal to wait. At inherited prior $P(-1)$, (24) implies that is optimal to wait. By (23), since under the conjectured strategy all other agents are investing, it is optimal to invest if the inherited prior is $P(1)$ or higher. This shows that $x_U = 1$ if and only if the inherited prior is greater than or equal to $P(1)$.

We now show that at an inherited prior of $P(-2)$ the newly informed agent sends an uninformative message. We will show that if this is not true at t then it gives the newly informed agent at $t-1$ an incentive to deviate from the conjectured strategy. To see this suppose by way of contradiction that at inherited prior $P(-2)$, truth-telling is part of the equilibrium at t. To show that this suppostion is false, consider period $t-1$ and suppose that the newly informed agent has inherited prior $P(-1)$ and receives a good signal. This agent's prior rises to $P(0)$ and he is supposed to invest (and send an uninformative message). Suppose that he deviates and waits. The uninformed agents' priors fall to $P(-2)$ at the beginning of period t. Under our posited strategy the newly informed agent at t sends a truthful message, the deviating agent's prior either rise to $P(1)$ or fall to $P(-1)$. By (25) this deviation is profitable. It follows at inherited prior $P(-2)$ the message must be uninformative. *Q.E.D.*

The communication game has nonstationary equilibria, in much the same way as the no communication game.

4. Conclusion

In his early work on herds, Banerjee (1992) argues that there are interesting unresolved strategic issues on how herds work when the timing of moves is endogenous. He also calls for an analysis of herds when agents can communicate. We have extended the analysis of herds in both of these directions. We have also extended the work of Lee (1995) to show that when the actions are continuous, moving from exogenous timing to endogenous timing actually makes herds more likely.

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