Job Market Paper:
Tacit Collusion under Interest Rate Fluctuations

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Abstract

In contrast to the existing literature on repeated games that assumes a fixed discount factor, I study an environment in which it is more realistic to assume a fluctuating discount factor. In a repeated oligopoly, as the interest rate changes, so too does the degree to which firms discount the future. I characterize the optimal tacit collusion equilibrium when the discount factor changes over time, under both price and quantity competition, and I show that collusive prices and profits depend not only on the level of the discount factor but also on its volatility. Collusive prices and profits increase with a higher discount factor level, but decrease with its volatility. These results have important implications for empirical studies of collusive pricing and the role that collusive pricing may play in economic cycles.

Keywords: tacit collusion, interest rate, random discount factor, repeated games.

JEL Classification: C7, D43, L13.

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1 Introduction

It is well known that oligopolies can use the threat of future price wars to sustain prices above perfectly competitive levels if firms care enough about the future (Friedman [7]). The extent to which firms care about the future depends primarily on the interest rate if the firms' objective is to maximize the present value of profits. The firms' discount factor may also depend on other (secondary) forces such as the probability that the product may became obsolete or that the government may nationalize the industry. Given that the interest rate and other variables that affect the discount factor are constantly changing, it is important to study tacit collusion under discount factor fluctuations.

I consider first the case in which the discount factor, identical for all firms, is randomly and independently drawn every period. I characterize the maximum symmetric tacit collusion prices and profits that can be supported in an environment in which firms are identical and they compete repeatedly on either price or quantity (the latter under certain conditions). The three main results derived from this characterization, with the third one the most surprising, are as follows.

First, the lower the discount factor in a given period, the lower the collusive prices and profits that can be supported in equilibrium in that period. The intuition behind this is simple. If the discount factor is low, the shadow of future retaliation for a present deviation is also low. Thus, to prevent deviations today, it may be necessary to charge a low price (produce a large quantity) and earn low profits today.

Second, the higher the probability of high discount factors, the higher the collusive prices and profits that can be supported in equilibrium. Again the intuition is simple. From the first result we know that the higher the realization of the discount factor, the higher collusive prices and profits will be. Hence, a shift in the distribution function to higher discount factors would result in an increase in the expected value of collusive profits and an increase in the threat of future punishment, allowing higher equilibrium prices and profits.

Third and more surprisingly, I show that the higher the volatility of the discount factor, the lower the collusive prices and profits that can be supported in equilibrium. The reason for this is two fold. First, the optimal tacit collusion must respect two constraints: the incentive compatibility constraint (for each discount factor profits have to be small enough so that firms have no incentives to deviate), and the feasibility constraint (profits can not be higher than monopoly profits). Given that the combination of these two constraints results in a concave profit function (as a function of the discount factor), an increase in volatility leads to a decrease in expected profits. Second, this decrease in expected profits reduces the size of future punishment and hence results in a decrease in equilibrium profits and prices.

This volatility effect is not secondary to the first two level effects. I show that it plays an important role in determining collusive prices and profits.

It is important to note that given that both a high discount factor today and in the future make it easy to support collusion, allowing for the more realistic case of positively correlated discount factors will not affect the main results. However, these results may be modified if changes in today's discount factor affect its future volatility.

The level and volatility effects have implications for both the empirical study of collusion and the study of macroeconomic fluctuations. With respect to the former, this paper presents several comparative static results that can be used to detect empirically the existence of collusive pricing. With respect to the latter, this paper stresses the role of interest rates and imperfect competition in aggregate fluctuations. Any change in policy, technology or preferences that affects the real interest rate (either in level or volatility) may have an impact on aggregate production through changes in collusive behavior.

Two other results of this paper are worth noting. First, I show that under quantity competition the optimal symmetric punishment has a simple stick-and-carrot characterization (the punishment takes only one period and is as big as possible in equilibrium), extending the results of Abreu [1] from the fixed discount factor case.

Second, I show that under price competition an increase in the number of firms reduces collusive prices and profits. The reason is that the greater the number of firms the greater the share of the market that can be captured by a deviation, and, hence, the lower equilibrium profits and prices must be to avoid deviations. In the case of quantity competition, more work is needed to asses the validity of this result, since not only do the incentives to deviate change with the number of firms, but so may the threat of future punishment.[1]

The rest of the paper is organized as follows. In Section 2, I relate this paper to the previous literature. In Sections 3 and 4, I study optimal tacit collusion under price and quantity competition, respectively. In Section 4, I analyze some extensions to the basic model. In Section 6 I conclude.

2 Related literature

The related literature falls into five categories: 1) studies of the effects of demand fluctuations on optimal tacit collusion, 2) studies of optimal punishment schemes under quantity competition, 3) repeated games with fixed discount factors, 4) empirical studies of collusive pricing, and 5) studies of the role of oligopolies in macroeconomic fluctuations.

Demand fluctuations and optimal tacit collusion: The well known paper by Rotemberg and Saloner [15] offers interesting results with respect to tacit collusion that also follow from changes in the relative importance of present and future profits. In their paper, however, changes in the relative importance of present and future profits are driven by changes in demand, not the discount factor. Given the expectation of future equilibrium profits, there is a maximum level of profits that can be supported in equilibrium. Hence, at this upper bound on profits, increases in demand do not result in increases in profits. In fact, if demand has increased and profits remain at the upper bound, prices must decrease. Therefore, Rotemberg and Saloner [15] find that oligopolies colluding

[1] To my knowledge, the effect of the number of firms on tacit collusive prices under quantity competition remains to be solved also for the case of fixed discount factors.

tacitly may behave more competitively in periods of high demand, that is, when the future is relatively less important. My paper also examines collusive behavior under changes in the relative importance of present and future profits, but unlike Rotemberg and Saloner [15] these changes are brought about through movements in the discount factor.

This difference in the source of the changes in the relative importance of future and present profits is not trivial, and indeed leads to different results. First, in this paper an increase in the discount factor always has a nonnegative effect on the equilibrium price, while in Rotemberg and Saloner [15] an increase in demand may result in either an increase or a decrease in price. For example, in Rotemberg and Saloner's model, if the demand is so low that the upper bound to profits is not binding, a small increase in demand will result in an increase in price. Second, while in this paper an increase in the volatility of the discount factor always results in a decrease in profits and prices, in Rotemberg and Saloner's model an increase in the volatility of demand is again ambiguous -it may result in an increase in profits and prices.[2] Therefore, in contrast to fluctuating demand, changes in the level or volatility of the discount factor have unambiguous effects.

The third difference between the two models lies in the effect that present and future shocks have on collusive prices. In Rotemberg and Saloner's model, a high demand today makes it *difficult* to support collusion since it offers greater incentives to deviate, while a high demand in future periods makes it *easier* to collude today given that it leads to an increased threat of future price wars. In contrast, in this model both high discount factors today and in the future make it *easy* to support collusion given that both increase the threat of future punishment.

The different effects that present and future levels of demand have on collusive pricing

[2] Rotemberg and Saloner [15] do not provide this comparative static result but straightforward examples can be obtained from their model. In their model the profit function may be convex in the fluctuating parameter so that an increase in volatility increases the expected profits and moves up the incentive compatibility constraint.

in Rotemberg and Saloner [15] led to several studies of whether their results were robust to correlation on demand shocks. Kandori [11] finds conditions under which demand correlation does not affect the result of countercyclical collusive pricing. Haltiwanger and Harrington [9] study tacit collusion under deterministic cyclic fluctuations of demand and find that higher collusive prices can be supported when demand is increasing than when it is decreasing. Bagwell and Staiger [3] study tacit collusion when demand shifts stochastically between high and low growth rates and find that collusive prices are higher for high rates of demand growth if demand growth rates are positively correlated through time.

Under discount factor fluctuations the issue of positive correlation is less important than under demand fluctuations, given that both high discount factors today and in the future have a positive impact on today's collusive prices. However, I show that the discount factor volatility may be important in understanding how more general discount factor fluctuation affect the basic results.

Optimal punishment schemes under quantity competition: Abreu [1] provides a simple stick-and-carrot characterization of optimal symmetric punishments for a fixed discount factor under quantity competition: "the most efficient way to provide low payoffs, in terms of incentives to cheat, is to combine a grim present with a credibly rosy future."[3] In this paper I show that the stick-and-carrot characterization extends to the case of discount factor fluctuations, with both the size of the stick and the size of the carrot depending on the realization of the discount factor.

The level effect and repeated games with fixed discount factors: It is well known that, for repeated games with fixed discount factors, the higher the discount factor, the bigger the set of equilibrium outcomes will be (see for example, Abreu, et al. [2]). In this paper I show that under discount factor fluctuations it is not only the level of the discount factor that matters, but also its volatility.

Empirical literature on collusive pricing: Based on the frameworks established by

[3] Abreu [1], pag. 206.

Rotemberg and Saloner [15] or Porter [13] and Green and Porter [8], there is an extensive literature that concentrates on changes in demand as sources of changes in collusive pricing. Those papers do not include the interest rate in their studies, see for example Porter [14], Slade [18], Ellison [6] and Borenstein and Shepard [4]. An exception can be found in Rotemberg and Woodford [16]'s study of markups and the economic cycle. Working with aggregate log-linearized data around the steady state of an intertemporal macroeconomics model, they use rates of return to "instrument" for the firm's expectations of future profits and find that high interest rates result in low markups. In this paper I present additional comparative static results arising from interest rate movements that may be used to study empirically collusive prices.

Collusive pricing and macroeconomic fluctuations: Previous literature has related tacit collusive pricing with macroeconomic fluctuations. For example, Rotemberg and Saloner [15] present a simple two-sector general equilibrium model in which one sector is oligopolistic and the other one is perfectly competitive. They show that exogenous shifts in demand towards the oligopolistic sector induce a decrease in collusive prices (since it increases the short run incentives to deviate) and may result in an increase in aggregate production. Rotemberg and Woodford [17] present a real business cycle model with tacitly colluding oligopolistic producers. In their model, an increase in government expenditure raises the short run incentives to deviate and results in a decrease in collusive prices. This, in turn, increases real wages, employment and output. In addition, the authors note that the increase in government expenditure may result in an increase in interest rates (since consumers must postpone consumption), which reinforces the first effect by lowering the threat of future punishments.

In this paper I present another way in which tacit collusion may result in aggregate fluctuations. Any change in policy, technology or preferences may have an impact on aggregate production through changes in collusive behavior, not only by affecting the real interest rate level, but also by affecting its volatility.

3 Price competition

Consider a market with N identical firms with a constant marginal cost of ς and facing a demand function $D(p)$ ($D'(p) < 0$). Firms compete repeatedly on price and the demand is divided equally among the firms charging the lowest price in each period. Firms only care about profits and are risk neutral and, hence, their objective is to maximize the discounted stream of profits. The distinctive feature of this model is that the discount factor δ_t, which discounts earnings from $t+1$ to t, is a continuous, independent and identically distributed random variable, between a and b, with p.d.f. $f(\delta_t)$ and c.d.f. $F(\delta_t)$.

The timing of the game in a given period t is as follows: the firms observe the realization of the discount factor, δ_t, *then* they choose the price for that period and finally they observe the market clearing price, quantities and payoffs. All the characteristics of the environment are common knowledge.

Given that firms can not commit to charge a given price or sign contracts amongst themselves or with third parties regarding prices, any equilibrium of the model must be a subgame perfect equilibrium of the infinitely repeated oligopoly game. I restrict my attention to equilibria in which all the firms charge the same price p. In this symmetric case, I can write the profits of each firm as $\pi(p) = \frac{(p-\varsigma)D(p)}{N}$ and total industry profits as $\Pi(\delta) = (p-\varsigma)D(p)$. I assume that there exists a price p^m that maximizes the total industry profits, that is, p^m is the monopoly (or perfect collusion) price. Denote $\pi^m = \pi(p^m)$ as the monopoly profit per firm and $\Pi^m = N\pi^m$ as total industry monopoly profit.

3.1 Optimal tacit collusion with a random discount factor

It is well known that in repeated oligopoly games, prices above the marginal cost can be supported in equilibrium if any price undercutting triggers future price wars. In the case of price competition, the best price war, in terms of punishment, is the reversion forever to the Bertrand equilibrium after any deviation. This punishment gives a discounted payoff of zero. Any other punishments that would result in a lower payoff are not enforceable given that any firm can make sure to earn zero profits by charging a price equal to the marginal cost in every period.

Given this punishment, I look for symmetric optimal tacit collusion strategies - strategies without price differences among firms and that in equilibrium support the maximum present value of profits. Since the environment in which the firms interact does not change over time, with the exception of the discount factor, the optimal tacit collusion solution will consist of the highest equilibrium price that the firms can charge in a period given the discount factor in that period. Therefore the solution will consist of a function $p^*(\delta) : [a, b] \to [\varsigma, p^m]$ which gives the highest equilibrium price that can be supported for each discount factor. This in turn defines a function $\pi^*(\delta) : [a, b] \to [0, \pi^m]$, which denotes the optimal tacit collusion equilibrium profits as a function of the period discount factor.

Fortunately, in the search for the optimal tacit collusion behavior it is enough to work with $\pi^*(\delta)$. As we see in Figure 1, a given level of profits, for example π_1, can result from different prices, such as p_1 and p_2. Given that I am interested in the optimal levels of profits that can be supported under tacit collusion and the fact that π_1 may be supported more easily by p_1 than by p_2,[4] I only consider the increasing part of the profit function. In this way, for every profit lower than π^m corresponds one and only one price lower than p^m. Therefore, I can define the function $\phi(\pi) = \pi^{-1}(\pi) : [0, \pi^m] \to [\varsigma, p^m]$, and once I solve for $\pi^*(\delta)$, I can recover $p^*(\delta)$ as $p^*(\delta) = \phi(\pi^*(\delta))$. Note that $\phi(\pi)$ is increasing on π.[5]

[4]As it will be clear soon, π_1 can be supported more easily by p_1 than by p_2 since the optimal deviation from p_1 yields $N\pi(p_1)$ which is lower than $N\pi(p^m)$ which can be obtain deviating from p_2.

[5]For simplicity, I will assume for the rest of the section that $\phi(\pi)$ is differentiable and, hence, $\phi'(\pi) > 0$.

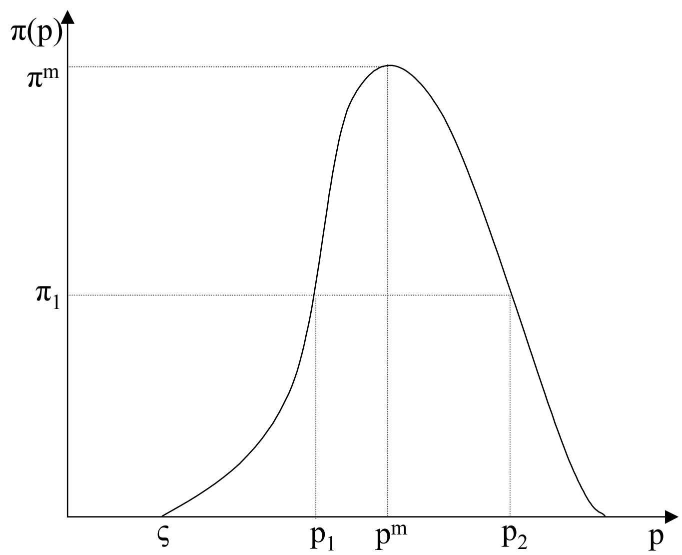

Figure 1: The profit function

Given the simplicity of the optimal punishment (reversion to Bertrand) and the fact that we are able to uniquely relate profits to prices, I concentrate on the characterization of the equilibrium optimal tacit collusion profits $\pi^*(\delta)$ without relying on the strategies that result in that equilibrium path. I study next the restriction on collusive profits for then characterizing the optimal tacit collusion solution.

Using the recursiveness of the problem, the present value at t of a firm stream of profits can be written as:

$$V(\delta_t) = \pi(\delta_t) + \delta_t \int_a^b V(\delta_{t+1}) f(\delta_{t+1}) d\delta_{t+1} \tag{1}$$

where $\pi(\delta_t)$ denotes the profits that the firms receive at time t if the discount factor is δ_t. Integrating over equation 1 and rearranging we have $\int_a^b V(\delta_t) f(\delta_t) d\delta_t = \frac{1}{1-\overline{\delta}} \int_a^b \pi(\delta_t) f(\delta_t) d\delta_t$, where $\overline{\delta}$ is the expected value of δ_t. Plugging this into (1), the present value of profits can be written as:

$$V(\delta_t) = \pi(\delta_t) + \frac{\delta_t}{1-\overline{\delta}} \int_a^b \pi(\delta_{t+1}) f(\delta_{t+1}) d\delta_{t+1} \tag{2}$$

Since these firms can not commit to a given price, in equilibrium they must be unwilling to charge a price different from the equilibrium price. How much can a firm gain from deviating? If all the firms are charging the same price above marginal cost, a single company can decrease its price by a penny and capture the whole market. Therefore, if the equilibrium profit is $\pi(\delta_t)$, a single company can gain $(N-1)\pi(\delta_t)$ by deviating (if we forget about pennies). For firms to be unwilling to deviate, punishment must follow a deviation. How much can a firm lose from being punished? As described before, the best punishment is to revert forever to the Bertrand equilibrium (the Nash equilibrium of the one stage game). Under this threat if one firm deviates it will earn the total industry profit the period of deviation but then it will earn zero profits forever. Then, for no firm to have an incentive to deviate, the following must hold:

$$\pi(\delta_t) \leq \frac{\delta_t}{(N-1)\left(1-\overline{\delta}\right)} \int_a^b \pi(\delta_{t+1}) f(\delta_{t+1}) d\delta_{t+1} \forall \delta_t \tag{3}$$

In addition, the profits per firm can not be greater than under monopoly pricing:

$$\pi(\delta_t) \leq \pi^m \tag{4}$$

Therefore it is clear that under the optimal symmetric tacit collusion equilibrium firms will choose profits as large as possible without violating the incentive compatibility constraint (3) and the feasibility constraint (4).[6] Then, dropping the subindexes for simplicity, the optimal tacit collusion profits levels $\pi^*(\delta)$ is a function from $[a, b]$ to $[0, \pi^m]$ subject to the following equation:

$$\pi^*(\delta) = \min \left\{ \frac{\delta}{(N-1)\left(1-\overline{\delta}\right)} \int_a^b \pi^*(\delta') f(\delta') d\delta', \pi^m \right\} \forall \delta \tag{5}$$

[6] It could be argued that that is not necessary since having profits lower than possible in a finite subset does not affect the expected value. But if we want the solution to be independent of the discount factor of the first period, profits must be as high as possible for every possible value of the discount factor.

Note that this equation does not provide the optimal tacit collusion profits since $\pi^*(\delta)$ appears in both sides of it. Equation (5) is just a necessary condition for optimal tacit collusion. In fact, choosing profits equal to zero for every discount factor solves this equation. From the possible many solution to equation (5), the one that provides the highest profit for each discount factor is the optimal tacit collusion solution: $\pi^*(\delta)$. The following proposition fully characterizes the function $\pi^*(\delta)$.

Proposition 1 *The function $\pi^*(\delta)$ depends on $f(\delta)$ and N in the following way:*

1) if $\overline{\delta} \geq 1 - \frac{a}{N-1}$, $\pi^*(\delta) = \pi^m$;

2) if $\frac{N-1}{N} \leq \overline{\delta} < 1 - \frac{a}{N-1}$, $\pi^*(\delta) = \pi^m$ for $\delta \geq c$ and $\pi^*(\delta) = \frac{\delta}{c}\pi^m$ for $\delta < c$, for a number $c \in (a, b]$ that solves the following equation: $c = (N-1)(1-\overline{\delta}) + \int\limits_a^c F(\delta)d\delta$;

3) if $\overline{\delta} < \frac{N-1}{N}$, $\pi^*(\delta) = 0$.

Proof. Case 1): $\overline{\delta} \geq 1 - \frac{a}{N-1}$ implies that $\pi^m \leq \frac{a}{(N-1)\left(1-\overline{\delta}\right)}\pi^m \leq \frac{\delta}{(N-1)\left(1-\overline{\delta}\right)}\pi^m \ \forall\delta$ and perfect collusion, $\pi^*(\delta) = \pi^m$, can be supported for every discount factor.

Case 2): Consider the case in which the two terms inside the brackets in equation (5) are binding for different ranges of δ. Given that the first term is increasing in δ, it would be binding for $\delta < c$, the second term would be binding for $\delta > c$, and both terms equal and binding for $\delta = c$, where $c \in [a, b]$. In this case, integrating over equation (5) and denoting the expected profit as A:

$$A = \int\limits_a^c \frac{\delta}{(N-1)\left(1-\overline{\delta}\right)} A f(\delta)d\delta + (1 - F(c))\,\pi^m$$

In addition, given that for $\delta = c$ both terms of equation (5) are equal, the expected profit can be also written as:

$$A = \frac{\pi^m (N-1)\left(1-\overline{\delta}\right)}{c}$$

Combining these two equations and by the fact that (integrating by parts) $\int\limits_a^s \delta f(\delta)d\delta =$

$sF(s)-\int\limits_a^s F(\delta)d\delta$, the number c solves the following equation:

$$c=(N-1)\left(1-\overline{\delta}\right)+\int\limits_a^c F(\delta)d\delta \tag{6}$$

It remains to be shown that, under the conditions of case 2), the number c that solves equation (6) exists and is unique. Write $H(r)=(N-1)\left(1-\overline{\delta}\right)+\int\limits_a^r F(\delta)d\delta-r$. Then $H(c)=0$. If $\frac{N-1}{N}<\overline{\delta}<1-\frac{a}{N-1}$, it can be easily seen that $H(a)=(N-1)\left(1-\overline{\delta}\right)-a>0$ and $H(b)=(N-1)\left(1-\overline{\delta}\right)-\overline{\delta}<0$. In addition, $H(r)$ is continuous and strictly decreasing ($\frac{\partial H(r)}{\partial r}=F(r)-1<0$ for $a\leq r<b$). Then, there exists a unique number c, between a and b, that makes $H(c)=0$. If $\overline{\delta}=\frac{N-1}{N}$, $H(b)=0$ and $c=b$ is the unique solution since $H(.)$ is strictly decreasing.

Case 3): From the analysis of the previous two cases follows that when $\overline{\delta}<\min\left\{\frac{N-1}{N},1-\frac{a}{N-1}\right\}$ neither a solution with perfect collusion for all or some discount factors is feasible, nor a solution with imperfect collusion is feasible. Then, the only possible solution to equation (5) is $\pi^*(\delta)=0$. Since $\frac{N-1}{N}$ can be greater than $1-\frac{a}{N-1}$ only if $a>\frac{N-1}{N}$, in which case $\overline{\delta}$ can never be lower than $\frac{N-1}{N}$, it follows that $\pi^*(\delta)=0$ if $\overline{\delta}<\frac{N-1}{N}$. ■

Proposition 1 shows that, depending on the distribution of the discount factor and the number of firms, there are three mutually exclusive cases that result in three different types of optimal tacit collusion. In case 1), $\overline{\delta}\geq 1-\frac{a}{N-1}$, any possible realization of the discount factor is high enough for each firm to value the future monopoly profits more than the one stage profits of deviation, and, hence, perfect collusion is an equilibrium for any discount factor. On the contrary, in case 3), $\overline{\delta}<\frac{N-1}{N}$, all the realizations of the discount factor are too low to be able to support any level of collusion. In between these two cases, case 2), perfect collusion can be supported for a range of high realizations of the discount factor while only lower levels of profits can be supported for a range of low realizations. The reason for this is that while for low discount factors it is not possible to support full collusion, it may still be possible to satisfy the incentive compatibility constraint by reducing the present incentives to deviate. For this, the present profits should be lowered so that no firm has an incentive to deviate. In this case, an increase in the discount factor results in an increase in the optimal tacit collusion profits and, hence, in prices. Given that in the other two cases changes in the discount factor have no effect on profits, the next theorem follows.

Theorem 2 $\frac{d\pi^*(\delta)}{d\delta} \geq 0$ *and* $\frac{dp^*(\delta)}{d\delta} \geq 0$.[7]

Note that the characterization of optimal tacit collusion under discount factor fluctuations includes the case of a fixed discount factor. For the fixed discount factor case, $a = b$, Proposition 1 coincides with the text book solution: perfect collusion if $\delta \geq \frac{N-1}{N}$ and no collusion otherwise.

3.2 The effects of changes in f(δ)

The characterization of the optimal tacit collusion equilibrium leads to interesting comparative statics results with respect to changes in the distribution function of the discount factor: 1) the higher the probability of high discount factors, the higher the equilibrium prices and profits, and 2) the higher the volatility of the discount factor, the lower the equilibrium prices and profits.

As an intermediate step to these results, I study first how changes in the distribution function modify the range of perfect collusion under case 2) of Proposition 1. For a cumulative distributions functions F define $\overline{\delta}_F$ as the expected discount factor and c_F as the solution limit to perfect collusion if case 2) applies.

Lemma 3 *Consider two cumulative distributions functions, F and G, such that $\frac{N-1}{N} < \overline{\delta}_{F,G} < 1 - \frac{a}{N-1}$ and F second-order stochastically dominates*[8] *G, then $c_F \leq c_G$.*

[7] I omit straightforward proofs.

[8] For two cumulative distributions functions $F(\delta)$ and $G(\delta)$, F second-order stochastic dominates G if for any r, $a \leq r \leq b$, $\int_a^r F(\delta)d\delta \leq \int_a^r G(\delta)d\delta$, and the inequality is strict in some range. In that case, it can be proven that $\overline{\delta}_F \geq \overline{\delta}_G$ and $\int_a^b u(\delta)f(\delta)d\delta \geq \int_a^b u(\delta)g(\delta)d\delta$, for any increasing concave twice-piecewise-differentiable function $u(\delta)$. See Hirshleifer and Riley [10].

Proof. From the definition of c_F: $H_F(c_F) = (N-1)\left(1-\overline{\delta}_F\right) + \int\limits_a^{c_F} F(\delta)d\delta - c_F = 0$. By second-order stochastic dominance $\int\limits_a^{c_F} F(\delta)d\delta \leq \int\limits_a^{c_F} G(\delta)d\delta$ and $\overline{\delta}_F \geq \overline{\delta}_G$. Therefore, $H_G(c_F) = (N-1)\left(1-\overline{\delta}_G\right) + \int\limits_a^{c_F} G(\delta)d\delta - c_F \geq 0$ and, given that $H_G(.)$ is strictly decreasing and the conditions on $\overline{\delta}_G$, there exists $c_G \in (c_F, b)$ such that $H_G(c_G) = 0$. ■

Denote $\pi_F^*(\delta)$, $E\pi_F^*$ and $p_F^*(\delta)$ as the optimal tacit collusion profit, it's expected value and optimal collusion prices under F, respectively.

Theorem 4 *Consider two cumulative distribution functions, F and G, such that F second-order stochastically dominates G, then $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ and $p_F^*(\delta) \geq p_G^*(\delta)$ for every δ. In addition, $E\pi_F^* \geq E\pi_G^*$.*

Proof. By second-order stochastic dominance $\overline{\delta}_F \geq \overline{\delta}_G$. So, from Proposition 1, we can see that if the solution under F belongs to case 1), the solution under G can belong to any of the three cases. If the solution under F belongs to case 2), the solution under G can belong to cases 2) or 3). And if the solution under F belongs to case 3), the solution under G must belong to the same case. For most of this combinations it is straight forward to see that $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ for every δ. The situation in which both the solution under F as under G belong to case 2) needs more analysis. Since F second-order stochastically dominates G, by Lemma 3, $c_F \leq c_G$. Then, $\pi_F^*(\delta) = \frac{\delta}{c_F}\pi^m \geq \pi_G^*(\delta) = \frac{\delta}{c_G}\pi^m$ if the incentive compatibility constraint is binding in both cases, $\pi_F^*(\delta) = \pi^m \geq \pi_G^*(\delta) = \frac{\delta}{c_G}\pi^m$, if the incentive compatibility constraint binds for G but not for F, and $\pi_F^*(\delta) = \pi_G^*(\delta) = \pi^m$ if it is not binding for any of the two. Therefore, $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ for every δ.

The result with respect to prices follows directly from the positive relationship between profits and prices.

Note that $\pi_F^*(\delta)$ is increasing and concave, hence, by second-order stochastic dominance and $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ for every δ we have that $E\pi_F^* = \int\limits_a^b \pi_F^*(\delta)f(\delta)d\delta \geq \int\limits_a^b \pi_F^*(\delta)g(\delta)d\delta \geq \int\limits_a^b \pi_G^*(\delta)g(\delta)d\delta = E\pi_G^*$. ■

The intuition of this result becomes clear if we consider two particular cases of second order stochastic dominance: when F first-order stochastically dominates[9] G and when G is a mean preserving spread of F.

From Theorem 2 we know that given a distribution of the discount factor, say G, equilibrium prices and profits are increasing in the realization of the discount factor. Then, a shift in the distribution function to higher values (which yields a cumulative distribution function F that first-order stochastically dominates G), would result in an increase in expected profits. This, in turn, increases the threat of future punishments and increases equilibrium prices an profits.

Corollary 5 *If F first-order stochastically dominates G, then $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ and $p_F^*(\delta) \geq p_G^*(\delta)$ for every δ. In addition, $E\pi_F^* \geq E\pi_G^*$.*

¿From Proposition 1 we know that given a distribution factor, say F, the optimal tacit collusion profit function is concave in the discount factor. Therefore, a mean preserving spread (which yields G), would result in a reduction in expected profits. This, in turn, reduces the threat of future punishment and results in lower equilibrium prices and profits.

Corollary 6 *If G is a mean preserving spread of F, then $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ and $p_F^*(\delta) \geq p_G^*(\delta)$ for every δ. In addition, $E\pi_F^* \geq E\pi_G^*$.*

Therefore, the volatility of the discount factor is inversely related to the firms' profits. This result might seem somewhat counterintuitive given that the firms are risk neutral, but the intuition is in fact simple. The combination of the incentive compatibility constraint with the feasibility constraint yields a profit function which is concave in the discount factor even when firms are risk neutral. Hence, an increase in volatility reduces

[9] For two cumulative distributions functions $F(\delta)$ and $G(\delta)$, F first-order stochastic dominates G if for all r, $a \leq r \leq b$, $F(r) \leq G(r)$, and the inequality is strict in some range. In that case, it can be proven that F second-order stochastic dominates G and $\int_a^b u(\delta)f(\delta)d\delta \geq \int_a^b u(\delta)g(\delta)d\delta$, for any increasing piecewise differential function $u(\delta)$. See Hirshleifer and Riley [10].

expected profits reducing the threat of future punishment and lowering equilibrium prices and profits.

3.3 The effects of changes in the number of firms

With N firms in the market a single firm may steal a fraction $\frac{N-1}{N}$ of the market by undercutting the price. Since this fraction is increasing in the number of firms, the higher the number of firms the higher is the present profit from deviation for a given profit, and the more difficult it will be to support collusion. In fact, it can be easily seen from Proposition 1 that for any distribution of the discount factor, there is large enough number of firms above which it is not possible to support any collusion.[10] Define $\pi_N^*(\delta)$ and $p_N^*(\delta)$ as the optimal tacit collusion profits and prices for N firms.

Theorem 7 *If $N > \frac{1}{1-\bar{\delta}}$, then $\pi_N^*(\delta) = 0$ and $p_N^*(\delta) = \varsigma$.*

In addition, it can be easily shown that increases in the number of firms reduce prices and profits (at both industry and firm levels). The next theorem follows from restatement Proposition 1 in terms of industry profits Π_N^* and noting that the range of perfect collusion in case 2) shrinks with increases in the number of firms.

Theorem 8 *Consider two different number of firms N and M, $N < M$, then $\Pi_N^*(\delta) \geq \Pi_M^*(\delta)$, $\pi_N^*(\delta) \geq \pi_M^*(\delta)$ and $p_N^*(\delta) \geq p_M^*(\delta)$ for every δ.*

3.4 Example with uniform distributions

The particular case in which the discount factor is distributed uniformly between a and b, $0 \leq a < b \leq 1$, provides clear examples of the previous results.

[10] It is interesting to note that this result does not depend on fixing the size of the market while changing the number of firms. If both the size of the market and the number of firms increase in the same proportion (that would consists on multiplying the demand function $D(p)$ and the number of firms N by a positive integer), the same result holds. Since an increase in the number of firms and size of the market leaves monopoly profits per firm unchanged but increases the incentives to deviate, the scope of collusion diminishes up to a point in which it disappears.

In the uniform case, taking into consideration that $\overline{\delta} = \frac{a+b}{2}$, I can restate Proposition 1 in the following way:

Proposition 9 *If $\delta \sim U(a,b)$, the function $\pi^*(\delta)$ depends on a, b and N in the following way:*

1) if $b \geq 2 - a\frac{N+1}{N-1}$, $\pi^*(\delta) = \pi^m$;

2) if $\frac{2(N-1)}{N} - a \leq b < 2 - a\frac{N+1}{N-1}$, $\pi^*(\delta) = \pi^m$ for $\delta \geq c$ and $\pi^*(\delta) = \frac{\delta}{c}\pi^m$ for $\delta < c$, with $c = b - \sqrt{N(b^2 - a^2) - 2(b-a)(N-1)}$;

3) if $b < \frac{2(N-1)}{N} - a$, $\pi^*(\delta) = 0$.

Therefore, in the case of uniform distribution of the discount factor, the level of profits for each discount factor depends on the magnitudes of a, b and N. If $b \geq 2 - a\frac{N+1}{N-1}$ perfect collusion can be supported for any realization of the discount factor. If $\frac{2(N-1)}{N} - a \leq b < 2 - a\frac{N+1}{N-1}$, perfect collusion can be supported only for high discount factors and only lower levels of profits can be supported for lower discount factors. Finally, if $b < \frac{2(N-1)}{N} - a$ no collusion can be supported.

Figure 2 shows the different ranges of a and b for the three cases of tacit collusion when $N = 2$.

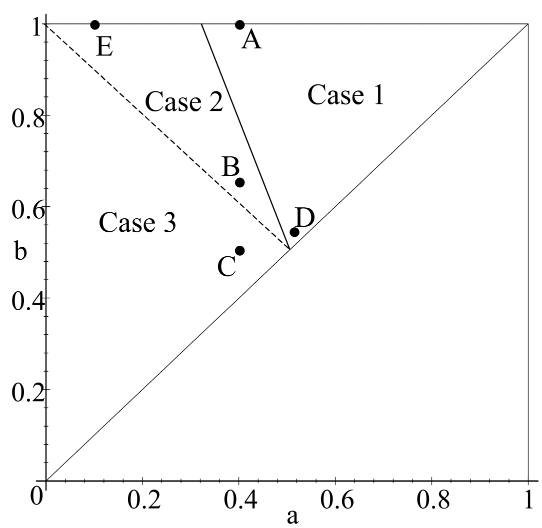

Figure 2: Ranges of tacit collusion

Since $b > a$, the relevant portion of the figure is above the 45 degree line. That part of the graph shows the ranges of a and b that result in different kinds of tacit collusion. For example, to the northeast of the solid black line are the combinations of a and b that results in perfect collusion (case 1) when there are two firms in the market. Between the solid and dashed black lines we see the combinations that result in perfect collusion for high discount factors and imperfect collusion for low discount factors (case 2), and below the dashed black line are the combinations that can not support any collusion (case 3).

Consider the distributions of δ represented in Figure 2 by the points A, B and C (the discount factor is distributed $U(0.4, 1)$, $U(0.4, 0.65)$ and $U(0.4, 0.5)$, respectively). Each of the points falls in a different region and hence will result in a different tacit collusion solution. The distribution denoted by point A results in perfect collusion, the distribution denoted by point B results in perfect collusion for high discount factors and imperfect collusion of lower discount factors and the distribution denoted by C results in no collusion at all. There are two additional things to note from this example. First, profits are (weakly) increasing in the realization of the discount factor, as Theorem 2 proves. While for A and C the tacit collusion profits do not depend on the realization of the discount factor, for B increases in realization of the discount factor may result in an increase of profits and prices. Second, the "more to the right" the distribution function is, the higher profits and prices are. Figure 3 shows that profits under A are larger than under B or C, as Corollary 5 proves.

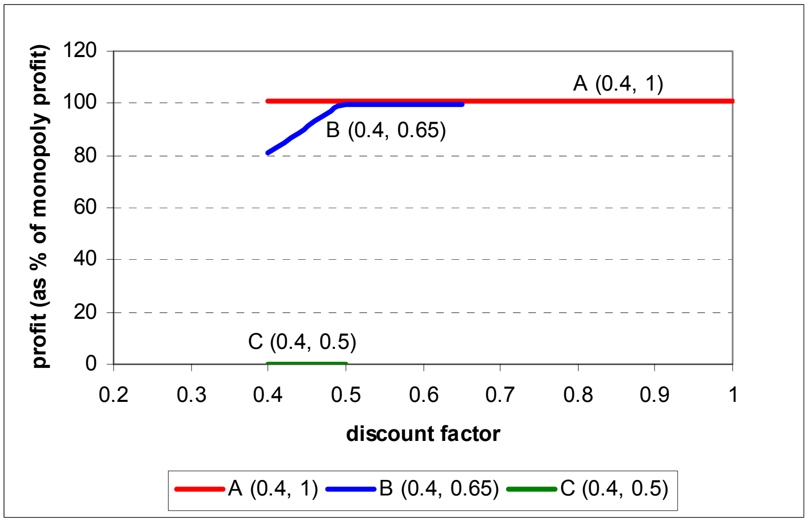

Figure 3: The effect of levels

Consider now the distribution function denoted by point D in Figure 2, $U(0.5, 0.55)$. This distribution function has the same expected value but a lower volatility than the distribution function denoted by point B. We can see from Figure 2 that if there are only two firms in the market, perfect collusion can be supported at point D, while perfect collusion can only be supported for a range of high discount factors for point B. Figure 4 shows the tacit collusion profit functions for these two cases as a percentage of monopoly profits per firm. Consistent with Corollary 6, Figure 4 shows that a mean preserving spread in the distribution of the discount factor reduces the expected value of profits.

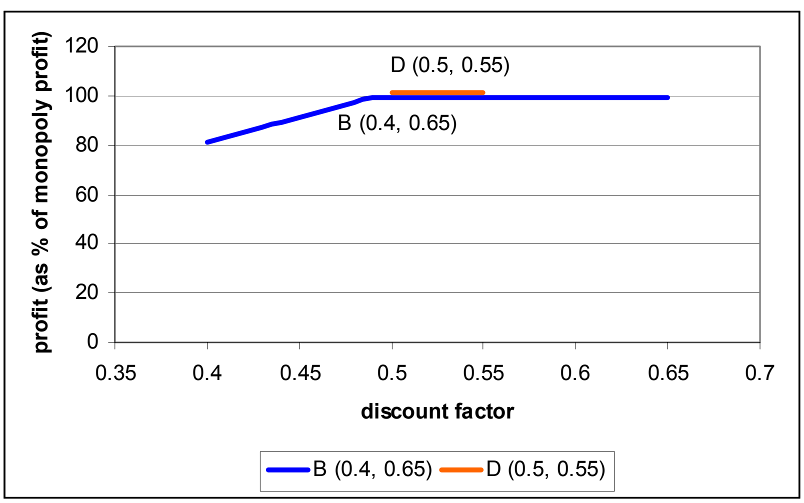

Figure 4: The effect of volatility

To make clear that the volatility effect is not a second order effect consider the distributions denoted by point E in Figure 2, $U(0.1, 1)$. This distribution has a higher expected discount factor than the distribution denoted by point D but it also has a higher volatility. Figure 5 shows that the distribution function with the highest expected discount factor and volatility results in lower collusive profits.

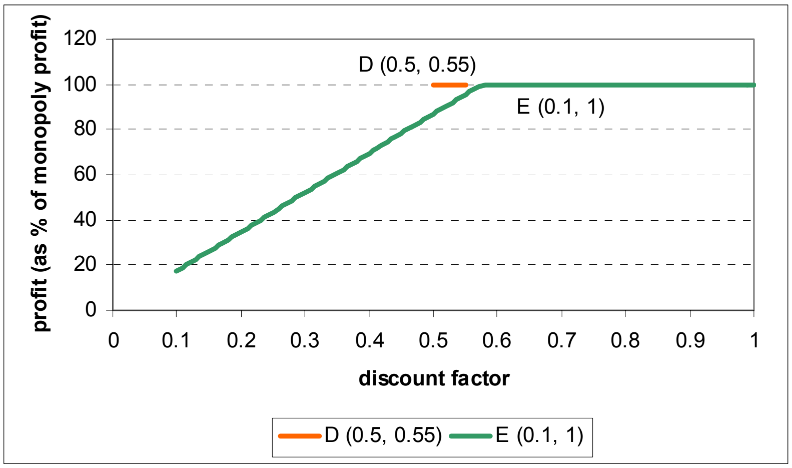

Figure 5: Volatility matters

Figure 6 shows the limits to the three cases of tacit collusion for $N = 2$, 4, 8 and 16. We see that the greater the number of firms, the smaller the set of distribution functions for which some collusion is possible.

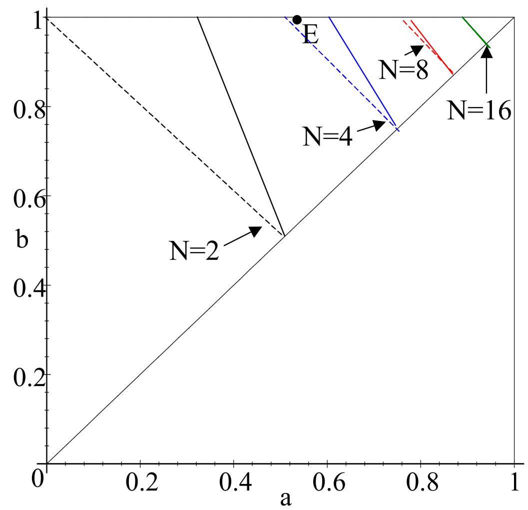

Figure 6: Ranges of tacit collusion for different N

Consider now the distribution of δ represented in Figure 6 by point E: the discount factor is distributed $U(0.52, 1)$. From Figure 6, we see that perfect collusion can be supported if $N = 2$, while perfect collusion can only be supported for a range of high discount factors if $N = 4$, and can not be supported at all for $N = 8$. Figure 7 shows the tacit collusion industry profits (as a percentage of industry monopoly profit) for these three cases and, consistent with Theorem 8, shows that the profits decrease with the number of firms.

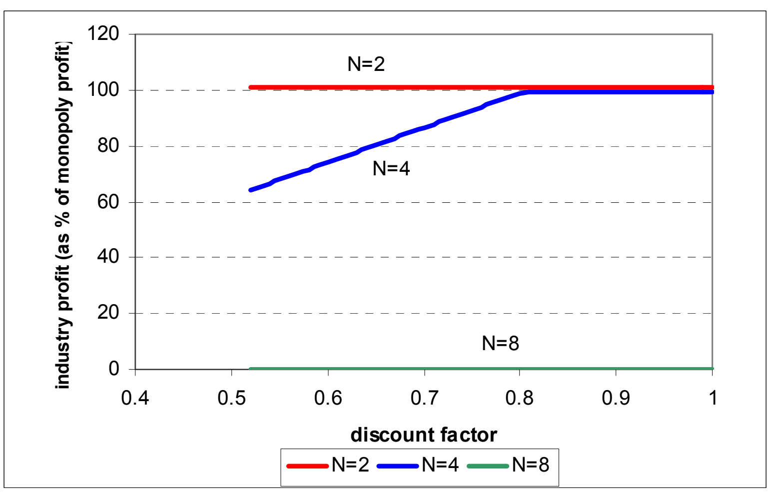

Figure 7: Tacit collusion and the number of firms

4 Quantity Competition

In this section I show that, under certain assumptions, the three main results that hold under price competition also hold under quantity competition. Namely, first, the higher the discount factor in a period, the higher the collusive prices and profits in that period, second, the higher the probability of high discount factors, the higher the collusive prices and profits, and third, the higher the volatility of the discount factor, the lower the collusive prices and profits that can be supported in equilibrium.

However, to prove this I have to characterize the optimal punishment scheme, which was not necessary under price competition. This is interesting because I show that, while punishment schemes can be extremely complex under quantity competition, the optimal punishment has a simple stick-and-carrot characterization.

I consider the same model of section 2 with one main difference: firms compete on quantities. In addition, and only for the sake of generality, I also assume that firms have a continuous and differentiable cost function $\varsigma(q)$ instead of the linear cost of section 2.

As in section 2, I restrict my attention to symmetric equilibria: all the firms produce at a given period the same quantity q. In this symmetric case, I can write the profits of each firm as $\pi(q) = P\left(Nq\right)q - \varsigma(q)$ and total industry profits as $\Pi(\delta) = N\pi(q)$. I assume that there exists a quantity q^m that maximizes the total industry profits, that is the perfect collusion quantity (q^m would be the Nth part of a monopolist optimal production if there are no fixed cost per factory and increasing returns to scale). Denote $\pi^m = \pi(q^m)$ as the perfect collusion profit per firm and $\Pi^m = N\pi^m$ as total industry perfect collusion profit.

4.1 Optimal tacit collusion with a random discount factor

In the case of quantity competition the Cournot reversion is not necessarily the best available punishment since it may be possible to generate subgame perfect threats that lower the profits below the Cournot level. Therefore, to characterize the optimal tacit collusion solution it is also necessary to define the optimal punishment scheme. In this section I characterize the optimal equilibrium punishment and collusion under certain assumptions. The first assumption is that there exists a symmetric Cournot equilibrium.

Assumption 1: There exists a quantity q^c that is the unique symmetric Cournot equilibrium.

In this equilibrium each firm earns a profit of π^c and it can be proven that $\pi^m > \pi^c$, and $q^m < q^c$.

The second assumption concerns the profits from deviation. In the case of quantity competition, if $N-1$ firms are each producing a quantity q, the remaining firm can obtain at most a profit of $\pi^d(q) = \max_{s \geq 0} \{P(s + (N-1)q)\, s - \varsigma(s)\}$ by producing some other quantity. The second simplifying assumption establishes that both $\pi^d(q)$ an $\pi(q)$ are decreasing with the former having a bigger slope than the latter, in absolute terms, for quantities below q^c while the opposite occurs for quantities above q^c.

Assumption 2: For $q \in [q^m, q^c)$, $\frac{d\pi^d}{dq} < \frac{d\pi}{dq} < 0$ and for $q \in (q^c, +\infty)$, $\frac{d\pi}{dq} < \frac{d\pi^d}{dq} \leq 0$.

These assumptions are valid, for example, in a market with a linear demand function and constant marginal cost. In addition, in the linear case there is a unique quantity that maximizes industry profits (q^m) and a unique and symmetric Cournot equilibrium (q^c). Hence there is no contradiction between the assumptions made in this section.[11]

As in section 2, the optimal symmetric tacit collusion equilibrium can be characterized by the maximum level of profits per firm that can be supported for each discount factor, which I denote $\pi^*(\delta) : [a, b] \to [\pi^c, \pi^m]$, abusing notation from section 2. Since assumption 2 ensures that there is a one to one relationship between profits and quantities produced in the relevant range, once $\pi^*(\delta)$ is obtained, the optimal tacit collusion quantities $q^*(\delta) : [a, b] \to [q^m, q^c]$ are also obtained. From the demand function we can

[11] In addition, these assumptions, as the assumption presented in the next subsection, could be obtained from assumptions regarding the demand and cost functions. Since those assumptions would be only sufficient ones and would not provide a better intuition I prefer to present conditions regarding $\pi(q)$ and $\pi^d(q)$ that yield the desired results.

obtain the optimal tacit collusion profits $p^*(\delta) = P(Nq^*(\delta))$.[12]

As in section 2, I use the recursiveness of the problem to write the present value of profits:

$$V(\delta) = \pi(\delta) + \frac{\delta}{1-\overline{\delta}} \int_a^b \pi(\delta')f(\delta')d\delta' \tag{7}$$

In addition, the feasibility condition can be written as:

$$\pi(\delta) \leq \pi^m \tag{8}$$

The incentive compatibility constraint differs from that in the previous section since neither the short run incentives to deviate nor the future punishments are the same. Under price competition, a firm can capture the whole market by a small price deviation, obtaining $(N-1)\pi(\delta)$ in profits from deviation. Under quantity competition, the maximum profit from deviation is $\pi^d\left(q\left(\pi(\delta)\right)\right) - \pi(\delta)$, where $q(\pi)$ is the quantity that every firm has to produce to get a per firm profit of π. In addition, the possible punishment from deviation may not be the same as in price competition. In price competition reverting to a situation of zero profits is a credible threat, since that is the Bertrand equilibrium. Instead, under quantity competition a punishment of zero profits forever may not be credible. What is credible depends on the biggest credible threat. This threat would consists of punishing the deviator with the lowest equilibrium discounted payoff, denoted by $\underline{V}(\delta)$, while rewarding compliance with the equilibrium with the highest equilibrium discounted payoff, denoted by $\overline{V}(\delta)$, if tomorrows discount factor is δ. Assume for now that the extreme discounted equilibrium payoff functions $\overline{V}(\delta)$ and $\underline{V}(\delta)$ exist, as it is proven later, and define their expected values as $E\overline{V}$ and $E\underline{V}$, respectively. Therefore, for $\pi(\delta)$ to be incentive compatible, it must be the case that no player has incentives to deviate if conforming is rewarded with the highest possible expected continuation payoff $E\overline{V}$ and deviating is punished with the lowest possible

[12]Note that this functions denote equilibrium outcomes and not strategies. The supporting strategies are not explicitly defined due to their lack of peculiarities.

expected continuation payoff $E\underline{V}$:

$$\pi^d\left(q\left(\pi(\delta)\right)\right)-\pi(\delta)\leq\delta\left[E\overline{V}-E\underline{V}\right]\ \forall\ \delta \tag{9}$$

For simplicity, write the left hand side of equation (9) as $\Phi(\pi(\delta))$ and denote $E\overline{V}-E\underline{V}$ on the right hand side of the equation as B. As such, for a given B, the incentive compatibility constraint can be written as

$$\Phi(\pi(\delta))\leq\delta B\ \forall\ \delta \tag{10}$$

Note that $\Phi(\pi^c)=0$ and that $\Phi(\pi)$ increases as π separates from π^c. Then, for a given amount of threat δB,there is a highest and lowest amount of profit that can be supported. Next I characterize the incentive compatible upper bound to profits, and its interaction with the feasibility constraint, and then characterize the incentive compatible lower bound to profits.

Lemma 10 *Under Assumptions 1 and 2, for a given B, the incentive compatible upper bound to profits is not binding for any δ if $aB>\Phi(\pi^m)$. If instead $aB\leq\Phi(\pi^m)$, there exists a number $c(B)\in[a,b]$ such that the upper bound can be written as $\pi(\delta)\leq\Phi_+^{-1}(\delta B)$ for $\delta\leq c(B)$, where $\Phi_+^{-1}(\delta B)$ is the inverse of $\Phi(\pi)$ if we restrict its domain to $[\pi^c,\pi^m]$, and it is not binding for $\delta>c(B)$. In addition, the incentive compatible upper bound is increasing in δ for $\delta\leq c(B)$, and $c(B)$ and $\Phi_+^{-1}(\delta B)$ are continuous.*

Proof. In Appendix. ■

Therefore, for low discount factors the maximum level of profits that can be supported is bounded by the incentive compatible upper bound, while for high values it is bounded by the feasibility constraint. Combining both we have the IC$^+$-F constraint:

$$\pi(\delta)\leq\begin{cases}\Phi_+^{-1}(\delta B) & if\ aB\leq\Phi(\pi^m)\ and\ \delta\leq c(B)\\ \pi^m & otherwise\end{cases} \tag{11}$$

In the optimal symmetric tacit collusion equilibrium, firms will choose profits as large as possible given the incentive compatible upper bound and the feasibility constraint.

In addition, given that conforming with the equilibrium strategy must be rewarded with the highest equilibrium payoff, the highest equilibrium discounted payoff $\overline{V}(\delta)$ has a simple relationship with the optimal tacit collusion solution. If $\pi^*(\delta)$ is the optimal tacit collusion profit function, then $\overline{V}(\delta) = \pi^*(\delta) + \frac{\delta}{1-\overline{\delta}} \int_a^b \pi^*(\delta')f(\delta')d\delta'$ and its expected value is $E\overline{V} = \frac{E\pi^*}{1-\overline{\delta}}$. Therefore, given the lowest expected equilibrium payoff $E\underline{V}$, the optimal tacit collusion solution is subject to the following equation:

$$\pi^*(\delta) = \begin{cases} \Phi_+^{-1}\left(\delta\left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) & if\ a\left(\delta\left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) \leq \Phi(\pi^m)\ and\ \delta \leq c\left(\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right) \\ \pi^m & otherwise \end{cases} \tag{12}$$

The following lemma characterizes the incentive compatible lower bound to profits.

Lemma 11 *Under Assumptions 1 and 2, for a given B, the incentive compatible lower bound to profits can be written as $\pi(\delta) \geq \Phi_-^{-1}(\delta B)$, where $\Phi_-^{-1}(\delta B)$ is the inverse of $\Phi(\pi)$ if we restrict its domain to $(-\infty, \pi^c]$. In addition, the incentive compatible lower bound is decreasing in δ, and $\Phi_-^{-1}(\delta B)$ is continuous.*

Proof. In Appendix. ■

Having characterized the incentive compatible lower bound to profits, I must still characterize the lower discounted continuation payoff $\underline{V}(\delta)$. I show that the optimal punishment scheme, which yields $\underline{V}(\delta)$, has a simple stick-and-carrot characterization (the punishment takes only one period and is as big as possible in equilibrium), extending the results of Abreu [1] from the fixed discount factor case.

Lemma 12 *Given $E\pi^*$ and $E\underline{V}$, the lowest equilibrium payoff function is $\underline{V}(\delta) = \Phi_-^{-1}\left(\delta\left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) + \frac{\delta}{1-\overline{\delta}}E\pi^*$.*

Proof. Consider any punishment scheme consisting of a profit of $\widetilde{\pi}(\delta)$ in the first period and an expected continuation payoff of $E\widetilde{V}$. Define the present value of the game in that case as $\underline{\widetilde{V}}(\delta) = \widetilde{\pi}(\delta) + \delta E\widetilde{V}$. For this punishment scheme to be credible

it must be the case that $\underline{\widetilde{V}}(\delta) \geq \pi^d \left(q(\widetilde{\pi}(\delta))\right) + \delta E\underline{V}$. Choose now the first payoff of a two phase punishment $\pi'(\delta)$ so that $\pi'(\delta) + \frac{\delta}{1-\overline{\delta}} E\pi^* = \underline{\widetilde{V}}(\delta)$. Given that $\frac{E\pi^*}{1-\overline{\delta}} \geq E\widetilde{V}$, $\pi'(\delta) \leq \widetilde{\pi}(\delta)$ and by $\pi^d \left(q(.)\right)$ being increasing, $\underline{\widetilde{V}}(\delta) \geq \pi^d \left(q(\pi'(\delta))\right) + \delta E\underline{V}$ and the two phase punishment is credible. Therefore any equilibrium punishment can be matched with a two phase punishment that yields the best continuation payoff in the second phase. Then, choosing the lowest equilibrium present payoff, I obtain the lowest equilibrium discounted payoff for a given discount factor, and $\underline{V}(\delta) = \Phi_-^{-1}\left(\delta \left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) + \frac{\delta}{1-\overline{\delta}} E\pi^*$.
■

Therefore, given the optimal tacit collusion solution, the lowest possible continuation payoffs are subject to the following equation:

$$\underline{V}(\delta) = \Phi_-^{-1}\left(\delta \left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) + \frac{\delta}{1-\overline{\delta}} E\pi^* \tag{13}$$

The solution to the problem of finding the optimal tacit collusion profits and the optimal punishment that support that collusion consists of finding the functions $\pi^*(\delta)$ and $\underline{V}(\delta)$ that solve equations (12) and (13) simultaneously and choosing the solution with the highest expect profit $E\pi^*$. The next proposition shows that this problem has a unique solution.

Proposition 13 *Under Assumptions 1 and 2, $\pi^*(\delta)$ and $\underline{V}(\delta)$ exist. In addition $\pi^*(\delta)$ is unique.*

Proof. Taking the expected value over (12), for any possible solution $\pi(\delta)$ it has to hold that:

$$E\pi = \int_a^{c\left(\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right)} \Phi_+^{-1}\left(\delta \left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) f(\delta) d\delta + \left(1 - F\left(c\left(\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right)\right)\right)\pi^m \tag{14}$$

In the same way, taking the expected value over (13), for any possible solution $\underline{V}(\delta)$ it has to hold that:

$$E\underline{V} = \int_a^b \Phi_-^{-1}\left(\delta \left[\frac{E\pi^*}{1-\overline{\delta}} - E\underline{V}\right]\right) f(\delta) d\delta + \frac{\overline{\delta}}{1-\overline{\delta}} E\pi^* \tag{15}$$

Note that there is a one to one relationship between the profit functions that satisfy equation (12) and the expected values that satisfy equation (14). That is, if $\pi^*(\delta)$ satisfies equation (12), then $E\pi^*$ must satisfy equation (14), and if the value $E\pi^*$ satisfies equation (14), $\pi^*(\delta)$ satisfies equation (12) with $E\pi^*$ in the right hand side. The same is true for equations (13) and (15). Therefore, we can find $\pi^*(\delta)$ and $\underline{V}(\delta)$ by choosing the solution to equations (14) and (15) with the highest $E\pi^*$. Note that $E\pi^* = \pi^c$ and $E\underline{V} = \frac{\pi^c}{1-\overline{\delta}}$ solve the pair of equations and, hence, there is at least one solution. Let

$$H(r,s) = \begin{cases} \int_a^{c\left(\frac{r}{1-\overline{\delta}}-s\right)} \Phi_+^{-1}\left(\left[\frac{r}{1-\overline{\delta}} - s\right]\right) f(\delta)d\delta + \left(1 - F\left(c\left(\frac{r}{1-\overline{\delta}} - s\right)\right)\right)\pi^m - r \\ \int_a^b \Phi_-^{-1}\left(\delta\left[\frac{r}{1-\overline{\delta}} - s\right]\right) f(\delta)d\delta + \frac{\overline{\delta}}{1-\overline{\delta}} r - s \end{cases}.$$

Since $c(.)$, $\Phi_+^{-1}(.)$, $\Phi_-^{-1}(.)$, and $F(.)$ are continuous, $H(r,s)$ is also a continuous function. Then, the set of numbers that make $H(r,s) = (0,0)$ is closed, given that the inverse images of closed sets are closed for continuous functions. In addition it must be bounded since $r \in [\pi^c, \pi^m]$ and $s \in \left[0, \frac{\pi^c}{1-\overline{\delta}}\right]$. Therefore, the set of solutions is non-empty, closed and bounded. Then, among the solutions there exists one with the highest r that gives $(E\pi^*, E\underline{V})$. Plugging this into equations (12) and (13) we obtain $\pi^*(\delta)$ and $\underline{V}(\delta)$. Uniqueness is clear from the fact that there is a one to one relationship between $E\pi^*$ and $\pi^*(\delta)$. ■

Optimal tacit collusion must fall in one of the following three cases, depending on which restriction is binding. First, it may be that only the feasibility constraint binds for every discount factor. In this case, the value of the future monopoly profits outweighs the profits from deviation, and perfect collusion is an equilibrium for any discount factor. Second, it may be possible that the incentive compatibility constraint binds for low discount factors while the feasibility constraint binds for high discount factors. Third, it may be possible that the incentive compatible upper bound to profits binds for every value of the discount factor. While in the first case changes in the discount factor do not affect profit and prices, in the last two cases, an increase in the discount factor results in an increase in collusive profit and prices. The reason for this is that a higher discount factor results in a higher threat of punishment, so that higher profits can be achieved without firms having incentives to deviate, and the next Theorem follows.

Theorem 14 *Under Assumption 1 and 2,* $\frac{d\pi^*(\delta)}{d\delta} \geq 0$ *and* $\frac{dp^*(\delta)}{d\delta} \geq 0.$

As in section 2, the equilibrium profits and prices are increasing in the discount factor.

4.2 The effects of changes in $f(\delta)$

In section 2, the comparative static results with respect to the distribution function of the discount factor depend on the optimal tacit collusion profit function being increasing and concave. If that is the case, shifts to the left of the distribution function or increments in volatility reduce the expectation of future profits and result in lower equilibrium profits and prices. Because under quantity competition the level of punishment is not independent of the discount factor, it is not enough to look at the shape of the optimal tacit collusion profits to obtain a comparative static result with respect the distribution function. What is important is the shape of the threat of future punishments: $\overline{V}(\delta) - \underline{V}(\delta)$.

The stick-and-carrot property of the optimal punishment implies that streams of payoffs leading to the highest and lowest discounted equilibrium payoff differ only in the first period. As a result, the threat of future punishment is simply the maximum difference in payoffs that can be supported in equilibrium in *one* period. Since I have already proved that the upper bound to profits is increasing and the lower bound to profits is decreasing, it only remains to be shown that the upper bound is concave while the lower bound is convex. The following assumption is a sufficient condition for that.

Assumption 3: For $q \in [q^m, +\infty)$, $\frac{d^2\pi^d}{dq^2} \geq 0 \geq \frac{d^2\pi}{dq^2}$.

As Assumptions 1 and 2, this assumption is valid in a market with a linear demand function and constant marginal cost. Hence, there is no contradiction among the assumptions made in this section.

Lemma 15 *Under Assumptions 1-3,* $\overline{V}(\delta) - \underline{V}(\delta)$ *is increasing and concave.*

Proof. In Appendix. ■

From the future threat being increasing and concave in the next period discount factor, the desired comparative static result with respect to the distribution function of the discount factor follows.

Theorem 16 *Consider two cumulative distribution functions, F and G, such that F second-order stochastically dominates G, then $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ and $p_F^*(\delta) \geq p_G^*(\delta)$ for every δ. In addition, $E_F \pi_F^* \geq E_G \pi_G^*$.*

Proof. Let $\overline{V}_j(\delta)$ and $\underline{V}_j(\delta)$, $j = F, G$, be the highest and lowest equilibrium discounted payoff under j. First, I show that $E_F\left(\overline{V}_F(\delta) - \underline{V}_F(\delta)\right) \geq E_G\left(\overline{V}_G(\delta) - \underline{V}_G(\delta)\right)$. Suppose not, then $E_G\left(\overline{V}_G(\delta) - \underline{V}_G(\delta)\right) \geq E_F\left(\overline{V}_F(\delta) - \underline{V}_F(\delta)\right)$. Since F second-order stochastically dominates G and $\overline{V}_G(\delta) - \underline{V}_G(\delta)$ is increasing and concave by Lemma 16, $E_F\left(\overline{V}_G(\delta) - \underline{V}_G(\delta)\right) \geq E_G\left(\overline{V}_G(\delta) - \underline{V}_G(\delta)\right) \geq E_F\left(\overline{V}_F(\delta) - \underline{V}_F(\delta)\right)$. But then, the strategies that yield $\pi_G^*(\delta)$ and $\underline{V}_G(\delta)$ under G do not violate the incentive and feasibility constraints under F, by Φ_+^{-1} being increasing and Φ_-^{-1} being decreasing on $\delta\left[\frac{E\pi^*}{1-\delta} - E\underline{V}\right]$. Therefore $\pi_F^*(\delta)$ is not an optimal tacit collusion solution under F, which is a contradiction.

Second, given that $E_F\left(\overline{V}_F(\delta) - \underline{V}_F(\delta)\right) \geq E_G\left(\overline{V}_G(\delta) - \underline{V}_G(\delta)\right)$ and Φ_+^{-1} is increasing, $\pi_F^*(\delta) \geq \pi_G^*(\delta)$ for every δ. The last two results follow from the positive relationship between profits and prices and the relationship between F and G, respectively. ■

The intuition behind this results is simple. Given that the threat of future punishment is increasing and concave in the discount factor, both increases in the probability of low discount factors and increases in its volatility reduce the expected value of the punishment and result in a reduction of collusive profits and prices.

4.3 The effects of changes in the number of firms

In section 2 I showed that under price competition, increases in the number of firms increase the incentives to deviate, decreasing equilibrium profits. This result may not be valid when firms compete on quantities since not only do the incentives to deviate change with the number of firms, but so may the threat of future punishment. In fact, the higher the number of firms the easier it is to support low profits -a consequence of which is that industry Cournot profits fall with the number of firms- and the higher the threat of punishment for deviation. Therefore, while under price competition it is enough to study the effect of the number of firms on the incentives to deviate, this is not sufficient under quantity competition.

While more work is needed to characterize general conditions under which increases in the number of firms decrease equilibrium profits and prices, the next subsection presents an example of such a situation.[13]

4.4 Example with uniform distributions

I study next the case in which the discount factor is distributed uniformly between a and b, $0 \leq a < b \leq 1$, and the inverse demand function -net of a constant marginal cost- is $P = 12 - Q$, and I provide clear examples of the previous results.

Three different types of optimal tacit collusion exist. If a and b are high, relative to the number of firms, perfect collusion can be supported for any realization of the discount factor. If a and b are low, relative to the number of firms, perfect collusion can not be supported for any realization of the discount factor, but in contrast to what

[13] To my knowledge this issue also remains to be solved for the case of a fixed discount factor. The closest related paper is Brock and Scheinkman [5] which studies the effect of the number of firms on tacit collusion for a fixed discount factor, price competition and an exogenous capacity per firm. They find that changes in the number of firms have a non-monotone effect on optimal collusive prices. Note that the capacity is exogenous and the link to quantity competition from Kreps and Scheinkman [12] does not apply.

happens under price competition, some collusion can still be supported. If a and b fall in a middle ground, perfect collusion can be supported only for high discount factors and only lower levels of profits can be supported for low discount factors.

Figure 8 shows the different ranges of a and b for the three cases of tacit collusion, for $N = 2$ and $N = 16$.

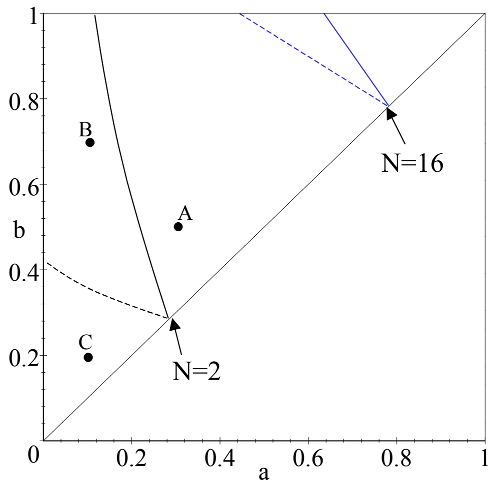

Figure 8: Ranges of tacit collusion

Since $b > a$, the relevant portion of the figure is above the 45 degree line. That part of the graph shows the ranges of a and b that result in different kinds of tacit collusion. For example, to the northeast of the solid black line are the combinations of a and b that result in perfect collusion when there are two firms in the market. Between the solid and dashed black lines are the combinations that result in perfect collusion for high discount factors and imperfect collusion for low discount factors, and below the dashed black line are the combinations that can not support perfect collusion. For example consider the distributions depicted by points A, B and C. While A results in perfect collusion, B can only support perfect collusion for high discount factors and lower profits for low discount factors. Finally, C can not support perfect collusion for any discount factor but can still support some collusion. See Figure 9.

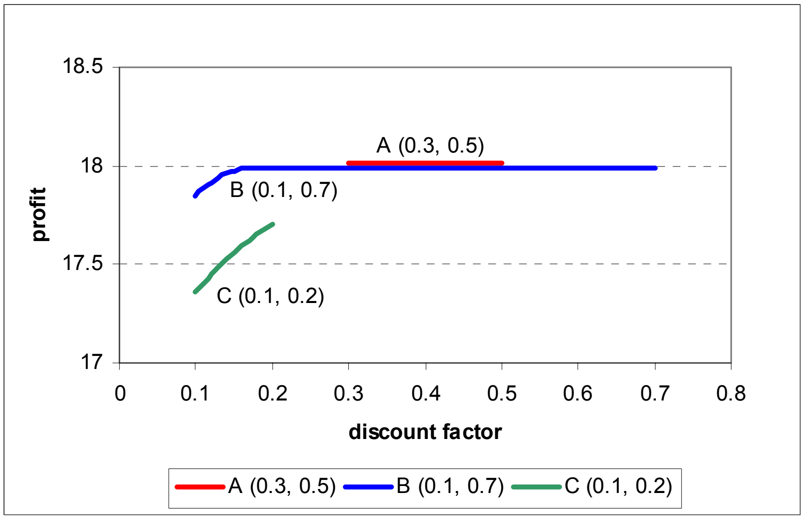

Figure 9: Tacit collusion profits with N=2

From Figure 9 it is clear that profits are increasing in the discount factor. The comparison between the optimal tacit collusion profits for points A and C is an example of the result that the higher the probability of high discount factor, the higher collusive profits and prices. The comparison between the collusive profits for points A and B is an example of the result that the higher the volatility of the discount factor, the lower profits and prices.

One can see the limits to the three types of tacit collusion for $N = 16$ in Figure 8. For the distribution function depicted by point A, perfect collusion can be supported if $N = 2$, but perfect collusion can not be supported at all -but lower levels of collusion can- for $N = 16$, therefore, profits must be lower in the latter case.

5 Extensions

In this section I analyze the restrictiveness of the assumption of symmetric equilibria and study some extensions. As an extension, I modify the assumption of independently distributed discount factor in two ways. I consider first deterministic discount factor cycles and show that increasing discount factors make easier to support collusion. Second, I consider the case in which the distribution of tomorrow's discount factor depends on today's value and show that an increase in the discount factor may result in a decrease in equilibrium prices and profits (since the increase in the discount factor may lead to an increase in its future volatility). Finally, I study the validity of the three main results of this paper for general repeated games.

5.1 Asymmetric equilibrium prices

In this paper I only consider symmetric equilibrium collusive prices and quantities. This assumption may not be that restrictive given that joint overall profits to firms are generally higher when all the firms charge the same price or produce the same quantity in equilibrium. The existence of asymmetries in firms' equilibrium collusive behavior can only reduce prices and total industry profits, since it is the incentive compatibility constraint of the less favored firm that binds.

In addition, in the case of price competition, this asymmetry effect is strengthened by an intrinsic discontinuity of the Bertrand model. With price competition, if firms offer different prices there will be a group of firms that will not provide goods to the market and will get zero profits. These firms will have large incentives to deviate. Thus, under price competition, the impact of even small price asymmetries on the incentive compatibility constraints can be significant.

Therefore, there is a compelling reason to restrict ourselves to symmetric equilibrium behavior in this paper: it is the equilibria that maximizes the industry's total profit. Introducing asymmetries would reduce the industry's profits by increasing the incentives to deviate for those less favored firms that get a small share of the market.[14]

[14]Nevertheless, under quantity competition, it may be useful to allow for asymmetric behavior off the equilibrium path. The optimal punishment schemes characterized in this paper may only be optimal under the restriction of symmetry off the equilibrium path. It is possible that asymmetries during the punishment stage generate bigger punishments and higher symmetric collusion, as it is the case under a fixed discount factor (Abreu [1]).

5.2 Deterministic discount factor cycles

In this section I consider deterministic discount factor cycles and show that higher collusive prices and profits can be supported when the discount factor is increasing. The reason is simple: the higher the future discount factors, the higher future collusive profits and the larger the threat of punishment. Hence, the higher the future discount factors, the higher present collusive prices and profits, as the next example shows.

Example 17 *An increasing discount factor facilitates collusion: For the discount factor cycle* $\{.55, .75, .55, .35\}$, *price competition and two firms in the market, the optimal tacit collusion solution is represented in Figure 10 as a percentage of monopoly profits. We can see that for* $\delta = 0.55$ *the optimal tacit collusion is higher when the discount factor is increasing (point A) than when it is decreasing (point B). Therefore, it is easier to support collusion for a given discount factor when the discount factor is increasing.*

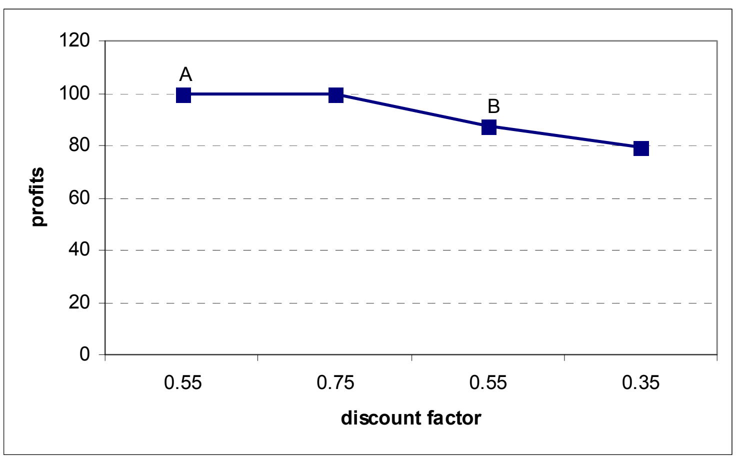

Figure 10: Profits under cyclical discount factor

Under cyclical discount factor fluctuations, both high discount factors today and in the future make it easy to support collusion given that both increase the threat of future price wars. In contrast, Haltiwanger and Harrington [9] find that under cyclical demand fluctuations, a high demand today makes it difficult to support collusion since it offers high incentives to deviate, while high demand in future periods makes it easy to collude today because it increases the threat of future price wars.

5.3 Correlated discount factor and the volatility effect

Given that both a high discount factor today and in the future make it easy to support collusion, allowing for the more realistic case of positively correlated discount factors will not affect the main results. But these results may be modified if changes in today's discount factor affect its future volatility. In this section I present an extension to the basic model to illustrate that an increase in the discount factor does not necessarily lead to higher collusive prices and profits if the increase in the discount factor also raises the volatility of future discount factors. When the value of the present discount factor affect the distribution of the future discount factor, the solution to the optimization problem can not be found easily. Nevertheless, under price competition and a discrete distribution of the discount factor, the problem can be solved as a linear programming problem (see apendix).

Example 18 *Consider the case in which the discount factor can take only three values* ($\frac{1}{4}$, $\frac{1}{2}$ *and* $\frac{3}{4}$)*, there are two firms and the monopoly profit per firm is* 18*. The distribution function of the discount factor depends on the past discount factor in the following way:*

$p(\delta_t \mid \delta_{t-1})$		δ_t		
		$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
δ_{t-1}	$\frac{1}{4}$	3/5	1/5	1/5
	$\frac{1}{2}$	0	1	0
	$\frac{3}{4}$	12/25	0	13/25

Solving the linear programming problem we find that the optimal symmetric tacit collusion equilibrium yields profits equal to 4.8*,* 18 *and* 15.8 *for the discount factor being* $\frac{1}{4}$, $\frac{1}{2}$ *and* $\frac{3}{4}$*, respectively.*

This example shows that an increase in the discount factor, while increasing the expectation of the future discount factor, may still result in a reduction of profits and prices. The reason is that not only does the expectation of future discount factors matter, but so does its volatility. In this case, given that future discount factors have higher volatility when $\delta = \frac{3}{4}$ than when $\delta = \frac{1}{2}$, equilibrium profits are lower under the former than under the latter.

5.4 General normal form games

In this section I study whether the main results of this paper can be extended to general infinitely repeated games with discount factor fluctuations. I consider an infinitely repeated simultaneous move game in which the discount factor is independently and identically distributed. As in the rest of this paper, players observe the realization of the discount factor before choosing an action.

In this more general environment the following results regarding discount factor levels can be shown: first, the higher the realization of the discount factor the larger the set of equilibrium outcomes, and second, the higher the probability of high discount factors the larger the set of equilibrium outcomes.[15] In contrast, it is not true that an increase in the volatility of the discount factor always results in a decrease in the set of equilibrium outcomes. The next example shows that an increase in the volatility of the discount factor may increase the set of equilibrium outcomes for some discount factors.

Example 19 *An increase in volatility of the discount factor may increase the set of equilibrium outcomes:*

Consider the following stage game:

[15]Given that the discount factor is i.i.d., before discounting it, the threat of future punishment is independent of the present realization of the discount factor. Hence, the higher the discount factor the more important that threat is and the bigger the set of equilibrium outcomes. Given this first result, shifts of the distribution function to the right result in increments in the threat of punishment and an increase in the set of outcomes for every realization of the discount factor.

			Column		
		a	*b*	*c*
Row	*A*	*5, 5*	*0, 0*	*-2, 10*
	B	*0, 0*	*4, 4*	*-2, 5*
	C	*10, -2*	*5, -2*	*0, 0*

In this stage game there is a unique Nash equilibrium (C,c), which is Pareto dominated by either (A,a) or (B,b). The infinite repetition of the stage game opens the possibility that these outcomes can be supported in equilibrium. Note that (A,a) yields a higher payoff than (B,b) but offers higher incentives to deviate. With discount factor fluctuations, we should expect that in the optimal symmetric equilibrium, (A,a) is played for realization of the discount factors that are high enough, (B,b) for lower ones and, finally, (C,c) when the realization of discount factor is too low to be able to support any cooperation by the threat of future punishment. In fact, if the discount factor is distributed U(0,1), the outcomes of the optimal symmetric equilibrium are the following[16]*: (A,a) if* $\delta \geq 0.65$*, (B,b) if* $0.13 \leq \delta < 0.65$*, (C,c) if* $\delta < 0.13$*, as shown in Figure 11. In this case the expected utility equals* 3.82*.*

Consider now a modification of the distribution of the discount factor. From the original U(0,1) distribution take the mass of the segment $[0.45, 0.55]$ *and add it to the area between* $[0.25, 0.3]$ *and* $[0.7, 0.75]$*. This modification adds volatility to the discount factor but at the same time adds weight to the discount factors that yield a high payoff. Hence, the change in the distribution function increases the equilibrium expected utility and relaxes the incentive compatibility constraint, increasing the set of discount factors for which (A,a) can be supported in equilibrium. In fact, under the modified distribution function, the outcomes of the equilibrium that maximizes the players expected utility is the following: (A,a) if* $\delta \geq 0.63$*, (B,b) if* $0.126 \leq \delta < 0.63$*, (C,c) if* $\delta < 0.126$*. In this case the expected utility equals* 3.97*.*

[16]The problem consists of finding the minimum discount factors for which (A,a) and (B,b) can be played in a symetric equilibrium.

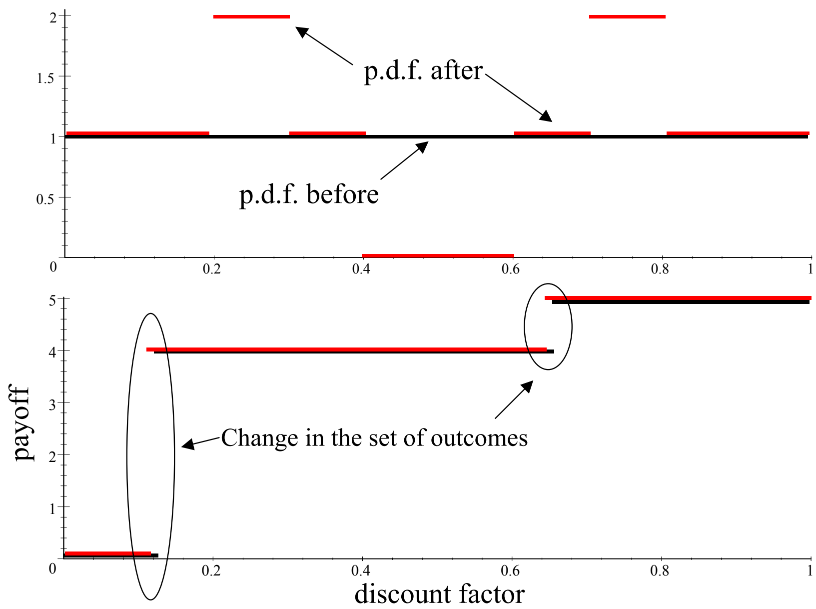

Figure 11: Volatility and general games

This example shows that a mean preserving spread of the discount factor may result in an increase on the expected payoff of the players and an expansion of the set of possible outcomes for some discount factors.

For repeated games in general it is no longer true that increases in the volatility of the discount factor reduce the set of equilibrium outcomes. The reason is that the maximum level of utility, as a function of the discount factor, is not necessarily a concave function (nor is the minimum level of utility necessarily convex). As result, increases in volatility may increase the expected equilibrium utility, increasing the threat of future punishment. This, in turn, increases the set of outcomes that can be supported for each discount factor.

6 Conclusions

To my knowledge, this paper represents the first effort to examine the effect of discount factor fluctuations in repeated games. In a repeated oligopoly, I characterized the optimal symmetric collusion and found that collusive prices and profits increase with both present and future discount factor levels and decrease with discount factor volatility. These results stress the importance that discount factor levels have on repeated games and introduce a new factor to the literature: the volatility of the discount factor.

This work has several important implications for future study. While most of the existing empirical literature on collusive pricing has largely ignored the role of the interest rate, this paper suggests that both the level and the volatility of the interest rate are important determinants of collusive pricing. Thus, to be complete, future empirical work should consider these forces.

This paper also has implications for the study of aggregate fluctuations. I show that any change in policy, preferences or technology may have an impact on the aggregate level of activity through changes in collusive behavior, not only by affecting the real interest rate, but also by affecting its volatility.

Finally, it would be interesting to study extensions of this work to general repeated games. While I show here that volatility reduces the scope of cooperation in repeated oligopolies, I also show that that is not necessarily true for general repeated games. Determining conditions under which higher volatility reduces the set of equilibrium outcomes for general repeated games remains for future work.

7 Apendix

Proof of Lemma 10: It is straight forward to see that if $aB > \Phi(\pi^m)$ perfect collusion can be supported for any discount factor. If that condition does not hold, define

$$c(B) = \begin{cases} \underset{\delta\in[a,b]}{min} \{\delta \mid \Phi(\pi^m) \leq \delta B\} \; if \; \Phi(\pi^m) \leq bB \\ b \; if \; \Phi(\pi^m) > bB \end{cases}$$

If $aB \leq \Phi(\pi^m) \leq bB$ there exists a number $c(B)$ that makes $\Phi(\pi^m) = c(B)B$ by continuity of a linear function. If $bB < \Phi(\pi^m)$, $c(B) = b$. Therefore, $c(B)$ exists (and is continuous).

When $\delta > c(B)$, the incentive compatibility constraint is not binding since π^m could be supported with even a lower discount factor. On the opposite case, when $\delta < c(B)$, the incentive compatibility constraint is binding given that under Assumption 2, $\frac{d\Phi}{d\pi} = \frac{d\pi^d}{dq}\frac{dq}{d\pi} - 1 > 0$.

Since $\Phi(\pi)$ is increasing for $\delta \leq c(B)$, its inverse exists in the relevant rage and the incentive compatibility constraint can be written as a function of δB, $\pi(\delta) \leq \Phi_+^{-1}(\delta B)$, for $\delta \leq c(B)$. Since $\frac{d\Phi}{d\pi} > 0$, this constraint is increasing in the discount factor. Finally, given that $\Phi(\pi)$ is continuous $\Phi_+^{-1}(\delta B)$ is also continuous. ■

Proof of Lemma 11: Since $\Phi(\pi)$ is decreasing in $(-\infty, \pi^c]$, its inverse exists in that range and the incentive compatibility constraint can be written as a function of δB, $\pi(\delta) \geq \Phi_-^{-1}(\delta B)$, which is decreasing in the discount factor. Finally, given that $\Phi(\pi)$ is continuous $\Phi_-^{-1}(\delta B)$ is also continuous. ■

Proof of Lemma 15: From the stick and carrot property of the optimal punishment, Lemma 13, the continuation payoffs of both the highest and lowest equilibrium discounted payoff coincide and $\overline{V}(\delta) - \underline{V}(\delta) = \pi^*(\delta) - \Phi_-^{-1}\left(\delta\left[\frac{E\pi^*}{1-\delta} - E\underline{V}\right]\right)$. From the characterization of $\overline{V}(\delta)$ and equation (12) we know that the shape of $\pi^*(\delta)$ depends on the shape of the IC$^+$-F constraint. In the IC range the concavity of the constraint is determined by the sign of $\frac{d^2\Phi_+^{-1}}{d\delta^2}$. By Assumptions 2 and 3, $\frac{d^2\Phi}{d\pi^2} = \frac{d^2\pi^d}{dq^2}\left(\frac{dq}{d\pi}\right)^2 + \frac{d\pi^d}{dq}\frac{d^2q}{d\pi^2} \geq 0$, and by Lemma 11, $\frac{d\Phi_+^{-1}}{d\delta} > 0$, then $\frac{d^2\Phi_+^{-1}}{d\delta^2} = \frac{-B^2}{\left(\frac{d\Phi}{d\pi}\right)^2}\frac{d^2\Phi}{d\pi^2}\frac{d\Phi_+^{-1}}{d\delta} \leq 0$ and Φ_+^{-1} is concave. The F range of the constraint is also concave, since it is a constant. Hence, given that the IC$^+$-F constraint is increasing and continuous, the IC$^+$-F constraint is increasing and concave and so is $\pi^*(\delta)$. From equation 13 we know that $\underline{V}(\delta)$ is convex if $\Phi_-^{-1}(.)$ is also convex. By Assumptions 2 and 3, $\frac{d^2\Phi}{d\pi^2} = \frac{d^2\pi^d}{dq^2}\left(\frac{dq}{d\pi}\right)^2 + \frac{d\pi^d}{dq}\frac{d^2q}{d\pi^2} \geq 0$, and by Lemma 12, $\frac{d\Phi_-^{-1}}{d\delta} < 0$, then $\frac{d^2\Phi_-^{-1}}{d\delta^2} = \frac{-B^2}{\left(\frac{d\Phi}{d\pi}\right)^2}\frac{d^2\Phi}{d\pi^2}\frac{d\Phi_-^{-1}}{d\delta} \geq 0$ and Φ_-^{-1} is decreasing and convex. Therefore, $\pi^*(\delta) - \Phi_-^{-1}\left(\delta\left[\frac{E\pi^*}{1-\delta} - E\underline{V}\right]\right)$ is increasing and concave on δ. ■

Optimal collusion and linear programming: Consider the case in which the discount factor takes in every period one of L values: $\delta_1, \delta_2, .., \delta_l, ..., \delta_L$. Denote as T the transition matrix, where t_{ls} denotes the probability that the future discount factor is δ_s given that today's is δ_l. Let V be the column vector of discounted continuation payoffs and Π the column vector of profits given the discount factor. Define $\widehat{T}$ as the matrix for which $\widehat{t}_{ls} = \delta_l t_{ls}$. Then, $V = \Pi + \widehat{T}V$, the incentive compatibility constraint is $(N-1)\Pi \leq \widehat{T}V$ and the feasibility constraint is $\Pi \leq \pi^m 1_L$, where 1_L is a column vector of ones. Then, the optimal tacit collusion profits result from the following problem:

$\max \alpha\Pi$ subject to: $\left[(N-1)\, I - \widehat{T}\left(I - \widehat{T}\right)^{-1}\right]\Pi \leq 0_L$ and $\Pi \leq \pi^m 1_L$, where α is any non-negative row vector.

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