Chapter 1

Introduction

1.1 Some Apparently Simple Questions

Consider the constant elasticity demand function

$$q = p^{-0.2}.$$

This is a function because for each price p there is an unique quantity demanded q. Given a hand-held calculator, any economist could easily compute the quantity demanded at any price.

An economist would also have little difficulty computing the price that clears the market of a given quantity, say, 2 units. Flipping the demand expression about the equality sign and raising each side to the power of -5, the economist would derive a closed-form expression for the inverse demand function

$$p = q^{-5}.$$

Using the calculator the economist would quickly compute a market clearing price of 0.031.

Suppose now that the economist is presented with a slightly different demand function

$$q = 0.5 \cdot p^{-0.2} + 0.5 \cdot p^{-0.5},$$

one that is the sum a domestic demand term and an export demand term. Using standard calculus, the economist could easily verify that the demand function is continuous, differentiable, and strictly decreasing. The economist once again could compute the quantity demanded at any price using a calculator and could easily and accurately draw an graph of the function.

Suppose that the economist again is asked to find the price that clears the market of a quantity of 2 units. The question is well-posed. A casual inspection of the graph of the demand function suggests that its inverse is well-defined, continuous, and strictly decreasing. A formal argument based on the Intermediate Value and Implicit Function Theorems would prove that this is so. An unique market clearing price clearly exists.

But what is the inverse demand function? And what price clears the market? After considerable effort, even the best trained economist will not find an answer using algebra and calculus. No apparent closed-form expression for the inverse demand function exists. The economist cannot answer the apparently simple question of what the market clearing price will be.

Consider now a simple model of an agricultural commodity market. In this market, acreage supply decisions are made before the per-acre yield and harvest price are known. Planting decisions are based on the price expected at harvest:

$$a = 0.5 + 0.5 \cdot Ep.$$

After the acreage is planted, a random yield y is realized, giving rise to a supply

$$q = a \cdot y.$$

The supply is entirely sold at a market clearing price

$$p = 3 - 2q.$$

Yield is exogenous and distributed normally with a mean of 1 and standard deviation of 0.1.

Most economists would have little difficulty deriving the rational expectations equilibrium of this market model. Substituting the first expression into the second, and then the second into the third, the economist would write

$$p = 3 - 2(0.5 + 0.5 \cdot Ep) \cdot y.$$

Taking expectations on both sides

$$Ep = 3 - 2(0.5 + 0.5 \cdot Ep),$$

she would solve for the equilibrium expected price $Ep = 1$. She would conclude that the equilibrium acreage is $a = 1$ and the equilibrium price distribution has a standard deviation of 0.2.

Suppose now that the economist is asked to assess the implications of a proposed government price support program. Under this program, the government guarantees each producer a minimum price, say 1. If the market price falls below this level, the government simply pays the producer the difference per unit produced. The producer thus receives an effective price of $\max(p, 1)$ where p is the prevailing market price. The government program transforms the acreage supply relation to

$$a = 0.5 + 0.5 \cdot E \max(p, 1).$$

Before proceeding with a formal mathematical analysis, the economist exercises a little economic intuition. The government support, she reasons, will stimulate acreage supply, raising acreage planted. This will shift the equilibrium price distribution to the left, reducing the expected market price below 1. Price would still occasionally rise above 1, however, implying that the expected producer price will exceed 1. The difference between the expected effective producer price and the expected price represents a positive expcted government subsidy.

The economist now attempts to formally solve for the rational expectations equilibrium of the revised market model. She performs the same substitutions as before and writes

$$p = 3 - 2(0.5 + 0.5 \cdot E \max(p, 1)) \cdot y.$$

As before, she takes expectations on both sides

$$Ep = 3 - 2(0.5 + 0.5 \cdot E \max(p, 1)).$$

The economist then solves the expression for the expected price by interchanging the max and E operators, replacing $E \max(p, 1)$ with $\max(Ep, 1)$. The resulting expression is easily solved for $Ep = 1$. This solution, however, asserts the expected price and acreage planted remain unchanged by the introduction of the government price support policy. This is inconsistent with the economist's intuition.

The economist quickly realizes her error. The expectation operator cannot be interchanged with the maximization operator because the latter is a nonlinear function. But if this operation is not valid, then what algebra or calculus operations would allow the economist to solve for the equilibrium expected price and acreage?

Again, after considerable effort, our economist is unable to find an answer using algebra and calculus. No apparent closed-form solution exists for the model. The economist cannot answer the apparently simple question of what the equilibrium acreage and expected price will be after the introduction of the government program.

1.2 An Alternative Analytic Framework

The two problems discussed in the preceding section illustrate how even simple economic models cannot always be solved using standard mathematical techniques. These problems, however, can easily be solved to a high degree of accuracy using numerical methods.

Consider the inverse demand problem. An economist who knows some elementary numerical methods and who can write basic Matlab code would have little difficulty solving the problem. The economist would simply write the following elementary Matlab program:

```
change = inf;
while abs(change) > 1.e-8;
  change = (.5*p^-.2+.5*p^-.5-2)/(.1*p^-1.2 + .25*p^-1.5);
  p = p + change;
end;
display (p);
```

He would then execute the program on a computer and, in an instant, compute the solution: the market clearing price is 0.154. The economist has used Newton's rootfinding method.

Consider now the rational expectations commodity market model with government intervention. The source of difficulty in solving this problem is obvious: the need to evaluate the truncated expectation of a continuous distribution.

An economist who knows some numerical analysis and who knows how to write basic Matlab code, however, would have little difficulty computing the rational expectation equilibrium of this model. The economist would replace the original lognormal yield distribution with a discrete distribution that has identical lower moments, say one that assumes values $y_1, y_2, \ldots, y_n$ with probabilities $w_1, w_2, \ldots, w_n$. After constructing the discrete distribution approximant, which would require only a single call to a library routine, the economist would code and execute the following elementary Matlab program:

```
a = 1; aold = inf;
while abs(a-aold)>1.e-8;
  aold = a;
  p = 3 - 2*a*y;
  a = 0.5 + 0.5*w'*max(p,1)
end
```

In an instant, the program would compute and display the rational expectations equilibrium acreage, 1.03, expected market price, 0.94, expected producer price, 1.06, the standard deviation of market price, 0.21, and the standard deviation of the producer price, 0.10. The economist has combined Gaussian quadrature techniques and function iteration methods to solve the problem.

1.3 Computational Economic Dynamics

Dynamic models arise in many areas of applied economics, including:

- Management - capital asset replacement, production-inventory management, consumption-investment analysis
- Finance - dynamic hedging, options and futures pricing, informal rural finance
- Market Analysis - commodity pricing and storage, agricultural policy and trade, dynamic oligopoly games
- Environmental Policy - renewable and nonrenewable resource management, optimal environmental regulation
- Development - dynamic general equilibrium models, optimal economic growth, longrun sustainability

Economists have traditionally preferred to analyze dynamic models using one of two techniques: calculus of variations and optimal control. Calculus of variations, originally developed by Euler and Lagrange in the 18th and 19th centuries, is based on generalizations of unconstrained optimization by calculus. Optimal control, originally developed by Pontryagin in the late 1950s, is based on The Maximum Principle, a continuous-time generalization of the Karush-Kuhn-Tucker Theorem.

The calculus of variations and optimal control are based on mathematically elegant theories. However, in applied Economic analysis, they have severe limitations. Because most dynamic economic models do not possess closed-form solutions, the methods, in most practical instances, can be used only to partially characterize a model's solution, not to fully derive it. The calculus of variations and optimal control are incapable of handling many of the structural complexities commonly encountered in applied economic work. The methods, moreover, are mainly designed for deterministic problems, and do not easily generalize to stochastic ones.

Due to the limitations of mathematical methods of analysis, economists have been turning increasingly to numerical methods to solve and simulate dynamic economic models. The most common numerical solution strategies have been based on the recursive dynamic programming principles first articulated by Bellman in the late 1950s. Numerical dynamic programming methods often generate only approximate solutions, but the approximations can often be made arbitrarily precise by increasing the computational effort. The biggest advantage of numerical dynamic programming is that it can handle many of the structural complexities that arise in real economic systems. Stochastic dynamic programming, moreover, is a natural generalization of deterministic dynamic programming.

Economists, particularly agricultural, financial, and managerial economics have been using numerical methods to solve and analyze certain classes of models for over fifty years. Linear and quadratic programming, for example, have been used to solve managerial models and partial equilibrium models since the late 1940s. Matrix numerical techniques have also been developed and used extensively to perform least squares estimation and its variants. Over the last twenty years, nonlinear programing has also been used increasingly by economists to perform maximum likelihood estimation and to solve general equilibrium models. Highly reliable commercial computer packages exist to perform all of these specialized functions, and economists have not required an in-depth understanding of numerical algorithms in order to use them in their work.

However, stochastic dynamic economic models vary greatly in structure and general purpose dynamic programming software packages have yet to become practicable. Economists wishing to solve stochastic dynamic models numerically must master a variety of basic numerical techniques. In addition to conventional mathematical programming techniques, an Economic analyst must acquire a working knowledge of basic linear equation, nonlinear equation, function approximation, and numerical integration techniques. The analyst must also master certain advanced numerical functional equation techniques, particularly those that are used to solve differential and integral equations. These techniques are rarely taught in undergraduate and graduate economics programs, perhaps due to the lack of books on numerical techniques aimed at economists. We hope that this book can help fill the void.