An Alternative Strategy for Estimation of A Nonlinear Model of the Term Structure of Interest Rates

by

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Abstract

This paper develops and tests a nonlinear general equilibrium model of the term structure of interest rates based on the framework of Cox, Ingersoll and Ross (CIR, 1985). The contributions of this paper to the literature are both theoretical and empirical. The theoretical advantages of the general equilibrium model developed in this paper over the CIR model are (a) the risk premium is endogenously derived as a nonlinear function of the instantaneous interest rate. (b) The nonlinear model shows that the term premium need not be strictly increasing in maturity as in CIR's model; it can be either increasing or humped, a result that is consistent with recent findings by Fama (1984) and McCulloch (1987). (c) Yields of different maturities are not perfectly correlated, but exhibit positive correlations. A partial differential equation for valuing the discount bond price is presented, and a closed-form expression is derived. The term structure of interest rates derived from this nonlinear model may be increasing, decreasing, humped or inverted, depending on parameter values.

In an empirical application of the model, we develop a strategy for estimation which permits analysis of the model's temporal stability. Our model–like that of CIR–expresses the underlying stochastic process as a highly nonlinear function of two fundamental, time-invariant parameters. Many researchers have found that general equilibrium models such as CIR's provide quite poor explanations of the evolution of the term structure of interest rates. As an alternative strategy to that of fitting the fundamental parameters, we employ nonlinear system estimation of the unrestricted reduced-form parameters with a moving-window strategy in order to capture the term structure volatility caused by factors other than the instantaneous interest rate. We purposefully do not impose any law of motion on the estimated volatilities. This methodology is shown to have strong predictive power for the observed term structure of interest rates, both in-sample and out-of-sample.

I. Introduction

The study of the term structure of interest rates has been of interest to economists and financial researchers for more than a decade. Modeling the movements of the term structure of interest rates in a world of uncertainty has become a primary concern because of its importance in the pricing of interest rate contingent claims and interest rate derivative securities. Although there is a sizable literature on term structure modeling, many existing models possess shortcomings that limit their ability to reflect the interrrelationship between financial markets and the macroeconomy. This paper presents an alternative nonlinear model of the term structure of interest rates which overcomes some of these limitations while maintaining consistency within a general equilibrium framework.

The model presented in this paper is developed within the well-known Cox, Ingersoll, Ross (CIR, 1985b) general equilibrium framework, in which negative interest rates are precluded, the solution is internally consistent with the underlying economy and rules out arbitrage opportunities. We allow the one state variable model to follow a stochastic process with constant drift and variance, and permit production returns to be nonlinearly related to technological change.

The main contributions of this paper are, first, that within the CIR general equilibrium framework, we derive the factor risk premium as a nonlinear function of the instantaneous interest rate r. This nonlinearity reflects how uncertainty in technological change in the given stochastic process affects interest rates and the risk premium. The closed form solution of our nonlinear general equilibrium model of the term structure of interest rates has several analytical improvements over Longstaff's (1989) model, and gives us realistic shapes of yield curves. Second, we recognise that models of the term structure of interest rates are theories of predicting the term structures for given parameters and stochastic processes, and the resulting theoretical term structures usually fail to match the actual term structures. A plausible explanation for this failure is time variation in the model's parameters, as Hull and White (1990) suggested. Thus, the primary focus of our empirical application is the estimation of time-varying parameters via a "moving window" nonlinear system strategy which places no constraints on the motion of parameters nor on their variance-covariance matrix. We find that these unrestricted reduced-form parameters are generally sensible, vis-a-vis their "fundamental" counterparts. The flexibility implied by the unconstrained moving-window approach allows us to illustrate the time-varying ability of the model to fit the observed term structure. The model's fit varies meaningfully over the postwar era, while the in-sample error variance appears to be correlated with common macroeconomic factors.

This paper is organized as follows. In Section II, we highlight the key assumptions of the CIR (1985b) and Longstaff (1989) models, and discuss some of the relevant results of their papers. Our version of the single state variable nonlinear model is derived in Section III, and a closed form solution is obtained for constant parameters. In Section IV, we discuss the properties of the resulting yield curves and term premiums. Section V presents our empirical application to interwar and postwar U.S. term structure data, while Section VI summarizes our results.

II. Review of the Literature

Economists have tried a number of different techniques to model the term structure of interest rates on discount bonds. Vasicek (1977) derived a general form of a partial equilibrium one-factor model of the term structure of interest rates, in which the instantaneous interest rate, r, is the only state variable, and follows a mean-reverting process of the form

$$dr = \kappa(\theta - r)dt + \sigma dz \qquad (1)$$

Arbitrage arguments are used to derive a partial differential equation which all default-free discount bond prices must satisfy in equilibrium. In economic terms, the excess expected return on the default free discount bond with maturity T must equal the risk premium of the same security. Extensions alone the same line were made by Dothan (1978), Richard (1978), Brennan and Schwartz (1979) and many others.

Richard (1978) uses a Black-Scholes type of arbitrage argument that assumes that a riskless portfolio can be formed using three default-free discount bonds with distinct maturities. That portfolio can be treated as a perfect substitute for any default free discount bond of other maturities.

Brennan and Schwartz (1979) developed a two-factor arbitrage model of the term structure of interest rates. They assume that at any point in time, the term structure can be written as a function of time and the yields on the default free discount bonds with the shortest and longest maturities. These two yields follow a joint Markov process in continuous time, as in (1). They estimate the joint stochastic process for the two yields and evaluate the predictive ability of the model to price a sample of Canadian government bonds. They find that the root mean square prediction error for bond prices is on the order of 1.5%.

These arbitrage models of the term structure of interest rates involve three major problems. First, the stochastic process governing the instantaneous interest rate fails to preclude negative values of r. Second, the principal partial differential equation which all default free discount bond prices must satisfy involves an excess expected return of a discount bond with maturity T. This excess expected return of the bond can take many unknown forms depending on the underlying real economic variables. As Cox, Ingersoll and Ross (CIR) (1985) described, these arbitrage models provide no way to guarantee that every choice of this unknown functional form will not violate the internal consistency of the real variables of the underlying economy. Third, there is also no way to ensure that every choice of this unknown functional form will result in the absence of arbitrage opportunities in the bond pricing model. As CIR show, to eliminate arbitrage opportunities, the excess expected return on a discount bond with maturity T must be expressed in the form □(r,t) $P_r(r,t,T)$, where □ is the unknown form and P_r is the partial derivative of the pricing function. This puts restrictions on the form of the excess expected return of a discount bond. CIR provide an example that some functional forms of □ are inherently inconsistent with the real variables of the underlying economy, and could result in arbitrage opportunities.[1]

Cox, Ingersoll and Ross (CIR) (1985a,b) developed a general equilibrium approach of the term structure of interest rates in a continuous-time, one state variable and linear production economy. The single factor Y in the CIR model is the state of

[1] The example in CIR (1985) is □(r,t) = □$_0$ + □r , where □ is the market price of interest rate risk.

technology. Interest rates are determined endogenously as a function of the state variable and their evolution can be expressed by the stochastic process

$$dr = \square(\square \; \square r)dt + \square\sqrt{r}\, dz(t) \tag{2}$$

where dz(t) follows a Wiener process.[2] This continuous time stochastic process precludes a negative interest rate. More importantly, based on assumptions about the economy, CIR derived a principal partial differential equation which all asset prices must satisfy. The solution of this partial differential equation automatically guarantees that the equilibrium discount bond pricing model will eliminate arbitrage opportunities and will also be consistent with the underlying economy.

Longstaff (1989) extended the CIR model through a nonlinear version of the term structure of interest rates. He introduced two forms of nonlinearity: first, he assumes that technological change affects production returns nonlinearly through a form of increasing return to scale; and second, he derives the instantaneous interest rate which can be expressed by the stochastic process

$$dr = \square(\square \; \square\sqrt{r})dt + \square\sqrt{r}\, dz(t) \tag{3}$$

where dz(t) follows a Wiener process.

Instead of using CIR's general equilibrium approach by internally deriving the form of excess expected returns on a default free discount bond, Longstaff assumes a linear form of the risk premium which is equal to that of the CIR model. This causes his model to suffer from the same drawbacks as the partial equilibrium model in lacking internal consistency and an absence of arbitrage opportunities. His bond pricing model also exhibits some counterfactual behavior. For example, his bond price is no longer a uniform decreasing function of the instantaneous interest rate, nor is it a convex function of that rate. Instead, bond price starts as an increasing and concave function of the instantaneous interest rate r when r is small, then become a decreasing and convex function of the instantaneous interest rate r when r exceeds a certain value. Empirically, Longstaff finds some support for his model versus CIR's in his estimates, which are constructed from 1-month through 12-month Treasury bond data of Fama (1984), but his model fails to predict or

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[2] ß□r(r) dz(t) ≡ □$\sqrt{Y}$ d□(t), where Y is the state variable and □(t) is a Wiener process.

approximate actual yield curves. As he noted (p. 217) "neither model completely captures the level and variation of Treasury bill yields during the study period."

This paper derives an alternative nonlinear bond pricing model within the CIR general equilibrium framework in which negative interest rates are precluded, the solution is internally consistent with the underlying economy and no arbitrage opportunities exist. This model allows technological change to follow a stochastic process with constant drift and variance, and permits production returns to be nonlinearly related to that process with increasing returns to scales, as suggested by Longstaff. In our model, the factor risk premium is <u>derived</u> as a nonlinear function of the instantaneous interest rate r, rather than relying on linear function that Longstaff assumes for his model. This nonlinearity reflects how uncertainty in technological change in the given stochastic process affects interest rates and the risk premium in a general equilibrium framework. The discount bond return over the risk free return is derived as a partial differential equation, and a closed form solution for discount bond price is provided. The result shows that the nonlinear general equilibrium model of the term structure of interest rates from this paper has several analytical improvements over other one-factor models in providing realistic shapes of the yield curve and term premium.

III. The Model

Cox, Ingersoll and Ross (1985b) develop an intertemporal general equilibrium model for the term structure of interest rates. It can be summarized as follows:

C1: There is a single good produced in the economy, which may be allocated to either consumption and investment. All values are measured in terms of units of this good.

C2: Homogeneous individuals maximize the expected logarithmic Von Neumann-Morgenstern utility function by choosing optimal consumption/ investment plans.

C3: Technological change, including production and investment opportunities, is expressed by a single state variable, which follows a diffusion process. In order to derive a closed form solution of the term structure of interest rates, CIR assume that this single state variable Y follows a stochastic process:

$$dY(t) = \left[\square Y + \square\right]dt + \square\sqrt{Y}\,dz(t) \tag{4}$$

where □, □ and □ are constants, and z(t) is a Wiener process.

C4: The production process is governed by

$$dX(t) = F(Y,t)dt + G(Y,t)dz(t) \tag{5}$$

Here CIR assume that the means and variances of the production rate of return are proportional to Y, so F(Y,t) = □Y and [G(Y,t)]'[G(Y,t)] = □Y, where □ and □ are constants. So the production process is simply

$$dX = \square Y dt + \sqrt{\square Y}\, dz \tag{6}$$

Then the equilibrium interest rate in CIR framework is r(Y,t) = cY, with c a constant. By Ito's lemma,

$$dr = \square(\square \square r)dt + \square\sqrt{r}\, dz \tag{7}$$

In this process, interest rates move elastically toward the long run value μ at adjustment speed □. The factor risk premium is the premium due to the single state variable Y, or the risk premium investors require for holding the risky security. In equilibrium, the factor risk premium should equal the excess expected return on that security over and above the risk free return, where the instantaneous risk-free rate is r. In CIR's model, the risky security is the pure discount bond with price P, so that the difference between the expected return on discount bond and rP is the risk premium.

Under assumptions C1 to C4, it can be shown that the factor risk premium in CIR (1985b) is a linear function of Y, c□P_rY, so it is a linear function of r, □rP_r. In equilibrium, CIR show that a discount bond price P(r,□) must satisfy the partial differential equation:

$$\square P_{\square} + P_r[\square\square \square (\square + \square)r] + \frac{1}{2}P_{rr}(\square^2 r) \square rP = 0 \tag{8}$$

where □ is the negative coefficient of the linear factor risk premium □rP_r, or the price of risk.

To derive the nonlinear model of the term structure of interest rates and obtain a closed form solution, Longstaff (1989) works within the CIR framework but assumes that the single state variable follows a process with random walk behavior with constant drift and constant variance:

$$dY = a\,dt + b\,dz(t) \tag{9}$$

Longstaff introduces nonlinearity through two different channels. First he assumes that the state variable Y affects production return nonlinearly. More specifically, the mean and variance of the production rate of return are proportional to Y^2, or

$$dX(t) = \square Y^2 dt + \square Y\,dz(t) \tag{10}$$

The instantaneous interest rate is then $r(Y,t) = cY^2$, a nonlinear relationship. This extension describes a different set of technologies than those in the linear case. It also induces mean reversion in the equilibrium interest rate, as discussed by Sundaresan (1984). But Longstaff does not derive the factor risk premium internally from the CIR framework; he merely assumes that the factor risk premium is linear in r, and equal to $\square r P_r$, the linear form chosen by CIR. This causes his model to suffer from the same drawbacks as partial equilibrium models. His model also exhibits some counterfactual behavior: e.g. the bond price is no longer uniformly decreasing in the instantaneous interest rate, nor is it a convex function of that rate.

In equilibrium the partial differential equation which a pure discount bond price $P(r,\square)$ must satisfy in Longstaff's model is

$$\square P_{\square} + P_r\left[\square\square\ \square\square\sqrt{r} + \square r\right] + \frac{1}{2} P_{rr}\square^2 r\ \square rP = 0 \tag{11}$$

This partial differential equation for $P(r,\square)$ differs from CIR's partial differential equation for $P(r,\square)$ in terms of $\square\sqrt{r}$. This is the result of assuming that the production rate of return exhibits increasing returns to scale with respect to technological change.

Longstaff introduces nonlinearity into the discount bond price formula by taking a trial solution

$$P(r,\square) = A(\square)\exp\left[B(\square)\,r + C(\square)\sqrt{r}\right] \tag{12}$$

The closed-form solution can be obtained by substituting equation (12) into equation (11). The nonlinear term $C\sqrt{r}$ is confirmed to be nonzero and it is the additional term added to the expression of the discount bond price in the CIR paper. This additional term links the nonlinear term $\sqrt{r}$ from the partial differential equation to the bond pricing formula, therefore widening the relationship between the interest rate r and the bond price $P(r,\tau)$, and provides a nonlinear version of yield with respect to the interest rate change.

This paper derives an alternative nonlinear model within the CIR general equilibrium asset pricing framework (1985a) based on four modified assumptions. Some of Longstaff's assumptions are maintained, with the important change that the factor risk premium is internally derived from the CIR general equilibrium asset pricing model and is shown to be proportional to Y and thus also proportional to $\sqrt{r}$. The factor risk premium has an expression $\lambda\sqrt{r}\,P_r$, where λ is the covariance of the rate of change in interest with the rate of change in the optimal investment. It is a negative constant.[3] This nonlinear factor risk premium, derived from the system, broadens the classes of risk to be considered, guarantees the absence of arbitrage opportunities in bond pricing, and ensures internal consistency with the real variables of the economy.

[3] As discussed in the CIR model, λ is the covariance of changes in current interest rate with changes in optimal investment, or the market portfolio. So if λ increases, the absolute value $|\lambda|$ decreases, or the covariance of changes in interest rate with changes in the market portfolio decreases. This implies that the market price of interest rate risk is smaller, so the factor risk premium should decrease. In our model, this result is guaranteed because the factor risk premium is $\lambda\sqrt{r}\,P_r$, where both λ and P_r are negative. When r increases, there is greater interest rate risk, therefore a higher factor risk premium, but the premium increases at a decreasing rate.

The following table gives us a brief comparison of assumptions and results for the three models:

CIR Model	Longstaff Nonlinear Model	Alternative Nonlinear Model
Mean and variance of production are proportional to Y	Mean and variance of production are proportional to Y^2	Mean and variance of production are proportional to Y^2
$r=cY$, c is a constant	$r=cY^2$, c is constant	$r=cY^2$, c is constant
Factor Risk Premium: $c\square Y=\square r$	Factor Risk Premium: $c\square Y^2=\square r$	Factor Risk Premium: $\sqrt{c}\ \square Y=\square\sqrt{r}$
$P=A\exp(B\,r)$	$P=A\exp(B\,r+C\sqrt{r})$	$P=A\exp(B\,r+C\sqrt{r})$
$P_r<0$	P_r is ambiguous	$P_r<0$
$P_{rr}>0$	P_{rr} is ambiguous	$P_{rr}>0$
$P_{\square}<0$	$P_{\square}<0$	$P_{\square}<0$
Unobserved parameters: □, μ and □	Unobserved parameters: □ and μ	Unobserved parameters: □ and μ
P is a decreasing and convex function of mean interest rate, μ^2	Ambiguous	P is a decreasing and convex function of mean interest rate, μ^2
The expected rate of return on a bond is $r+\square Br$	The expected rate of return on a bond is $r+2\square Br+\square C\sqrt{r}$	The expected rate of return on a bond is $r+\square B\sqrt{r}+\frac{\square C}{2}$
Variance of returns on a bond is $(B\square)^2 r$	Variance of returns on a bond is $(B\sqrt{r}+\frac{C}{2})^2\square^2$	Variance of returns on a bond is $(B\sqrt{r}+\frac{C}{2})^2\square^2$
Term Premium (TP) is $\square Br$; TP=0 as $\square=0$ or $r=0$ TP>0 as □□	Term Premium (TP) is $2\square Br+\square C\sqrt{r}$; TP=0 as $\square=0$ or $r=0$ TP ambiguous as □□	Term Premium (TP) is $\square B\sqrt{r}+\frac{\square C}{2}$; TP=0 as $\square=0$ TP=$\frac{\square C}{2}$ as $r=0$ TP>0 as □□

The alternative nonlinear model rests on the following four assumptions:

A1: In a continuous time competitive economy, there is a single good produced which can be allocated to either consumption and investment. All values are measured in terms of units of this good.

A2: Homogeneous individuals maximize expected constant relative risk aversion (CRRA) utility functions. The utility function is independent of the state variable Y in term of choosing consumption and investment plans.

$$E\int_{t_1}^{t_2} U(C(s),Y(s),s)\,ds \tag{13}$$

where

$$\mathrm{U(C(s),Y(s),s)} = e^{-\rho(s-t)}\left[\frac{C(s)^{\gamma}-1}{\gamma}\right] \tag{14}$$

and γ is constant , t< s< T and (1-γ) is the coefficient of relative risk aversion. The indirect utility function has the form

$$J(W,Y,s) = f(Y,s)U(W,s) + g(Y,s) \tag{15}$$

The specified utility function (14) implies a convenient separability property for the indirect utility function which simplifies the solution of the consumption and investment problem.

A3: Y represents the state of the technology and is itself changing randomly over time. Y follows a stochastic process of the form

$$dY = a\,dt + b\,dz(t) \tag{16}$$ [4]

where a, b are constants and a<0. The mean and variance of dY, the change in the state of technology, are a and b^2, respectively.

[4] This is also called Reflected Brownian Motion: when Y reaches zero, the process returns immediately to positive values. This stochastic process has a long run stationary distribution $(-\frac{2a}{b^2})Exp(\frac{2aY}{b^2})$, Y>0.

A4: The production return process exhibits increasing returns to scale with respect to technological change. Specifically, it is expressed by a single state variable Y, and governed by the stochastic process in the form of

$$dX(t) = \square Y^2 dt + \square Y\, dz(t) \tag{17}$$

where $\square$, $\square$ are constants and $\square > 0$. The mean and variance of dX, the change in the rate of production return, are proportional to Y^2.

Based on these assumptions and the explicit forms of stochastic differential equations, this paper solves CIR's endogenous equilibrium interest rate of equation (3) in their paper (1985b) to a simple and explicit form:

$$r(Y,t) = a^{*'} F + a^{*'} GG' a^{*} W \square\frac{J_{WW}}{J_W}\square + a^{*'} GS' \square\frac{J_{WY}}{J_W}\square \tag{18}$$

where

$$F(Y,t) = \square Y^2$$

$$[G(Y,t)][G(Y,t)]' = \square^2 Y^2$$

$$\square W \square\frac{J_{WW}}{J_W}\square = 1 \square \square$$

$$[G(Y,t)][S(Y,t)]' = (\square Y) b$$

$$\square\square \frac{J_{WY}}{J_W}\square = \square\square \frac{f_Y}{f}\square$$

$$a^{*} = \left(\frac{1}{\square^2 Y^2}\right)(\square Y^2) + \left(1 \square \frac{\square Y^2}{\square^2 Y^2}\right) = 1$$

The right-hand side of equation (18) is the expected market rate of return minus the market return variance and covariance, as discussed by CIR. Further we assume the special case of a logarithmic utility function with $\square = 0$. It can then be shown that $f(Y,s) = \frac{1 \square e^{\square\square(s \square t)}}{\square}$, so that $f_Y = 0$. Then the equilibrium interest rate has the form:

$$r(Y) = a^{*} \square Y^2 + (a^{*})^2 \square^2 Y^2 = cY^2 \tag{19}$$

where $a^{*} = 1$, c is a positive constant and Y follows the stochastic process (15). As a result of Ito's lemma, the dynamics of the equilibrium interest rate from the general equilibrium framework follow the stochastic differential equation :

$$dr = \square\left(\square \square \sqrt{r}\right)dt + \square\sqrt{r}\, dz \tag{20}$$

where $\square = \square 2a\sqrt{c}$, $\square = \frac{\square b^2\sqrt{c}}{2a}$, $\square = 2b\sqrt{c}$, $\square\square = b^2 c$.

This characterization has the following properties:

(1) The instantaneous interest rate has a upward drift $\square(\mu\text{-}\sqrt{r}\ )$ when $\sqrt{r} < \mu$ and a downward drift when $\sqrt{r} > \mu$. It has a variance $\square^2 r$, which depends on r, and $\square>0$, so the variance increases as the instantaneous interest rate increases.

(2) At r=0, the variance is zero and drift is $\square\mu > 0$, so that negative interest rates are precluded.

(3) The boundary study of the stochastic process of the interest rate based on Feller (1951), and Karlin and Taylor (1981) by Longstaff shows that if the initial interest rate is nonnegative, then subsequent interest rates from the process will be nonnegative.

(4) Even if the interest rate reaches zero, since $\mu>0$, the interest rate will subsequently become positive.

(5) The interest rate dynamics represented above are mean reverting towards μ^2 at the speed of adjustment □.

(6) The interest rate movement can be described by only two parameters □ and $\square^2$, since $\mu = \frac{\square^2}{4\square}$, a function of □ and $\square^2$.

Let the equilibrium price of a pure discount bond at time t be P(□,r), where □ = (T-t). It pays one dollar at maturity T. P(□, r) follows the geometric Brownian Motion

$$\frac{dP}{P} = m(\square,r)\,dt + n(\square,r)\,dz(t) \qquad (21)$$

where

$$m(\square,r) = \frac{1}{P}\square P_{\square} + P_r\square\left(\square\square\sqrt{r}\right) + \frac{1}{2}P_{rr}\left(\square^2 r\right)\square$$

$$n(\square,r) = \frac{1}{P}\left[P_r\square\sqrt{r}\right]$$

The expected rate of return on this bond is

$$m(\square, r) = \frac{1}{P}\left[\square P_{\square} + P_r \square\left(\square \square \sqrt{r}\right) + \frac{1}{2} P_{rr}\left(\square^2 r\right)\right] \tag{22}$$

The excess expected return on this pure discount bond over the risk-free return equals

$$\square P_{\square} + P_r \square\left(\square \square \sqrt{r}\right) + \frac{1}{2} P_{rr}\left(\square^2 r\right) \square r P \tag{23}$$

In equilibrium, the excess expected return on a security over the risk free return rP must equal the risk premium that purchasers of the security demand:

$$\left[\square P_{\square} + P_r \square\left(\square \square \sqrt{r}\right) + \frac{1}{2} P_{rr}\left(\square^2 r\right) \square r P\right] = \square_y P_r \tag{24}$$

where $\square_y P_r$ is the general form for the risk premium.

In the CIR framework modified by our assumptions, the single factor risk premium has the form

$$\square_y = \left(\square \frac{J_{WW}}{J_W}\right) a^{*'} \left[G(Y,t)\right]\left[S(Y,t)\right]' W + \left(\square \frac{J_{WY}}{J_W}\right)\left[S(Y,t)\right]\left[S(Y,t)\right]' \tag{25}$$

$$\square W\left(\frac{J_{WW}}{J_W}\right) = 1 \square \square$$

$$\left[G(Y,t)\right]\left[S(Y,t)\right]' = \left(\square Y\right) b$$

where

$$\left(\square \frac{J_{WY}}{J_W}\right) = \left(\square \frac{f_y}{f}\right)$$

$$a^* = \left(\frac{1}{\square^2 Y^2}\right)\left(\square Y^2\right) + \left(1 \square \frac{\square Y^2}{\square^2 Y^2}\right) = 1$$

For the logarithmic utility function with $\square = 0$, equation (25) becomes

(26)

where $\lambda = \frac{b\phi}{\sqrt{c}}$ is a constant. This additional nonlinearity provides a more realistic class of risk premium.

In equilibrium the excess expected return on a security must be equal to the risk premium that purchasers of the security demand, so that

$$\left[-P_{\tau} + P_r \beta\left(\mu - \sqrt{r}\right) + \frac{1}{2} P_{rr}\left(\sigma^2 r\right) - r P\right] = \left(\lambda \sqrt{r}\right) P_r \quad (27)$$

Thus we have the stochastic differential equation of the term structure of interest rates:

$$-P_{\tau} + P_r\left[\kappa_0 - \kappa_1 \sqrt{r}\right] + \frac{1}{2} P_{rr}\left(\sigma^2 r\right) - r P = 0 \quad (28)$$

with boundary condition P(r,0) = 1, and where τ=T-t, $\kappa_0=\beta\mu$, $\kappa_1=\beta+\lambda$, $\mu=\frac{\sigma^2}{4\beta}$, $\beta=-2a\sqrt{c}$ and $\sigma^2=4b^2c>0$.

The equation above can be rewritten as

$$rP + P_r \lambda \sqrt{r} = -P_{\tau} + P_r \beta\left(\mu - \sqrt{r}\right) + \frac{1}{2} P_{rr}\left(\sigma^2 r\right) \quad (29)$$

The right-hand side is the expected return on the discount bond from Ito's lemma. Then the instantaneous rate of return can be written as $\frac{rP+P_r\lambda\sqrt{r}}{P} = r + \lambda\frac{P_r\sqrt{r}}{P}$ which is no longer proportional to the interest elasticity with respect to bond price as in the CIR model. The parameter λ is the covariance of rate of change in interest with the rate of change in optimal investment, or the market portfolio. CIR call λ the "market price of interest rate risk." Since $P_r < 0$ (which will be shown later), a negative covariance $\lambda < 0$ implies a positive risk premium $\lambda\frac{P_r\sqrt{r}}{P}$ > 0.

To derive a closed form solution, the following expression is hypothesized for the price of a pure discount bond:

$$P(\tau,r) = A(\tau)\exp\left[B(\tau)r + C(\tau)\sqrt{r}\right] \tag{30}$$

In contrast, the bond pricing formula for the CIR model is

$$P(\tau,r) = A(\tau)\exp\left[B(\tau)r\right] \tag{31}$$

The additional nonlinear term $C(\tau)\sqrt{r}$ in equation (30) shows how the nonlinearity in the term structure of interest rate movement affects the discount bond price, and ultimately the yield curve. To test whether the nonlinear term $C(\tau)\sqrt{r}$ belongs to the discount bond price formula, the partial derivatives of the expression (30) is calculated and substituted into the partial differential equation

$$-P_\tau + P_r\left[\alpha_0 \,\square\, \alpha_1\sqrt{r}\right] + \frac{1}{2}P_{rr}\left(\sigma^2 r\right) - r\,P = 0 \tag{32}$$

with boundary condition P(r,0) = 1. This test confirms that $A(\tau)$, $B(\tau)$ and $C(\tau)$ are nonzero. By using variable separation methods, we derived the following closed form solution:

$$P(\tau,r) = A(\tau)\exp\left[B(\tau)r + C(\tau)\sqrt{r}\right] \tag{33}$$ [5]

where

$$A(\tau) = \exp\left[\lambda_1\tau - \frac{1}{2}\ln\left(1 + e^{\sqrt{2}\sigma\tau}\right) - \lambda_2\left(\frac{1}{1 + e^{\sqrt{2}\sigma\tau}}\right) + \square\right] \tag{34}$$

$$B(\tau) = \frac{\sqrt{2}\left(1 - e^{\sqrt{2}\sigma\tau}\right)}{\sigma\left(1 + e^{\sqrt{2}\sigma\tau}\right)} \tag{35}$$

$$C(\tau) = \frac{2\alpha_1\left[1 - \exp\left(\frac{\sigma}{\sqrt{2}}\tau\right)\right]^2}{\sigma^2\left[1 + e^{\sqrt{2}\sigma\tau}\right]} \tag{36}$$

$$\lambda_1 = \frac{2\sqrt{2}\,\sigma\alpha_0 - \alpha_1^2}{2\sigma^2} \tag{37}$$

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[5] Derivations are given in a mathematical appendix available from the authors.

$$\lambda_2 = \frac{\sqrt{2}\,\square_1^2}{\square^3} \tag{38}$$

$$\tau = T - t \tag{39}$$

$$\square = \frac{1}{2}\ln(2) + \frac{\lambda_2}{2} \tag{40}$$

$$\square_0 = \square\square \quad \text{and} \quad \square_1 = \square + \square \tag{41}$$

P(τ, r) satisfies the boundary condition P(r, 0) =1 and is a function of r (the instantaneous interest rate) and τ (term to maturity), where A(τ)>0, B(τ)<0 and C(τ) can be either positive or negative.

There are two constraints for this alternative nonlinear model:

(1) $\ln A(\tau) + B(\tau)r + C(\tau)\sqrt{r} \le 0$. This constraint is a necessary and sufficient condition for our discount bond pricing model to ensure that the present value of \$1 paid at maturity satisfies $0<P(r,\tau)\le 1$. This is also a necessary and sufficient condition for a non-negative yield, since the yield curve is $Y(\tau) = -\frac{1}{\tau}(\ln(A) + Br + C\sqrt{r})$.

(2) To make our model consistent with economic reality, we further assume that $(B + \frac{C}{2\sqrt{r}}) \le 0$ and $0 < A \le 1$.

There are several important properties of this bond price function:

(1) P(τ,r) is a decreasing and convex function of the instantaneous interest rate r. This is consistent with the general solution for equilibrium bond pricing.

(2) The bond price is also a decreasing function of term to maturity τ. The longer the term to maturity, the lower the discount bond price, because it is the present value of \$1 which you receive at maturity.

(3) At the limit as r□ , or □□ , we have $\lim_{r□}$ P(r,□) = 0, or $\lim_{□□}$ P(□, r) = 0 respectively. These are realistic solutions since the present value of \$1 approaches zero if r□ or □□ .

(4) Since $\lim_{r□}$ B(□) = 0 and $\lim_{r□}$ C(□) = 0, $\lim_{r□}$ P(□r) = A(□), where A(□) is a function of term to maturity, which is greater than zero for all □. When r = 0, the systematic risk in the interest rate process $□\sqrt{r}$ become zero, the interest rate changes become certain, so that P(□0) = A(□), only a function of □. Then A(□) must be the interest-rate-risk-free rate of return on a bond for term to maturity □. Thus in equilibrium, the price for a discount bond is P(□, r) = A(□) exp [B(□) r + C(□)$\sqrt{r}$], which is positive, and will equal the interest-rate-risk-free rate of return on a bond for □ if the instantaneous interest rate r equals zero.

(5) The bond price is a decreasing and convex function of the long term mean interest rate μ. That is, $P_\mu < 0$ and $P_{\mu\mu} > 0$. As the long term mean interest rate increases, the bond price decreases, or vice versa as the bond price increases, the long term mean interest rate decreases. This result indicates that (i) the long term interest rate is drawn to a central value μ on average, and the effect of the long term mean interest rate on discount bond prices has the similar properties as the short term interest rate r on discount bond prices, that is $P_r < 0$ and $P_{rr} > 0$; (ii) bond prices have negative effects on the expected future interest rate, since as the price of a bond declines, the long term interest rate rises. This implies that the expected future interest rate should also rise. Bond prices can be both increasing and decreasing in the speed of adjustment coefficient □ and the 'market price of risk' □.

IV. The Nonlinear Yield Curve and Term Premium

The yield of a discount bond with price formula (33) can be expressed as

$$R(□) = -□^{-1} \ln \frac{dP}{P} = -□^{-1} (\ln A + Br + C\sqrt{r})$$

which is nonlinear in r. This is a direct application from the bond pricing formula with an arbitrary extra term exp[C(□)$\sqrt{r}$]. It reflects how the nonlinearity in the term structure of interest rates model (a function of $\sqrt{r}$) affects the discount bond yield. This nonlinearity of yield curves will provide a much broader range of shapes for theoretical and empirical studies.

From the model derived in Section III, the yield to maturity of a pure discount bond has the following closed form solution:

$$R(\tau, r) = -\frac{1}{\tau}\left[\ln A(\tau) + B(\tau) r + C(\tau)\sqrt{r}\right]$$

$$= \frac{1}{\tau}\left[l_1 \tau - l_2 \ln\left(1 + e^{-\sqrt{2}\sigma\tau}\right) - \frac{l_3}{\left(1 + e^{-\sqrt{2}\sigma\tau}\right)} + \right.$$

$$\left. \frac{\sqrt{2}\left(1 - e^{-\sqrt{2}\sigma\tau}\right)}{\sigma\left(1 + e^{-\sqrt{2}\sigma\tau}\right)} r + \frac{2\lambda_1}{\sigma^2} \frac{\left[1 - \exp\left(-\frac{\sigma\tau}{\sqrt{2}}\right)\right]^2}{\left(1 + e^{-\sqrt{2}\sigma\tau}\right)} \sqrt{r} + \lambda \right] \tag{42}$$

This is a nonlinear function of both the term to maturity τ and the instantaneous interest rate r. The square root nonlinearity in r will give us empirically realistic shapes, such as humped and inverted yield curves, for certain combinations of parameters.

<u>Properties of the Yield Curve</u>

(i) At the limit $\tau \to 0$, as t→ T, the yield to maturity converges to the instantaneous interest rate at maturity: $\lim_{\tau \to 0} R(\tau, r) = r_T$. As $\tau \to \infty$, the yield to maturity converges to a positive constant, independent of the current interest rate: $\lim_{\tau \to \infty} R(\tau, r) = \frac{2\sqrt{2}\sigma\lambda_0 + \lambda_1^2}{2\sigma^2}$ >0. Between these two limits, yield curves have many possible complex shapes. These results are consistent with the factors that yield is equal to the spot interest rate at maturity; that the yield curve has a flattened tail for bonds with long maturities, and that yield volatility is less for a long term bond than for a short term bond. The dependence of the model's discount yield on tenor and the instantaneous interest rate is illustrated in Figures 1 and 2.

(ii) As the short term interest rate r increases, the yield increases ($\frac{dR}{dr} > 0$) and the rate of change can be both positive or negative ($\frac{d^2R}{dr^2}$ can have either sign). So the yield can increase both at increasing or decreasing rate. If we graph the yield curve as a function of term to maturity τ, the yield curve is shifted upward as the instantaneous interest rate r increases. As r further increases, the upward shift could be either larger or smaller. At $\tau=0$, the upward shift of the yield curve is equal to the change in r, since $\lim_{\tau \to 0} \frac{dR}{dr} = 1$. As $\tau \to \infty$, the upward shift of the yield curve approaches zero, since $\lim_{\tau \to \infty} \frac{dR}{dr} = 0$. Also as the long term mean interest rate increases, the yield curve is shifted upward ($\frac{dR}{d(\mu^2)} > 0$) and the rate of change decreases ($\frac{d^2R}{(d\mu^2)^2} < 0$).

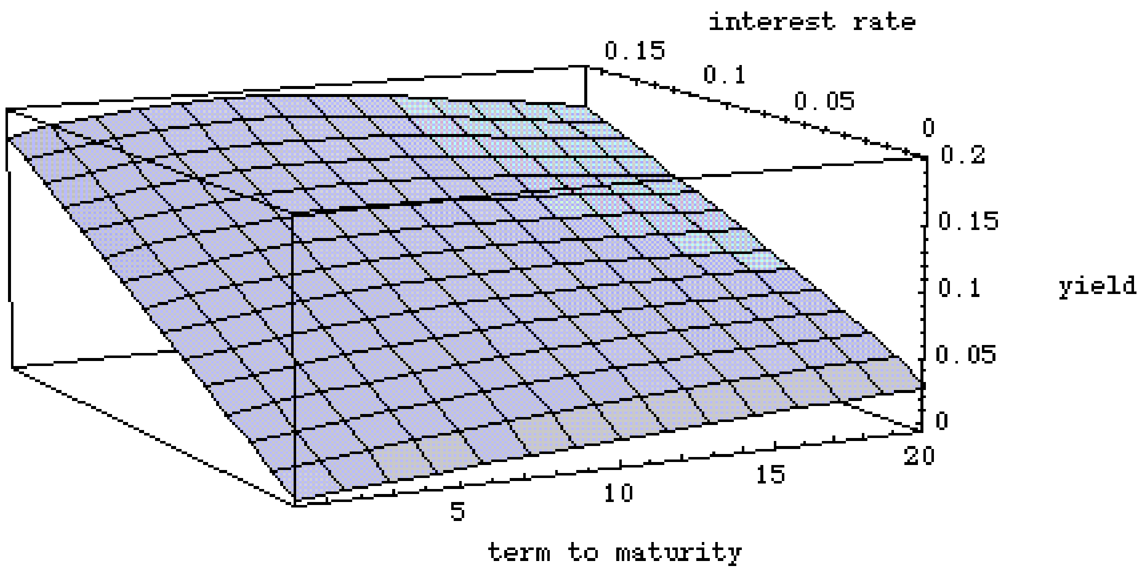

Fig. 1(a) represents the relation between discount bond yield, tenor, and instantaneous rate with $\square_0$ = 0.003073 and $\square_1$ = -0.01688.

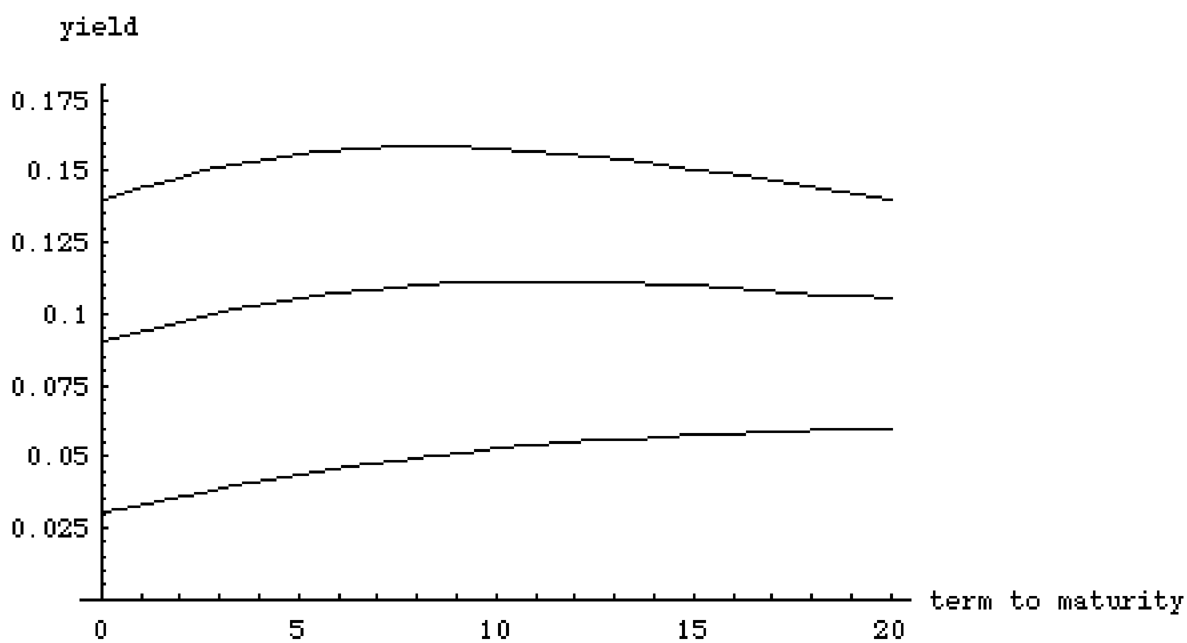

Fig. 1(b) represents discount bond yield with $\square_0$ = 0.003073 and $\square_1$ = -0.01688.

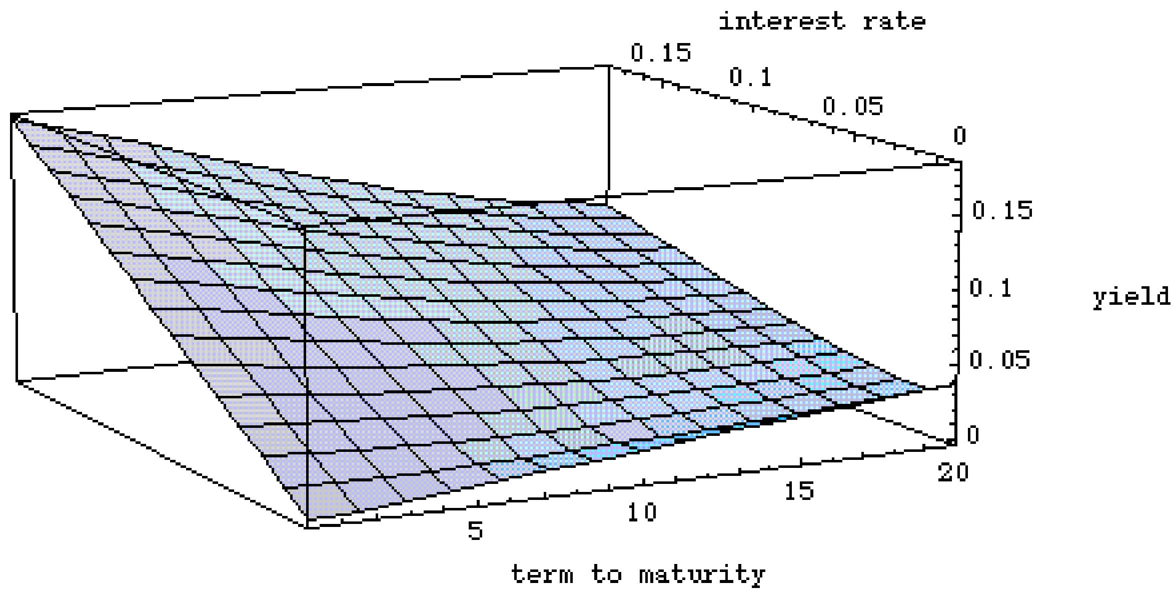

Fig. 2(a) represents the relation between discount bond yield, tenor, and instantaneous rate with $\square_0$ = 0.002699 and $\square_1$ = 0.03026.

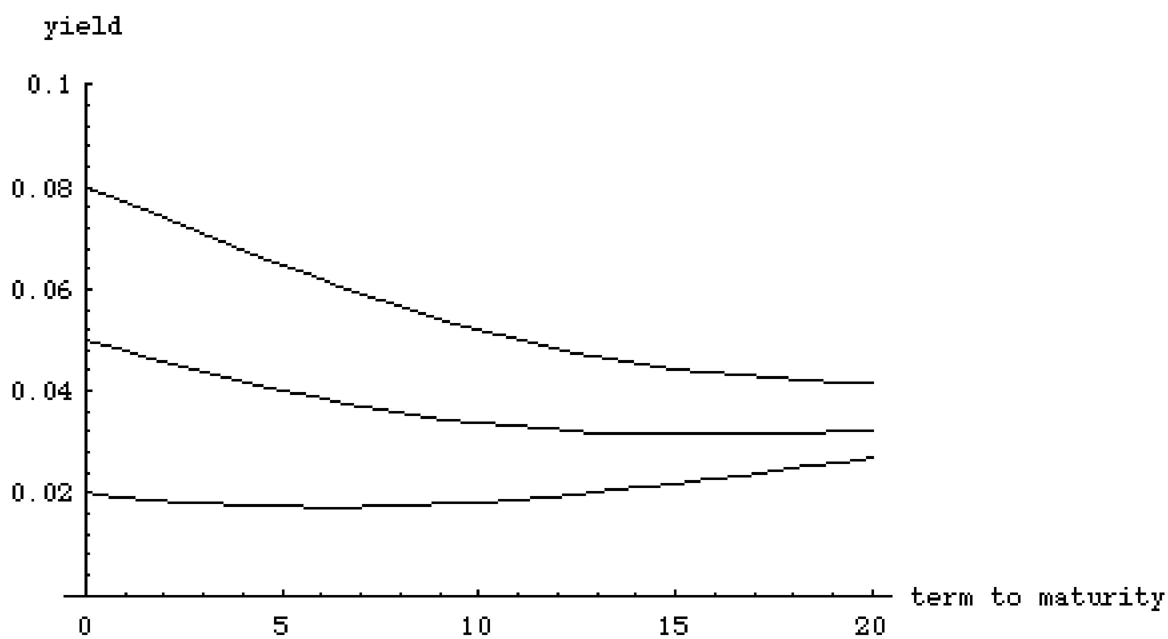

Fig. 2(b) represents discount bond yield with $\square_0$ = 0.002699 and $\square_1$ = 0.03026.

Properties of Bond Return

The dynamics of discount bond prices of our nonlinear model are governed by the following stochastic partial differential equation:

$$\frac{dP}{P} = \left[r + \lambda B(\tau)\sqrt{r} + \frac{\lambda}{2}C(\tau) \right] dt + \left[B(\tau)\sqrt{r} + \frac{C(\tau)}{2} \right] \sigma dz \qquad (43)$$

The instantaneous expected rate of return on a discount bond with maturity τ is:

$$E\left[\frac{dP}{P}\right] = r + \lambda B(\tau)\sqrt{r} + \frac{\lambda}{2}C(\tau) \qquad (44)$$

The variance of the discount bond with maturity τ is:

$$\nu = \left[B^2 r + BC\sqrt{r} + \frac{C^2}{4} \right] \sigma^2 \qquad (45)$$

Both are functions of r and τ. As r increases, the return variance increases, indicating a larger risk premium. As τ increases, the return variance can rise or fall, reflecting the possibility of both positive and negative term premium. In the limit, we have $\lim_{\tau \to 0} E(\frac{dP}{P}) = r$ and $\lim_{\tau \to 0} \nu = 0$, so the instantaneous expected rate of return is risk free as $\tau \to 0$. As the term to maturity approaches infinity, we have $\lim_{\tau \to \infty} E(\frac{dP}{P}) = (r - \frac{\sqrt{2}\square}{\square}\sqrt{r} + \frac{\square\square_1}{\square^2}) > 0$ and $\lim_{\tau \to \infty} \nu = (2r - \frac{2\sqrt{2}\square_1}{\square}\sqrt{r} + \frac{\square_1^2}{\square^2}) > 0$; both are positive constants. As the instantaneous interest rate approaches zero, we have $\lim_{r \to 0} E(\frac{dP}{P}) = \frac{\lambda}{2} C$ and $\lim_{r \to 0} \nu = \frac{C^2}{4}\sigma^2$. This is another important property from our nonlinear model which differs from both CIR and Longstaff's. As $r \to 0$, the variance in the interest rate process approaches zero, so interest rate changes become certain. But the expected rate of return and variance of returns on a bond are $\frac{\lambda C}{2}$ and $\frac{C^2}{4}\sigma^2$ respectively, not zero, because the rate of return and variance on a discount bond are also functions of term to maturity τ. If τ does not approach zero, or the bond is not near to maturity, then as we discussed earlier in property (4) of interest rate dynamics, there are still possibilities that interest rates will move away from zero and become positive again.

Properties of the Term Premium

The instantaneous term premium is the instantaneous expected return of the bond over the instantaneous return of the risk-free asset. It has the following properties:

(i) The instantaneous term premium of a discount bond with maturity □is TP(□) = $[□□B(□)\sqrt{r} + \frac{□}{2} C(□)] > 0$, which is a function of both r and □. The term premium converges to zero as □□ 0, reflects returns with no risk. As □□ , we have $\lim_{□□} TP = \frac{□}{□^2} [□1 - \sqrt{2r}\, □]$, a constant. As □increases, TP may either increase or decrease. The term premium is an increasing and concave function of r (with a limiting value $\frac{□}{2} C(□) > 0$ as r□ 0). Finally it is easy to show that the term premium is a decreasing function of □,[6] ($\frac{dTP}{d□} = b\sqrt{r} + \frac{C}{2} \leq 0$), a measure of the "market price of risk." As □ increases, TP decreases. But since □< 0, an increase in □ means a decrease in the "market price of risk" or a decrease in the covariance of changes in consumption with changes in the interest rate. So the risk is lower as □ increases, and therefore so is the term premium. The dependence of the term premium on the tenor and instantaneous interest rate is illustrated in Figures 3 and 4.

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[6] Since □ is the covariance of the rate of change in interest with the rate of change in optimally invested wealth, □<0. This implies that the term premium is greater than zero. As □ increases, |□| decreases. The covariance of the rate of change in the current interest with the rate of change in the optimal investment in the market portfolio is actually decreasing, causing the risk premium to decline.

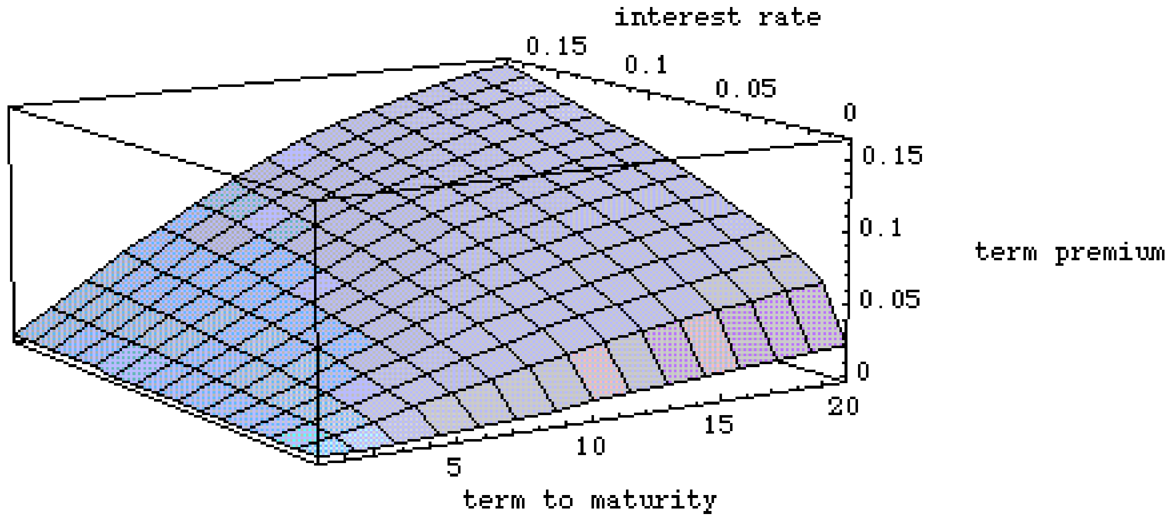

Fig. 3(a) represents the relationship between the term premium, tenor and instantaneous interest rate with $\square_0$=0.003073 and $\square_1$=-0.01688.

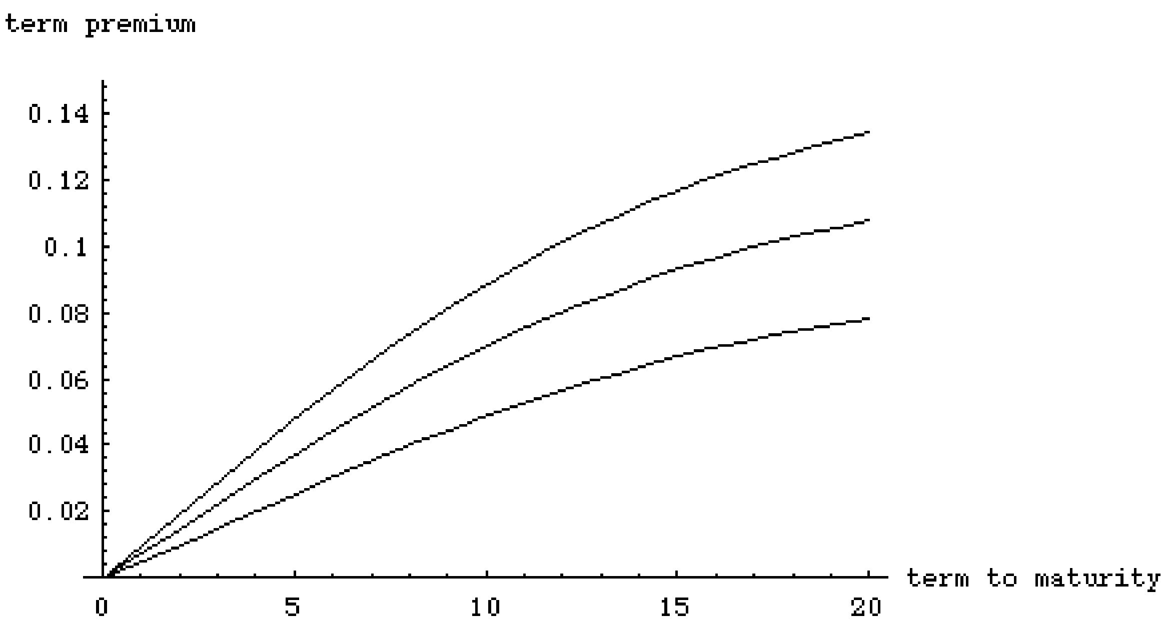

Fig. 3(b) represents term premia TP($\square$) with instantaneous interest rates r_1 = 5%, r_2 = 7% and r_3 = 12% respectively. Parameters are $\square_0$=0.003073 and $\square_1$=-0.01688.

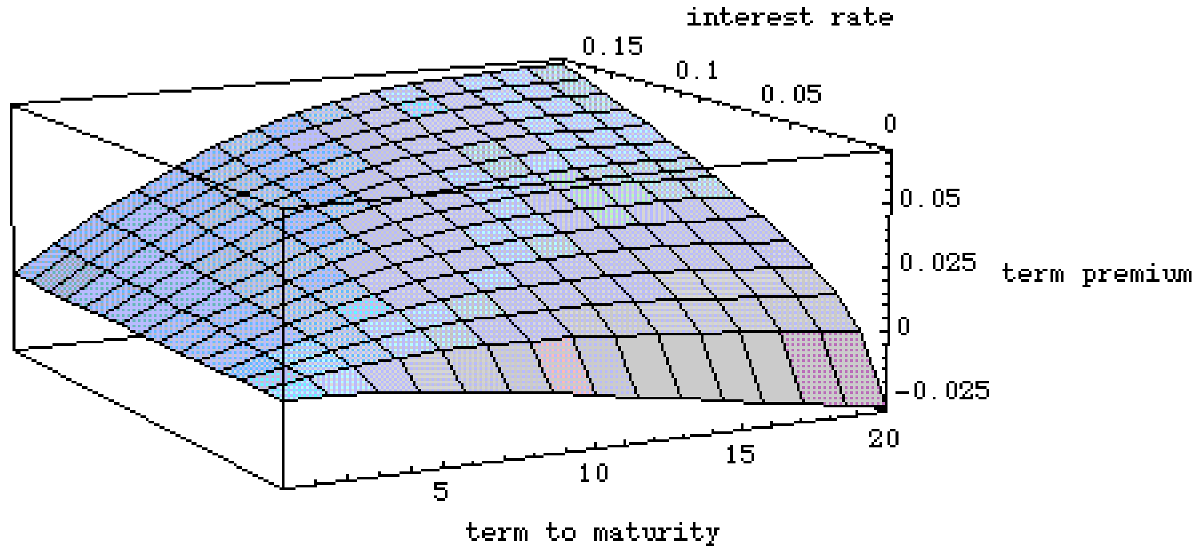

Fig. 4(a) illustrates the relationship between the term premium, tenor and instantaneous interest rates with $\square_0$ = 0.002699 and $\square_1$ = 0.03026.

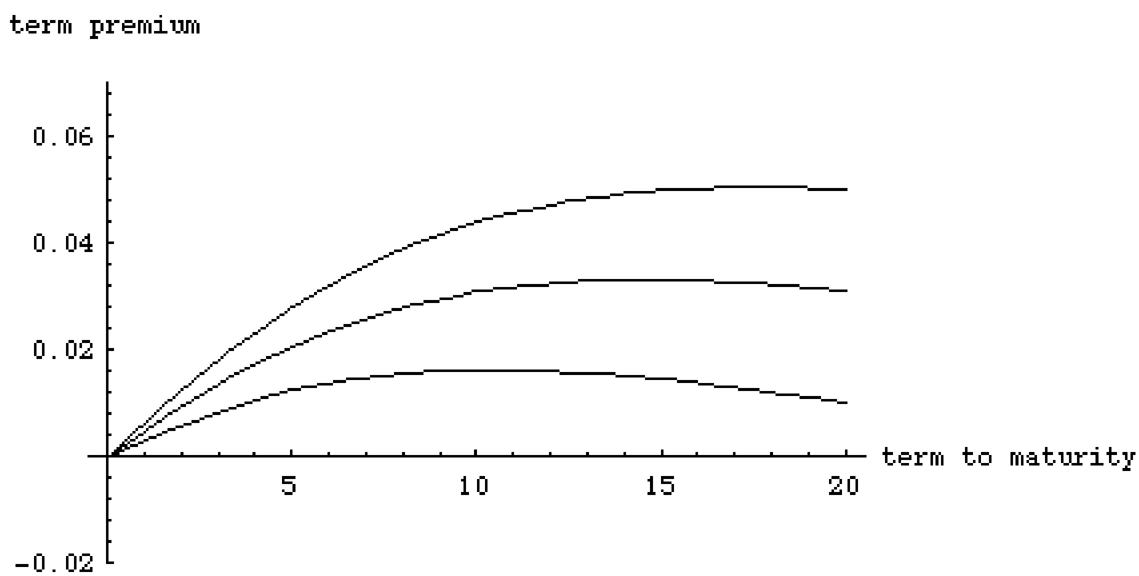

Fig. 4(b) represents term premia TP($\square$) with instantaneous interest rates $r_1 = 3\%$, $r_2 = 5\%$ and $r_3 = 10\%$ respectively. Parameters are $\square_0$=0.002699 and $\square_1$=0.03026.

(ii) In our model, if $0 < \sqrt{2r}\square < -\square_1$, $[B\sqrt{r} + \frac{C}{2}] = 0$ for term $\square = 0$ or $\square = \frac{\sqrt{2}}{\square}\ln[\frac{\square_1 + \sqrt{2r}\square}{\square_1 - \sqrt{2r}\square}]$, so that the variance of returns on a discount bond $\square = [B\sqrt{r} + \frac{C}{2}]^2\square^2$ is zero. But $0 < \frac{\square_1 + \sqrt{2r}\square}{f\square_1 - \sqrt{2r}\square} < 1$ for $0 < \sqrt{2r}\square < -\square_1$, so that $\ln[\frac{\square_1 + \sqrt{2r}\square}{\square_1 - \sqrt{2r}\square}] < 0$. Therefore the rate of return on a bond is certain and equal to r□only at $\square = 0$. That is:

$$\frac{dP}{P} = \left[r + \square B(\square)\sqrt{r} + \frac{\square}{2}C(\square)\right]dt + \left[B(\square)\sqrt{r} + \frac{C(\square)}{2}\right]\square\, dz \qquad (46)$$
$$= r\,dt$$

and the expected rate of return on an instantaneously maturing bond is certain, or $E(\frac{dP}{P}) = r\,dt$. Investors can form a risk-free portfolio only by repeatedly investing in a series of instantaneously maturing bonds.

V. Empirical Findings

The model to be empirically tested in this section is the equilibrium discount bond price equation (33):

$$P(\square, r) = A(\square)\exp\left[B(\square)r + C(\square)\sqrt{r}\right]$$

It is a highly nonlinear function of two variables, the instantaneous interest rate r and term to maturity, □, and two fundamental parameters: $\square_0$ and $\square_1$. Conceptually, equation (33) should hold for all tenors and values of r, enabling us to price any default-risk-free bond given consistent parameter estimates of the fundamental terms. In empirical testing of the CIR model, many researchers have found that the model does a relatively poor job of pricing, and that its ability to model actual bond prices varies over both tenor and time. We have found that our alternative model, expressed in terms of estimates of the fundamental parameters $\square_0$ and $\square_1$, possesses a similar weakness in terms of price forecasts. It might be expected that any single-factor model of the term structure–including our alternative model–would do a better job of pricing short-term bonds than longer-term bonds, given that the single fundamental factor is (a proxy for) the instantaneous interest rate. But the lack of temporal stability suggests that there may be other factors involved which cannot be captured by the single-factor framework. Some researchers have responded by building more complex models of the equilibrium term structure which explicitly consider a second factor, such as the volatility of rates.

In our empirical analysis, we have taken a different approach: retaining the single-factor characteristic (and the relative simplicity) of our alternative model, while allowing the data to yield time-varying parameters, including the volatility of forecasted bond prices (and derived spot rates). We have done so by eschewing the direct estimation of $\square_0$ and $\square_1$, focusing instead on the intermediate parameters A($\square$), B($\square$) and C($\square$) of equations (34-36). We derive unrestricted estimates of these parameters (that is, not incorporating the restrictions defining A($\square$), B($\square$) and C($\square$) in terms of $\square_0$ and $\square_1$),[7] which simplifies the estimation problem considerably, while allowing us to utilize a nonlinear systems estimator which jointly models the prices of bonds of varying tenors, incorporating all the information in the current term structure. Discount bond price equations for various tenors are likely to be correlated because of the interrelated nature of the markets for bonds with similar terms to maturity. A nonlinear system estimator which takes advantage of cross-equation correlations is thus appropriate. These estimates are derived from ten-year samples of monthly data via a moving window approach, with no constraints imposed on successive window estimates. Thus, we are able to directly analyze the temporal stability of parameters A($\square$), B($\square$) and C($\square$), as well as the variance-covariance matrix of equation errors.

To implement equation (33) in this manner, we require prices of zero-coupon bonds of various tenors, as well as a proxy for the instantaneous interest rate. We utilize the zero-coupon bond prices constructed by Coleman, Fisher and Ibbotson (1989, 1993) for the postwar era. We make use of their monthly estimates of zero coupon bond prices for 12 tenors: 1-, 3-, 6-, 9-, 12- and 18-months, as well as for 2, 3, 4, 5, 7, and 10 years. CFI provide monthly quotations on price and yield of zero-coupon bonds for 1955 through 1992. Their one-month rate is used as a proxy for the instantaneous short rate. A sequence of ten-year windows is constructed, with each window dropping the earliest month and adding the following month relative to its predecessor.

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[7] We consider whether the constraints on the bond price function presented above are satisfied by the parameter estimates, although they are not applied in the estimation. We found that almost all sets of point estimates satisfy both constraints.

Figure 5 illustrates the coefficients A(□), B(□) and C(□) in point and interval form (for a one standard error band) for the 12-month tenor. The A(□) coefficient (the level term in the bond pricing model) is relatively stable around unity until the 1973-1982 period, when it rises by almost ten per cent (presumably reflecting the stress places on the financial system by the oil shocks), and then falls substantially in the 1978-1987 period. The B(□) coefficient–which multiplies the level of the instantaneous rate––falls sharply in the 1959-1968 period, then rises to a level of -0.5 in the 1962-1971 and following samples. Its value rises to the zero range in 1974-1982, and falls off again in the 1978-1987 sample. Coefficient C(□)–the coefficient of the square root term–is positive for the early years and the later years, becoming significantly negative in the intermediate range of 1972-1981.

Figures 6 and 7 provide similar estimates for the three-year and ten-year tenors, respectively. All coefficients are less precisely estimated for these longer tenors, as might be expected from the form of the single-factor model. The A(□) coefficients exhibit greater fluctuation around unity, with the effect of the 1973-1982 period again evident in the graphs. At the longer tenor, the increase in standard error of estimate in that period is also quite evident. The variations in coefficients B(□) and C(□) are also quite marked, with both coefficients taking on larger values at these longer tenors. The fall in B(□) in the 1960-1969 sample is quite marked at the three-year tenor, with an associated increase in the C(□) coefficient in that interval. In the ten-year tenor, B(□) falls markedly in 1960-1969 as well as the 1966-1975 period and again in the last sample. Coefficient C(□) is positive for the first third of the period, then falls sharply in both 1969-1978 and 1977-1986 before returning to positive values.

While the unconstrained movement of these parameter estimates is of interest, we might have much more concern for the time-varying ability of the model to predict bond prices–or, as a more challenging goal, to predict discount yields at the various tenors. Since the estimated coefficients of the nonlinear system are chosen to best predict bond price, and not its nonlinear transformation yield, we might expect that yield forecasts would be of lower quality than their price counterparts. To evaluate the performance of our model, we consider both in-sample forecast accuracy–as gauged by the standard errors of estimate–and ex ante forecast performance. The latter is judged in terms of mean absolute error, root mean square error (RMSE) and

Theil's U for one-, three-, six- and twelve-month ahead forecasts of discount yield. The Theil's U statistic evaluates the estimated equations' ability to outperform a naive (no-change) model over those horizons. Values less than unity indicate that this model is superior to the naive model in terms of RMSE.

Figure 8 illustrates the sizable variation in standard errors of estimate of short-term rate equations over the postwar era. The in-sample forecast accuracy of short-term rates jumps dramatically in the 1970-1979 interval, increasing almost fivefold for the one-month rate, and doubling at the nine month tenor. Figures 9 and 10 present standard errors of estimate for medium and longer tenors. The 3-10 year tenors demonstrate that in-sample accuracy first deteriorates in the 1963-1972 period, with the same pattern of a sharp increase in the early 1970s' windows. These measures of forecast accuracy indicate that the fit of the alternative nonlinear model varies quite considerably over the postwar era.

Finally, we examine the variation in ex ante forecast accuracy in Figures 11, 12 and 13, which refer to forecasts of the bellwether three-month spot rate for 3, 6, and 12 steps ahead, respectively.[8] Figure 11 presents the mean absolute error (MAE) for these forecast horizons. Again, the forecast error remains relatively low until the 1970-1979 sample, rising sharply in the late 1970s. The graph in Figure 12 of root-mean-square error (RMSE) shows an even more dramatic increase in the 1970 period (although, surprisingly, it does not seem to affect the 12-month-ahead forecast) and again in the 1977-1986 sample. Theil's U remains below the naive benchmark of unity for most of the period studied, rising to almost 1.5 for 3- and 6-step ahead forecasts in the 1969-1978 sample. The model appears to forecast reasonably well during the early 1970s samples, weakening again in the late 1970s. Overall, these ex ante forecast statistics–generated from forecasts of yield, a nonlinear transformation of the model's dependent variable–indicate that the alternative nonlinear model does a reasonably good job of short term interest rate forecasting, both in- and out-of-sample.

VI. Conclusions

[8] Similar statistics for other tenors and forecast horizons are available on request.

This paper develops and tests a nonlinear one-factor general equilibrium model of the term structure of interest rate within the framework of Cox, Ingersoll and Ross (CIR, 1985). The advantage of this general equilibrium approach over the partial equilibrium counterpart is that the stochastic properties of the instantaneous interest rate are endogenously determined. Therefore, the general equilibrium model precludes negative interest rates, is consistent with the underlying real economic variables, and rules out arbitrage opportunities. A partial differential equation for valuing discount bond price is presented, and a closed-form expression is derived. This extension broadens classes of risk premiums and term structures which can be empirically evaluated. The model shows that the risk premium need not be strictly increasing in maturity, and yields of the discount bond may be increasing, decreasing, humped or inverted, depending on parameter values.

In an empirical application of the model, we develop a strategy for estimation which permits analysis of the model's temporal stability. We employ nonlinear system estimation of the unrestricted reduced-form parameters with a moving-window strategy in order to capture the term structure volatility caused by factors other than the instantaneous interest rate. We purposefully do not impose any law of motion on the estimated volatilities. This methodology is shown to have strong predictive power for the observed term structure of interest rates, both in-sample and out-of-sample.

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Figure 5
Moving window point and interval estimates for 12 month tenor

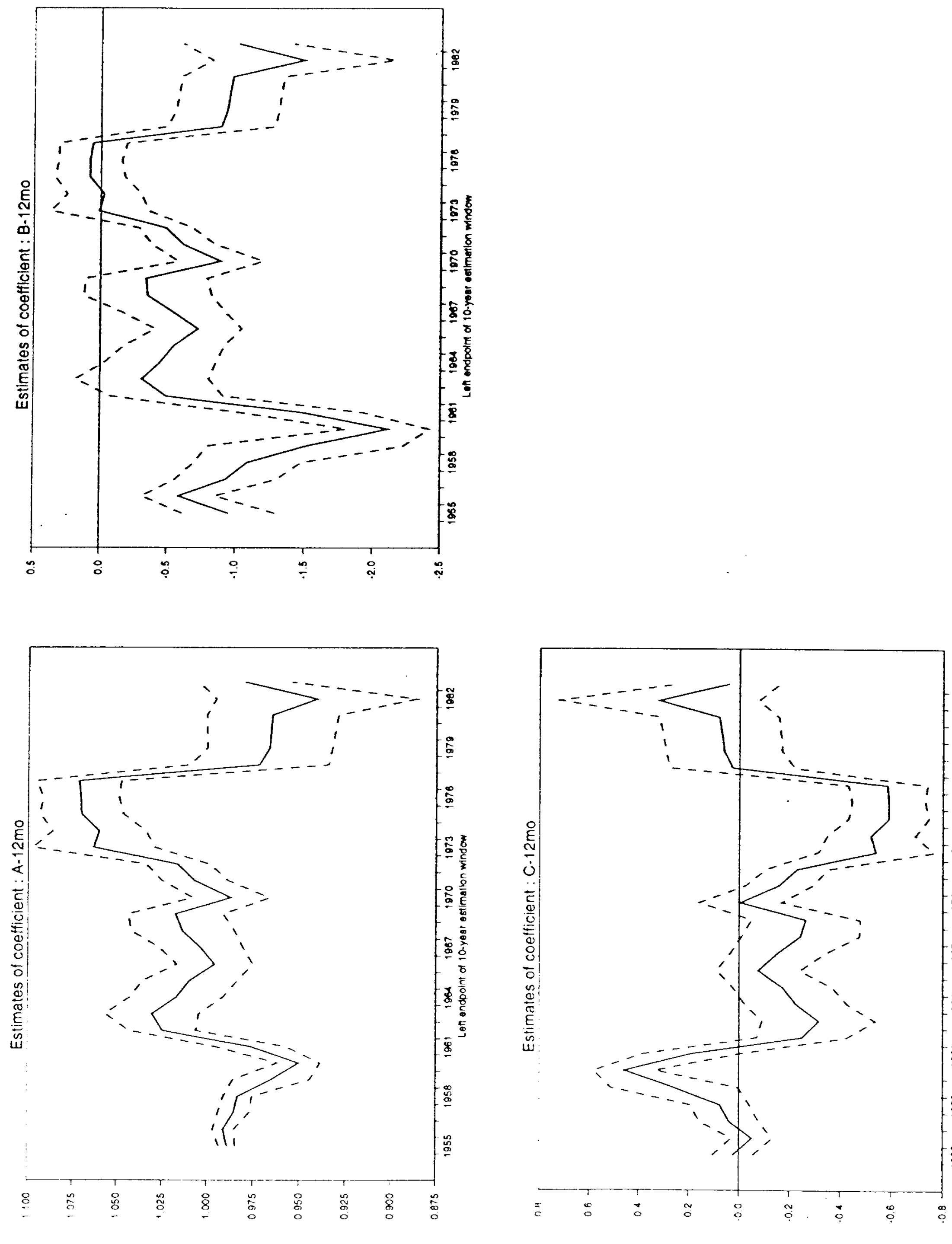


Figure 6
Moving window point and interval estimates for 3 year tenor

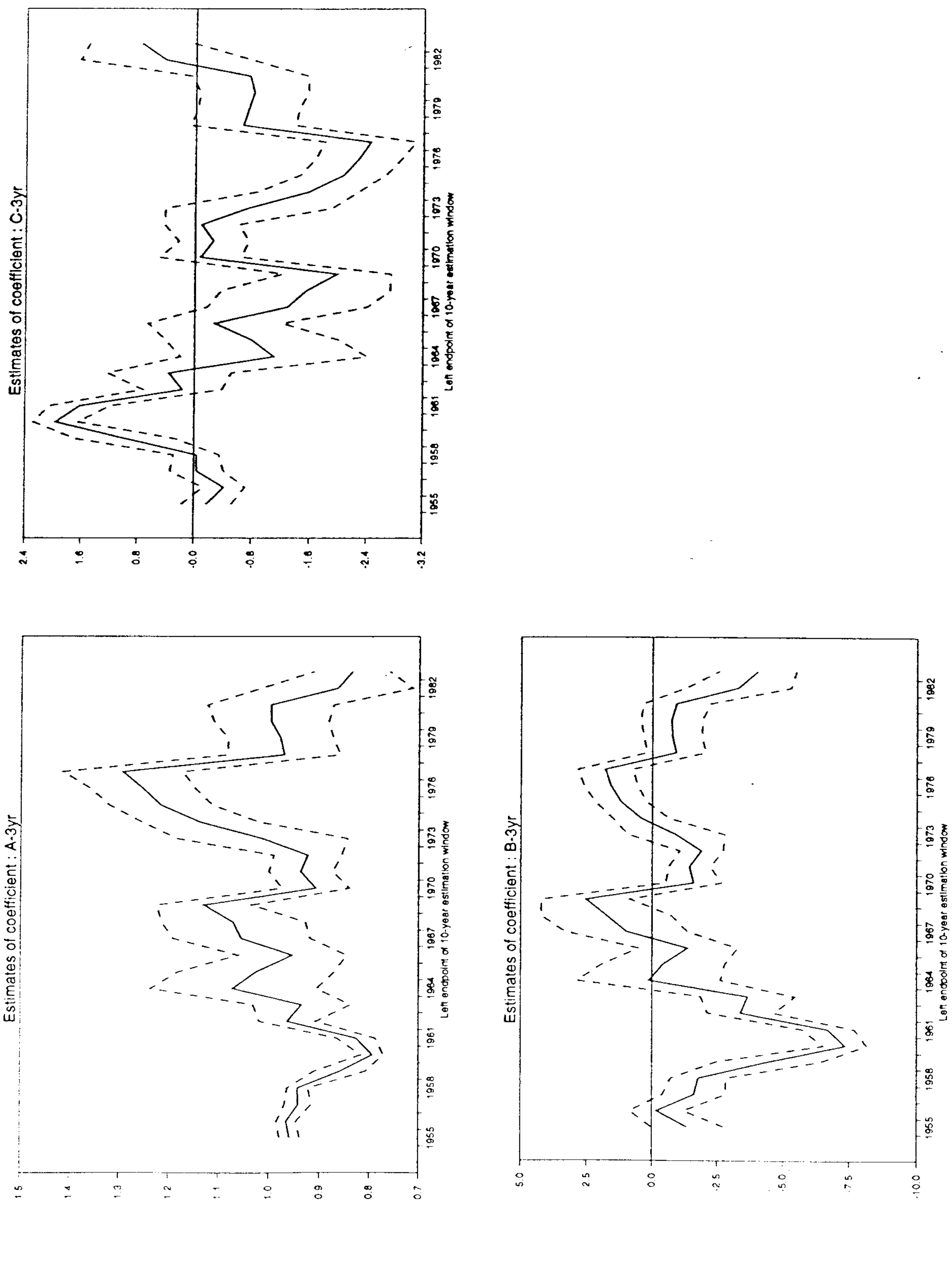


Figure 7
Moving window point and interval estimates for 10 year tenor

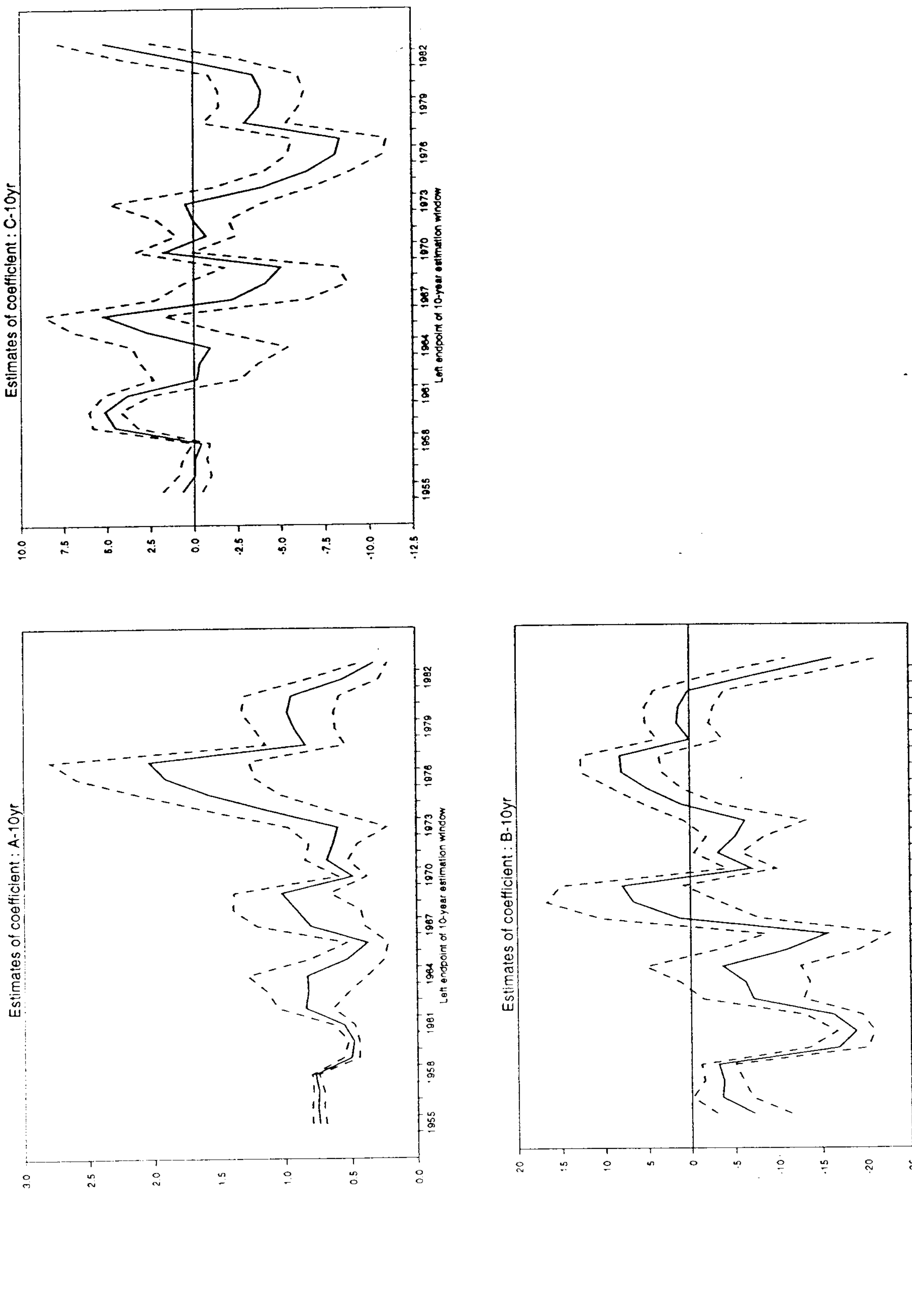


Figure 8
Moving window standard errors of estimate, 1-9 month tenors

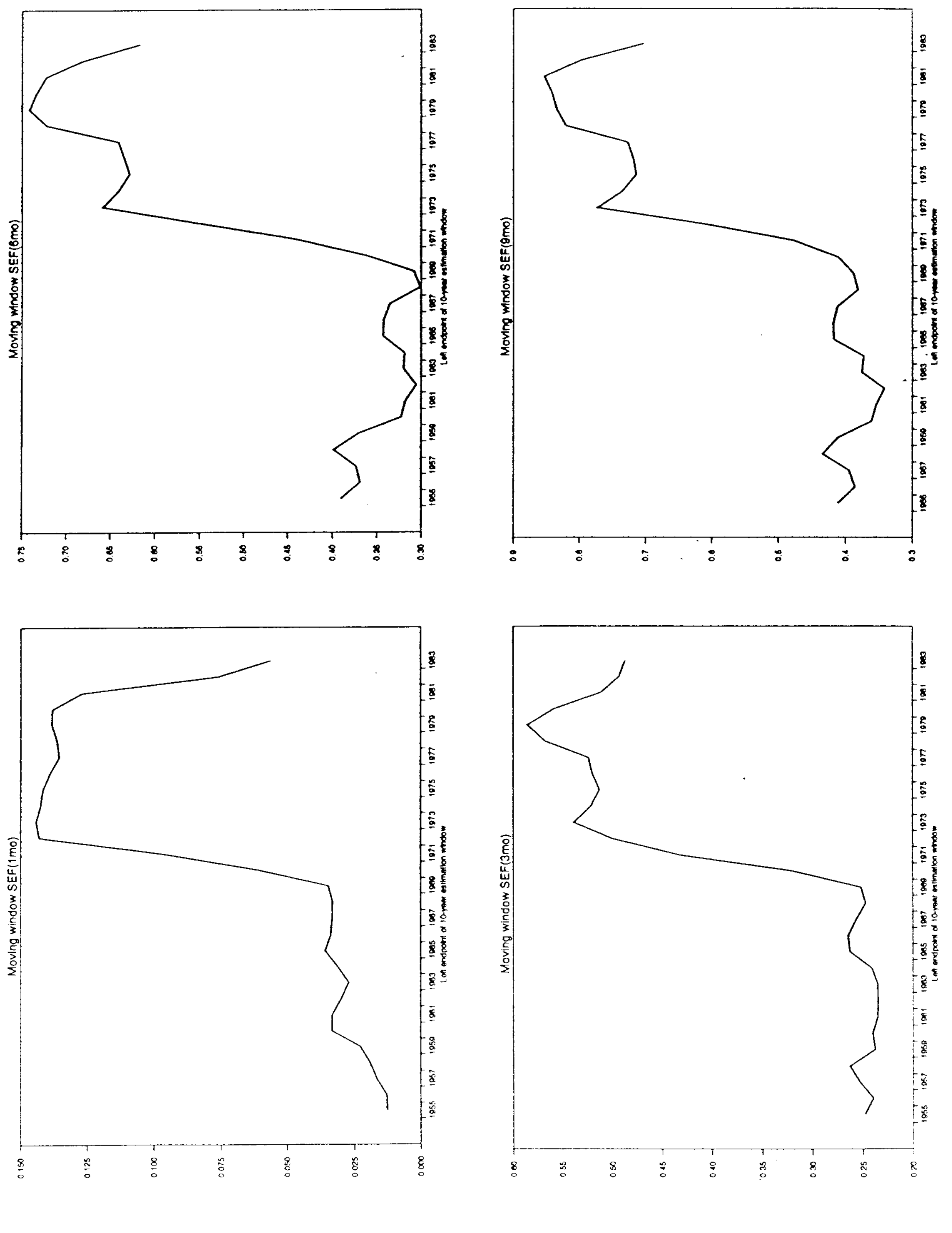


Figure 9
Moving window standard errors of estimate, 1-3 year tenors

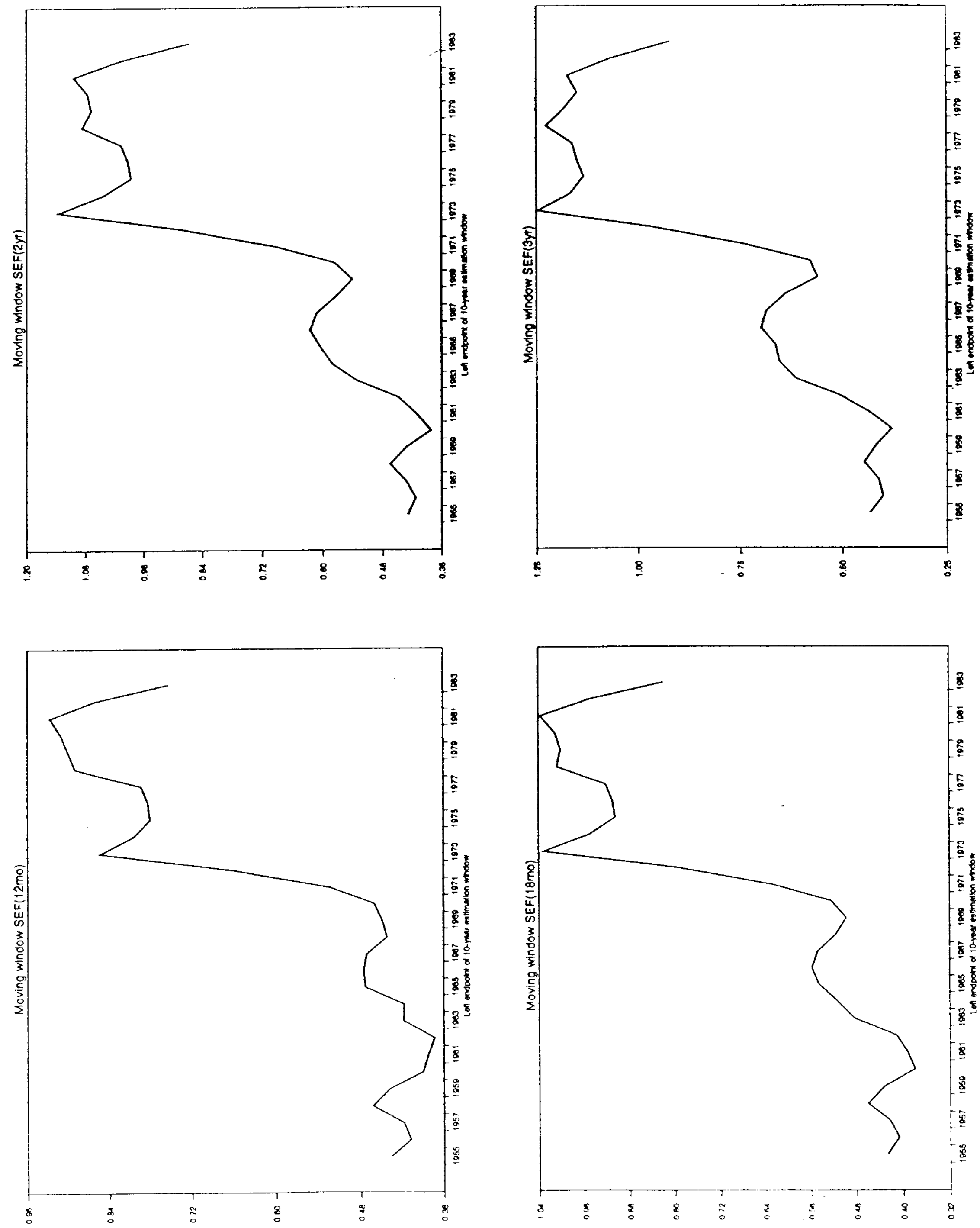


Figure 10
Moving window standard errors of estimate, 4-10 year tenors

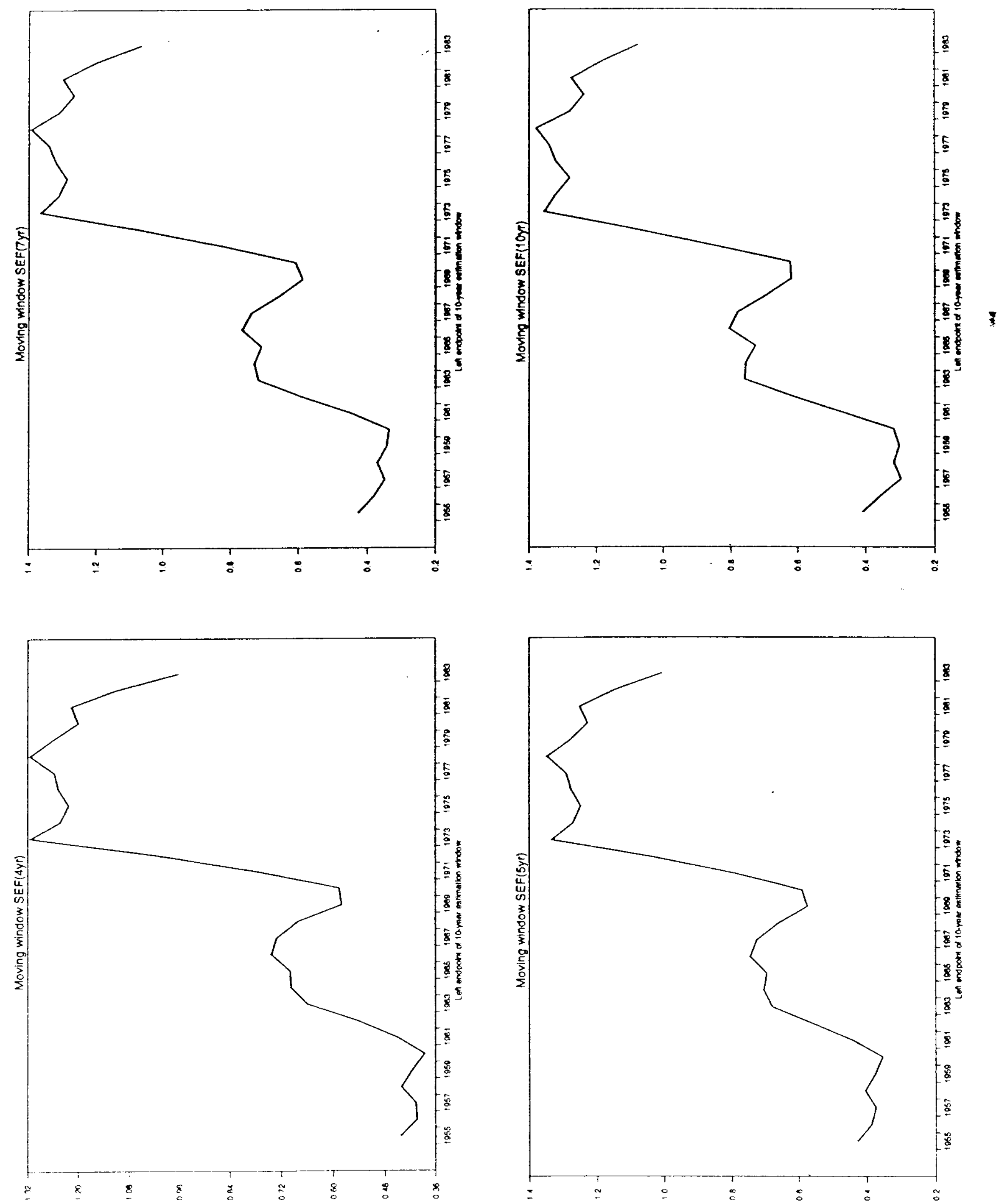


Figure 11: MAE of 3-month spot rate ex ante forecasts

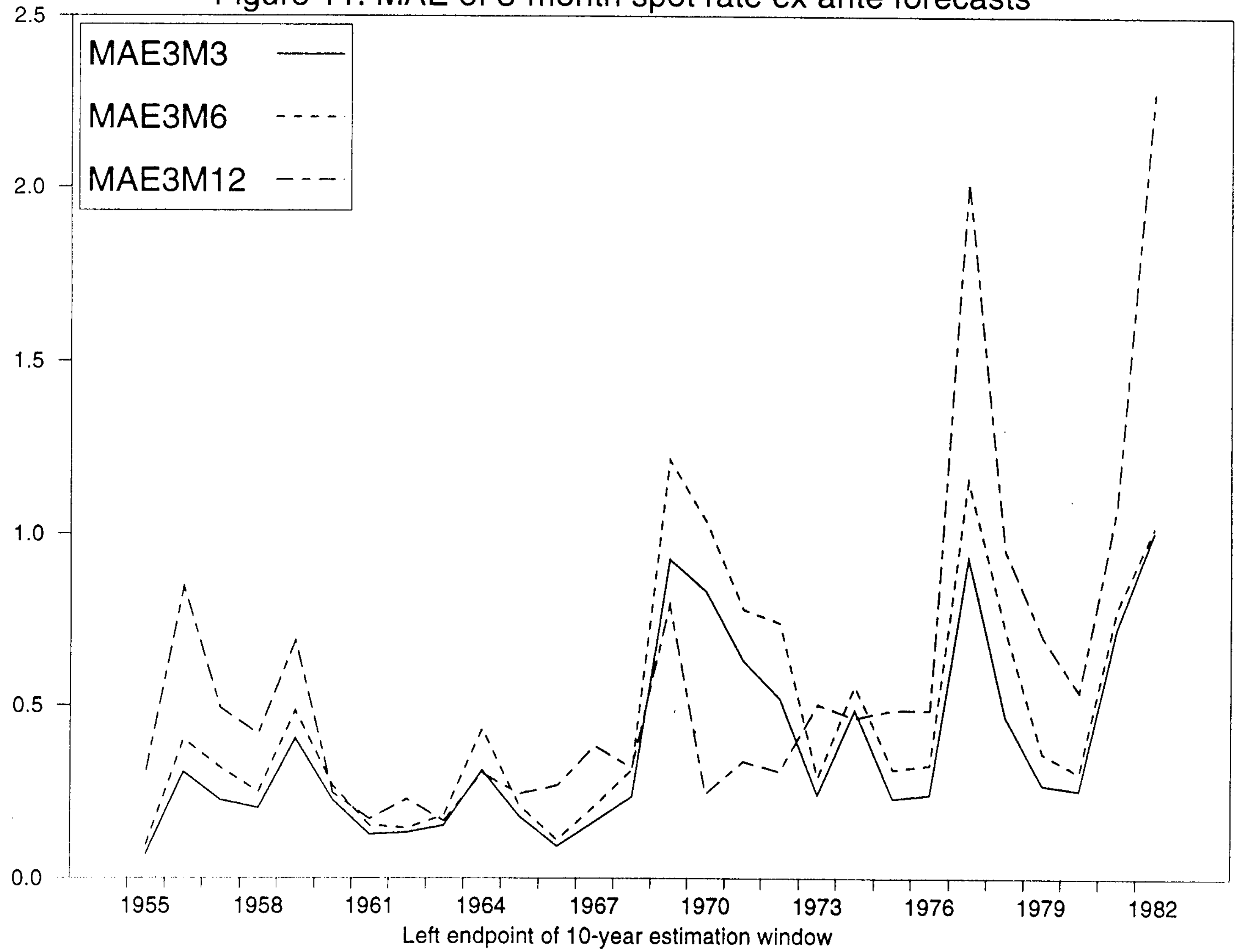
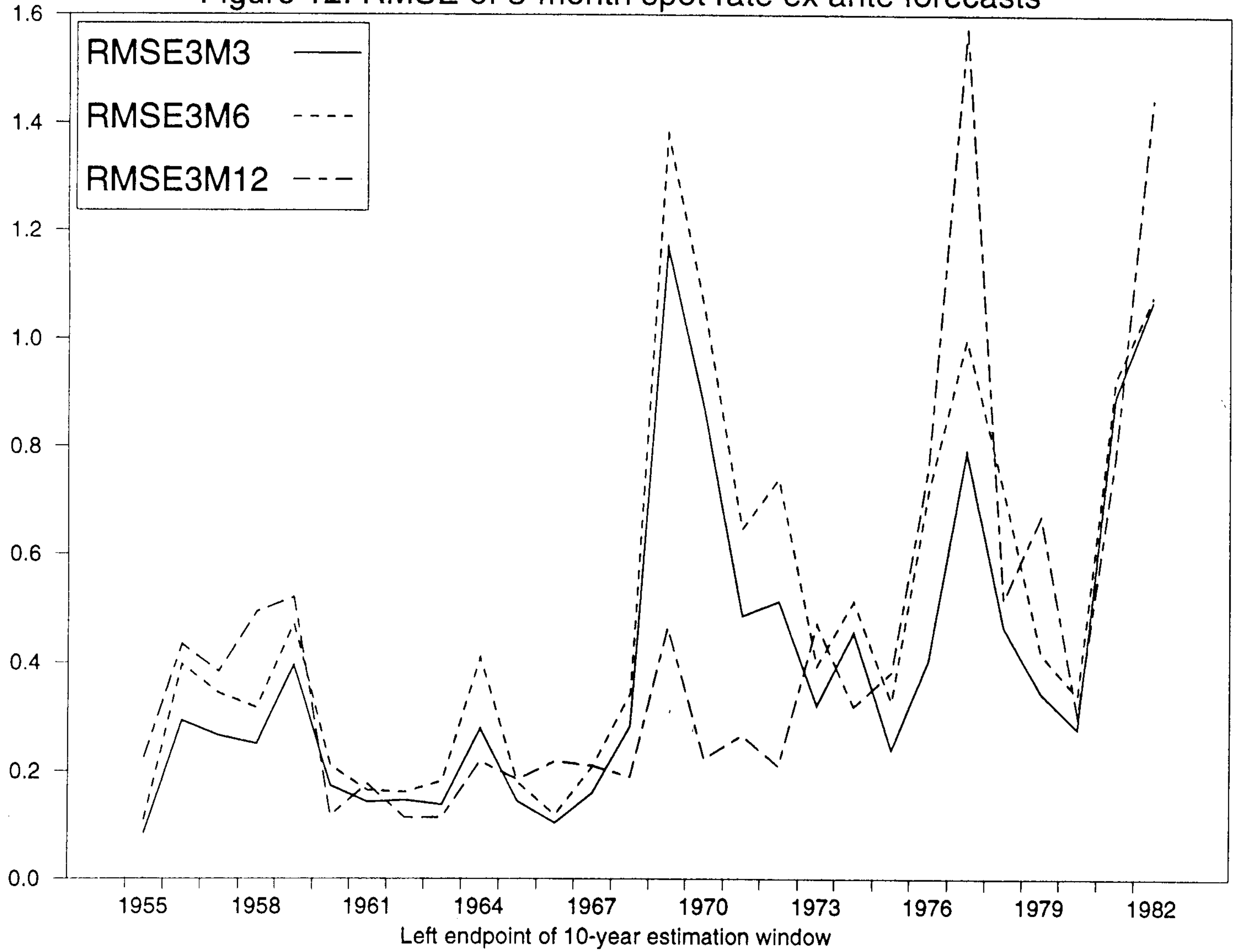


Figure 12: RMSE of 3-month spot rate ex ante forecasts



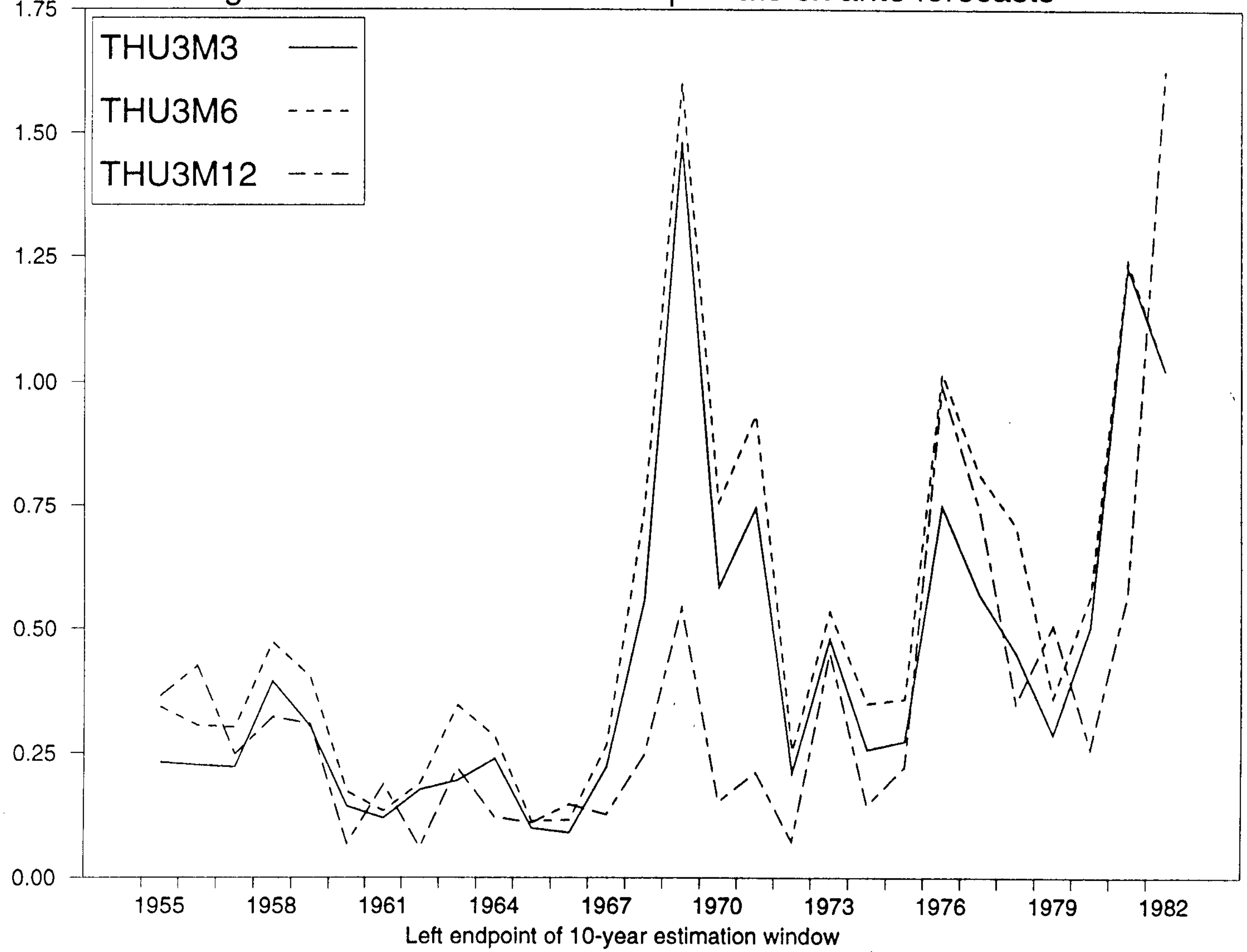


Figure 13: Theil U of 3-month spot rate ex ante forecasts