Testing for Structural Change in Conditional Models

Bruce E. Hansen*
Boston College†

January 1996
Revised: May 1998

Abstract

In the past decade, we have seen the development of a new set of tests for structural change of unknown timing in regression models, most notably the SupF statistic of Andrews (1993), the ExpF and AveF statistics of Andrews-Ploberger (1994), and the L statistic of Nyblom (1989). The distribution theory used for these tests is primarily asymptotic, and has been derived under the maintained assumption that the regressors are stationary. This excludes structural change in the marginal distribution of the regressors. As a result, these tests technically cannot discriminate between structural change in the conditional and marginal distributions. This paper attempts to remedy this deficiency by deriving the large sample distributions of the test statistics allowing for structural change in the marginal distribution of the regressors. We find that the asymptotic distributions of the SupF, ExpF, AveF and L statistics are not invariant to structural change in the regressors. To solve the size problem, we introduce a "fixed regressor bootstrap" which achieves the first-order asymptotic distribution, and appears to possess reasonable size properties in small samples. Our bootstrap theory allows for arbitrary structural change in the regressors, including structural shifts, polynomial trends, and exogenous stochastic trends. It allows for lagged dependent variables and heteroskedastic error processes.

*This research was supported by a grant from the National Science Foundation and a Alfred P. Sloan Foundation Research Fellowship. Thanks go to Byeongseon Seo for excellent research assistance, to James Stock and Mark Watson for useful conversations, and to four referees for helpful comments.

†Department of Economics, Boston College, Chestnut Hill MA 02167-3806. bruce.hansen@bc.edu. Homepage: http://www2.bc.edu/~hansenb/

1 Introduction

There has been a recent surge of interest in tests for constancy of parameters in dynamic econometric models. The classic approach of assuming that the date of structural change is known has been replaced by testing procedures which do not presuppose such knowledge. Particularly important contributions include Nyblom's (1989) test for martingale parameter variation, Andrews' (1993) asymptotic theory for Quandt's (1960) test for a one-time parameter shift, and the exponentially-weighted tests of Andrews and Ploberger (1994). There appears to be considerable interest in the practical implications of these tests for econometric practice; see the recent exploratory work of Stock and Watson (1996).

The distribution theory referenced above has been derived under the assumption that the conditioning variables are stationary. This might not be a desirable assumption in practice. Consider the standard linear regression model

$$y_{ni} = x_{ni}'\beta_{ni} + e_{ni}, \qquad i = 1, ..., n, \tag{1}$$

with $\sigma^2 = E(e_{ni}^2) < \infty$. When we test for structural change in equation (1), we are typically interested in whether or not β_{ni} is constant, and not particularly concerned with the distribution of x_{ni}. Thus the question of structural change in (1) is conceptually distinct from the question of whether or not x_{ni} is stationary. Note that we have written the variables in (1) using array notation. This will facilitate large sample distribution assumptions allowing for non-stationarity in these processes, but otherwise has no important content.

In fact, it is often of particular interest to test for structural change in the conditional relationship (1) when it is known that the distribution of the conditioning variable x_{ni} has experienced a structural change. Indeed, constancy of (1) in the presence of a shift in the marginal distribution of x_{ni} is part of the definition of *super exogeneity* proposed by Engle, Hendry and Richard (1983). Tests of this hypothesis have been discussed by Hendry (1988) and Engle and Hendry (1993).

If the distributions of tests for constancy of β_{ni} are robust to structural change in x_{ni}, then we would have nothing to worry about, as a significant test statistic could be unambiguously interpreted as evidence for structural instability in β_{ni}. But this may not be the case. If the null distribution is affected by a structural change in x_{ni}, then a significant test statistic could indicate that there is instability in either β_{ni} or x_{ni} (a conclusion which is not of much interest), and if the distribution under the alternative is adversely affected, power could suffer.

In this paper we carefully explore the asymptotic distributions allowing for structural change in the regressors. In section 2 we describe the model and test statistics, allowing for a one-time structural change in the regression parameters. In section 3 we describe the first-order asymptotic theory. We consider both asymptotically stationary and non-stationary processes, where the latter includes structural change in the regressors. We find that the distributions are different for non-stationary regressors. To quantify these differences, section 4 explores the important special case of a single structural change in the marginal process. We find that the size and power distortions can be quite large. In particular, Nyblom's L statistic is asymptotically conservative, and the size distortion of Quandt's SupF statistic is potentially unbounded. In section 5 we discuss asymptotically valid inference based on a bootstrap distribution. In a simulation experiment, we find that this bootstrap technique works quite well when compared with the conventional asymptotic tests. Section 6 concludes. Proofs are presented in an Appendix.

Throughout the paper, $I(\cdot)$ denotes the indicator function, $[\cdot]$ denotes integer part, and "$\Rightarrow$" denotes weak convergence with respect to the uniform metric over $r \in [0,1]$.

2 Model and Tests

The conditional distribution of y_{ni} given x_{ni} takes the form of a linear regression (1) where y_{ni} is real-valued and x_{ni} is an m-vector, and structural change in the conditional distribution arises through the coefficient β_{ni}. The structural change in β_{ni} takes the form

$$\beta_{ni} = \begin{cases} \beta, & i < t_0 \\ \beta + \theta_n, & i \geq t_0. \end{cases} \tag{2}$$

The parameter $t_0 \in [t_1, t_2]$ indexes the relative timing of the structural shift, and θ_n indexes the magnitude of the shift.

We are interested in tests of $H_0 : \theta_n = 0$ against $H_1 : \theta_n \neq 0$. To examine asymptotic local power we will specify H_1 as local to H_0. Specifically, we assume that θ_n takes the form

$$\theta_n = \delta\sigma/\sqrt{n} \tag{3}$$

with δ fixed as $n \to \infty$. The parameter δ indexes the degree of structural change under the local alternative H_1. For concreteness, we collect our maintained assumptions:

Assumption 1 *The linear regression model is (1)-(2)-(3). The error e_{ni} is a martingale difference: $E(e_{ni} \mid \Im_{ni-1}) = 0$ where $\Im_{ni-1}$ is the sigma-field generated by current x_{ni} and lagged values of (x_{ni}, e_{ni}). The sequence e_{ni}^2 satisfies a WLLN, so that $n^{-1}\sum_{i=1}^{n} e_{ni}^2 \to_p \sigma^2$. The parameters $\tau_0 = t_0/n$, $\pi_1 = t_1/n > 0$ and $\pi_2 = t_2/n < 1$ are fixed as $n \to \infty$.*

Sometimes we will add the assumption that e_{ni} is conditionally homoskedastic:

$$E\left(e_{ni}^2 \mid \Im_{ni-1}\right) = \sigma^2, \qquad a.s. \tag{4}$$

Under H_0, model (1)-(2) reduces to $y_{ni} = x_{ni}'\beta + e_{ni}$ which does not depend on t_0. Denote the ordinary least squares (OLS) estimator $\hat{\beta}$, residuals $\hat{e}_i$, and variance estimate $\hat{\sigma}^2 = (n-m)^{-1}\sum_{i=1}^{n}\hat{e}_i^2$. Under the alternative $H_1 : \theta_n \neq 0$, the model can be written as

$$y_{ni} = x_{ni}'\beta + x_{ni}'\theta_n I(i \geq t_0) + e_{ni}. \tag{5}$$

For any fixed t, (5) can be estimated by OLS, yielding estimates $(\hat{\beta}_t, \hat{\theta}_t)$, residuals $\hat{e}_{it}$ and variance estimate $\hat{\sigma}_t^2 = (n-2m)^{-1}\sum_{i=1}^{n}\hat{e}_{it}^2$. Let $\hat{t} = \text{argmin}\,\hat{\sigma}_t^2$ denote the least squares estimate of the breakdate and set $\tilde{\beta} = \hat{\beta}_{\hat{t}}$ and $\tilde{e}_i = \hat{e}_{i\hat{t}}$.

The standard test for H_0 against H_1 for known t (e.g., Chow (1960)) is the Wald statistic:

$$F_t = \frac{(n-m)\hat{\sigma}^2 - (n-2m)\hat{\sigma}_t^2}{\hat{\sigma}_t^2}. \tag{6}$$

The Wald statistic F_t is equivalent to the likelihood ratio statistic when e_{ni} is iid $N(0, \sigma^2)$.

We are interested in tests of H_0 when the true changepoint t_0 is unknown. Quandt (1960) proposed the likelihood ratio test which is equivalent to $\text{SupF}_n = \sup_t F_t$, where the supremum is taken over $t \in (t_1, t_2)$. Andrews and Ploberger (1994) developed a theory of optimal testing, and suggested a related family of tests, including an exponentially weighted Wald test (optimal against distant alternatives) $\text{ExpF}_n = \ln \int \exp(\text{F}_t/2) dw(t)$ and the average F test (optimal against very local alternatives) $\text{AveF}_n = \int_t \text{F}_t dw(t)$, where w is a measure putting weight $1/(t_2 - t_1)$ on each integer t in the interval $[t_1, t_2]$. The Quandt and Andrews-Ploberger statistics assume that θ_n and t_0 are unknown parameters. Nyblom (1989) instead considered random structural change. Let θ_n and t_0 be random variables such that $E[\theta_n \{i = t_0\}] = 0$ and $E[\theta_n \theta_n' \{i = t_0\}] = \left(\sum_{t=1}^n x_{ni} x_{ni}'\right)^{-1} \phi^2$ for some real number ϕ. Nyblom's (1989) Lagrange Multiplier (LM) test for $\phi^2 = 0$ against $\phi^2 > 0$ rejects for large values of

$$\text{L}_n = \frac{1}{n\hat{\sigma}^2} \sum_{i=1}^n \left(\sum_{t=1}^i x_{nt} \hat{e}_t \right)' \left(\sum_{t=1}^n x_{nt} x_{nt}' \right)^{-1} \left(\sum_{t=1}^i x_{nt} \hat{e}_t \right). \tag{7}$$

3 Asymptotic Theory

3.1 Asymptotically Stationary Process

Andrews (1993), Andrews-Ploberger (1994) and Nyblom (1989) assumed that the data are stationary. We now show that their distribution theory holds somewhat more broadly.

Definition 1 *An array a_{ni} is asymptotically mse-stationary if*

$$\frac{1}{n} \sum_{i=1}^{[nr]} a_{ni} a_{ni}' \Rightarrow rA \tag{8}$$

where $A = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^n E\left(a_{ni} a_{ni}'\right)$.

Intuitively, a process is asymptotically mse-stationary if its second moments are constant, while the behavior of higher-order moments is irrelevant. Strictly stationary, ergodic, and square integrable processes are asymptotically stationary, as are certain deterministic cyclic processes, seasonal dummies, and "stationary" forms of heteroskedasticity.

An interesting example is an AR(1) subject to small structural change:

$$y_{ni} = \beta_{ni} y_{ni-1} + e_{ni} \tag{9}$$

with e_{ni} iid, and β_{ni} following (2)-(3). Since β_{ni} changes from β to $\beta + \theta_n$ at time t_0, y_{ni} is non-stationary. Yet since $\theta_n \to 0$ as $n \to \infty$, this non-stationarity may not be important in large samples. Indeed, we can calculate that as $n \to \infty$,

$$E\left(\frac{1}{n}\sum_{i=1}^{[nr]} y_{ni}^2\right) = \sigma^2 \frac{1}{n}\sum_{i=1}^{[nr]}\sum_{j=1}^{\infty} \beta_{ni+1-j}^2 \to rA$$

where $A = \sigma^2 / \left(1-\beta^2\right)$. It follows by standard arguments that (8) holds for y_{ni}, so that y_{ni} is asymptotically mse-stationary.

We now explore the distribution theory for our model under this assumption.

Theorem 1 *If Assumption 1 holds, x_{ni} is asymptotically mse-stationary, (4) holds, and for some $q > 1$,*

$$\sup_{n \geq 1, i \leq n} E\left|x_{ni} e_{ni}\right|^{2q} < \infty \tag{10}$$

then the asymptotic null distributions of SupF_n, ExpF_n, AveF_n, *and* L_n *are those given in Andrews (1993), Andrews and Ploberger (1994) and Nyblom (1989).*

Theorem 1 shows that the key assumption for the asymptotic theory is the asymptotic constancy of second moments. Intuitively, in linear regression information is reflected in the second moments of the data, and when the data are asymptotically mse-stationary information is accumulated linearly.

3.2 Asymptotically Non-Stationary Processes

The key condition for Theorem 1 is that the second moments of the cumulated data grow linearly. In this section we explore the consequences of violations of this assumption. We now consider a set of high-level conditions on the sample moments. We will give a list of standard examples which satisfy these conditions at the end of this section.

Assumption 2 *As $n \to \infty$*

$$\frac{1}{n}\sum_{i=1}^{[nr]} x_{ni} x_{ni}' \Rightarrow M(r) \tag{11}$$

$$\frac{1}{n}\sum_{i=1}^{[nr]} x_{ni} x_{ni}' e_{ni}^2 / \sigma^2 \Rightarrow V(r) \tag{12}$$

and

$$\frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{[nr]} x_{ni}e_{ni} \Rightarrow N(r), \tag{13}$$

where $M(r)$, $V(r)$ and $N(r)$ are random $m \times m$, $m \times m$, and $m \times 1$ matrix processes, respectively, such that $M(r)$ and $V(r)$ are continuous in r almost surely, for any $r > 0$, $M(r) > 0$ and $V(r) > 0$ a.s., and conditionally on $\{V(s) : 0 \leq s \leq 1\}$, $N(r)$ is a mean-zero Gaussian process with conditional covariance kernel

$$E\left(N(r_1)N(r_2)'\right) = V\left(\min\left(r_1, r_2\right)\right). \tag{14}$$

Assumption 2 is a generalization of asymptotic mse-stationarity, because if x_{ni} and $x_{ni}e_{ni}$ are asymptotically mse-stationary, (11)-(12)-(13) are satisfied with $M(r)$ and $V(r)$ linear matrix functionals, and $N(r)$ a Brownian motion (this is the essence of the proof of Theorem 1). Below, we discuss a set of examples which satisfy Assumption 2, and section 4 explores in detail a particular example. First we give the asymptotic distributions of the structural change tests under local alternatives.

Theorem 2 *Under Assumptions 1 and 2,*

$$\begin{aligned} F_{[nr]} &\Rightarrow \left(N^*(r) - Q(r)\delta\right)' M^*(r)^{-1}\left(N^*(r) - Q(r)\delta\right) \\ &\equiv F(r \mid \delta) \end{aligned} \tag{15}$$

where

$$N^*(r) = N(r) - M(r)M(1)^{-1}N(1) \tag{16}$$

$$Q(r) = \left(M(r) - M(r)M(1)^{-1}M(\tau_0)\right) - \left(M(r) - M(\tau_0)\right) I(r \geq \tau_0),$$

and

$$M^*(r) = M(r) - M(r)M(1)^{-1}M(r). \tag{17}$$

Hence

1. $\mathrm{SupF}_n \rightarrow_d \sup_{\pi_1 \leq r \leq \pi_2} F(r \mid \delta)$;
2. $\mathrm{ExpF}_n \rightarrow_d \ln\left[\frac{1}{\pi_2 - \pi_1}\int_{\pi_1}^{\pi_2} \exp\left(\tfrac{1}{2}F(r \mid \delta)\right) dr\right]$;

3. $\text{AveF}_n \to_d \dfrac{1}{\pi_2 - \pi_1} \displaystyle\int_{\pi_1}^{\pi_2} F(r \mid \delta) dr;$

4. $\text{L}_n \to_d \int_0^1 \left(N^*(r) - Q(r)\delta\right)' M(1)^{-1} \left(N^*(r) - Q(r)\delta\right) dr.$

Theorem 2 gives the asymptotic distributions under local departures from H_0. To find the null distribution, set $\delta = 0$, in which case we find

$$F(r) \equiv F(r \mid 0) = N^*(r)' \left(M(r) - M(r)M(1)^{-1}M(r)\right)^{-1} N^*(r).$$

Unless $M(r)$ is linear in r, $N^*(r)$ will not be a Brownian bridge and $F(r)$ will not equal the squared tied-down Bessell process that appears in Andrews (1993). It follows that the asymptotic distributions of the test statistics given in Theorem 2 are different than the distributions tabulated by Andrews (1993), Andrews and Ploberger (1994) and Nyblom (1989). Apparently, these tests for structural change are not asymptotically pivotal when we allow for asymptotic non-stationarity[1].

The null sampling distributions of the test statistics examined in Theorem 2 appear to depend on the functions $M(r)$ and $V(r)$. No other nuisance parameters enter the asymptotic distributions. Under the alternative hypothesis ($\delta \neq 0$), the only additional nuisance parameter is τ_0, the true timing of the structural change in regression parameter.

Theorem 2 applies in many interesting examples. In the following examples, for simplicity we assume that the error e_{ni} satisfies (4).

Example 1: Linear Trend: $x_{ni} = i/n$. Then $M(r) = r^3/3$ and $N(r) = \int_0^r s dW(s)$.

Example 2: Trend in Variance: $x_{ni} = \sqrt{i/n}\, v_i$ with v_i iid(0,1). Then $M(r) = r^2/2$ and $N(r) = \int_0^r s^{1/2} dW(s)$.

Example 3: Stochastic Trend: $x_{ni} = n^{-1/2} z_i$, where Δz_i is $I(0)$ and independent of e_{ni+j} for all j. Then letting W_2 and W_1 denote independent Brownian motions, $M(r) = \int_0^r W_2(s)^2 ds$ and $N(r) = \int_0^r W_2(s) dW_1(s)$. Using similar reasoning, Assumption 2 applies to cointegrated regression models estimated using the leads-and-lags technique of Saikkonen (1991), Phillips and Loretan (1991) and Stock and Watson (1993).

[1] The results of Theorem 2 stand in stark contrast to those for the Chow (1960) test for structural change of known timing. From (15), one can deduce that the asymptotic null distribution of the Chow F statistic is chi-square and hence it is asymptotically pivotal.

In all three examples, $M(r)$ is non-linear and $N(r)$ is not a Brownian motion. This non-linearity implies that the tabulated critical values in Andrews (1993), Andrews and Ploberger (1994) and Nyblom (1989) will be inappropriate in these contexts. We now turn to a more thorough examination of a particular example of interest.

4 Structural Change in the Marginal Distribution

To illustrate the possible divergence between the distribution results of Theorem 1 and Theorem 2, we examine the case where there is a single structural change in the marginal distribution of x_{ni} at date k. We extend Definition 1 slightly by saying that a_i is asymptotically mse-stationary over the region (k_1, k_2) with $k_1 = [n\kappa_1]$, $k_2 = [n\kappa_2]$ if $\frac{1}{n}\sum_{i=k_1}^{k_2} a_i a_i' \Rightarrow (\kappa_2 - \kappa_1)A_{12}$ for some matrix A_{12}.

Assumption 3 *Let $k = [n\kappa]$ and $\kappa \in (0,1)$. The variables x_{ni} is asymptotically mse-stationary for $i < k$, and x_{ni} is asymptotically mse-stationary for $i \geq k$. (4) and (10) hold.*

Assumption 3 specifies that the marginal distribution of x_t is asymptotically stationary before and after the date k, but allows an arbitrary structural change at observation k. The change may occur in the mean, variance and/or serial correlation in the regressor. Note that Assumption 3 allows for lagged dependent variables as discussed in section 3.1. It is straightforward to verify that Assumption 3 implies Assumption 2 with

$$M(r) = M_1 r + (M_2 - M_1)(r - \kappa) I(r \geq \kappa), \tag{18}$$

and $V(r) = M(r)$, where

$$M_1 = \lim_{n\to\infty} \frac{1}{n\kappa} \sum_{i=1}^{[n\kappa]-1} E\left(x_{ni} x_{ni}'\right), \tag{19}$$

and

$$M_2 = \lim_{n\to\infty} \frac{1}{n(1-\kappa)} \sum_{i=[n\kappa]}^{n} E\left(x_{ni} x_{ni}'\right). \tag{20}$$

Since Assumption 2 holds, the distribution theory of Theorem 2 applies.

Under Assumption 3 the function $M(r)$ is a piece-wise linear function in r with a kink at $r = \kappa$. We see that the accumulation of information is non-linear, rather than linear.

To illustrate the impact this non-linearity has on the asymptotic distributions, we consider the leading case of one regressor ($m = 1$) so that M_2 and M_1 are scalars. For this case we can make some interesting analytical observations about the asymptotic null distributions of the SupF and L statistics. Set $\omega = M_2/M_1 - 1$ and

$$v(r) = \frac{r + \omega\,(r - \kappa)\,I\,(r \geq \kappa)}{1 + \omega(1 - \kappa)}. \tag{21}$$

Let

$$\text{SupF} = \sup_{\pi_1 \leq r \leq \pi_2} \frac{N^*(r)^2}{M^*(r)} \tag{22}$$

denote the asymptotic null distribution of SupF_n under Assumption 3 (from Theorem 2), where $N^*(r)$, $M^*(r)$ and $M(r)$ are defined in (16), (17) and (18), respectively.

Theorem 3

$$\text{SupF} \equiv \sup_{\pi_1^* \leq r \leq \pi_2^*} \frac{W^*(r)^2}{r\,(1 - r)}$$

where $W^*(r)$ is a standard Brownian bridge, $\pi_1^* = v(\pi_1)$, and $\pi_2^* = v(\pi_2)$.

The distribution in Theorem 3 is identical to that found by Andrews (1993) for stationary processes, but depends on the index

$$\lambda^* = \frac{\pi_2^*\,(1 - \pi_1^*)}{\pi_1^*\,(1 - \pi_2^*)}$$

rather than $\lambda = \pi_2\,(1 - \pi_1)\,/\,[\pi_1\,(1 - \pi_2)]$ as found by Andrews. Thus the relevant measure of "spread" between π_1 and π_2 is not based on the linear measure r implicit in the definition of λ, but should be based on the non-linear measure $v(r)$ reflecting the actual rate of accumulation of sample information. Note that λ^* is a function only of π_1^* and π_2^*, the "relative information" cumulated in the regressors at the times π_1 and π_2.

From Table 1 of Andrews (1993), we see that the asymptotic critical values for SupF_n are increasing in λ. Hence if $\lambda^* > \lambda$ the statistic SupF_n will tend to reject too frequently if Andrews' critical values are used. On the other hand, if $\lambda^* < \lambda$, then SupF_n will tend to reject too infrequently, reducing power. Setting $\pi_1 = .15$ and $\pi_2 = .85$, Figure 1 plots λ^* as a function of κ for four positive values of ω. For each ω, λ^* is maximized at $\kappa = \pi_1$ and minimized at $\kappa = \pi_2$. As ω increases, λ^* can become arbitrarily large. The sampling implication is that the Quandt-Andrews SupF_n statistic can have arbitrarily large size distortion.

While we cannot find a general analytic expression for the L distribution, we can find its limiting behavior for large ω. Again for $m = 1$ let

$$\mathrm{L}(\omega, \kappa) = \frac{\int_0^1 N^*(r)^2 dr}{M(1)}$$

denote the asymptotic null distribution of L_n under Assumption 3.

Theorem 4

$$\operatorname*{plim}_{\omega \to \infty} \mathrm{L}(\omega, \kappa) = (1 - \kappa) \int_0^1 W^*(r)^2,$$

where $W^*(r)$ is a standard Brownian bridge.

Note that the conventional Nyblom asymptotic distribution is $\mathrm{L}(0, \kappa) \equiv \int_0^1 W^*(r)^2$. Theorem 4 shows that if ω is large, then the Nyblom statistic will be asymptotically conservative, in the sense that the asymptotic distribution of the test statistic will be dominated by the conventional distribution. This suggests that Nyblom's test will substantially underreject the null hypothesis when ω and κ are large.

Figure 2 plots numerical estimates[2] of the asymptotic Type I error[3] (at the 10% nominal level) for the four asymptotic distributions (SupF, ExpF, AveF, and L) as a function of ω, for $\kappa = .25$, .50, .75, and .95[4]. The SupF_n, ExpF_n, and AveF_n tests set $\pi_1 = .15$ and $\pi_2 = .85$. The most striking feature of the plots is that the Nyblom test substantially underrejects for large ω (as predicted by Theorem 4). The second most noticeable feature is that SupF_n over-rejects for $\kappa = .25$ and under-rejects for $\kappa = .95$ (as predicted by Theorem 3). A very interesting finding is that the Andrews-Ploberger ExpF_n test has virtually zero size distortion for any κ and ω.

Figure 3 plots estimated[5] asymptotic local power functions for the four test statistics (not size-corrected) for the case of a large structural change in the marginal distribution,

[2]Generated using random normal samples of size 1000 with 50,000 replications for each $\omega = 1, 2, ..., 16$.

[3]Rejection rates using asymptotic critical values and fixed parameters.

[4]Only graphs for positive ω are shown as they are symmetric in the transformation $(\omega, \kappa) \to (-\omega, 1 - \kappa)$.

[5]Generated using random normal samples of size 1000 with 20,000 replications for $\delta = 1, 2, ..., 10$.

$\omega = 10$. The plots are as a function of δ, for $\kappa = .25$, .50, .75, and .95, and τ_0 is set to equal κ. The plots show that for $\kappa \leq .75$, the power of the Nyblom test is adversely affected by the shift in the marginal distribution. In the case $\kappa = .25$, the power loss is dramatic. The AveF test also suffers a mild power loss in some cases. In contrast, the ExpF and SupF tests appear to have the best power, with the exception of the case $\kappa = .95$ (where the Nyblom test is over-sized).

In summary, the numerical analysis of the asymptotic distributions of the test statistics suggest that at least for $m = 1$ the Nyblom test is quite poorly behaved in the presence of shifts in the marginal distribution, yet the Andrews-Ploberger ExpF_n statistic is essentially unaffected by such shifts. The popular Quandt-Andrews SupF_n statistic suffers from a mild size distortion, but with no noticeable effect on power.

5 Bootstrapping

5.1 The Fixed Regressor Bootstrap

We have seen in the previous sections that non-stationarity or structural change in the marginal distribution affects the asymptotic distributions of the test statistics in complicated ways. An alternative is to consider a bootstrap distribution. The term "bootstrap" was introduced by Efron (1989) and has since spawned a large literature. Most of the theory and techniques require random samples. Extensions to dependent data have been confined to strictly stationary processes, including the parametric bootstrap of Bose (1988) and the block resampling bootstraps of Carlstein (1986) and Künsch (1989). To my knowledge, there is no theoretical literature[6] which applies to the present context – involving non-standard test statistics and explicitly non-stationary data processes. A priori, it is not clear if bootstrap methods will work, as we know that standard bootstrap techniques fail in the context of non-stationary autoregressions, see Basawa, et. al. (1991).

In our model, appropriate application of the bootstrap is not obvious. Under Assumption 2, x_{ni} need not be stationary, so block re-sampling is inappropriate. Parametric bootstrap

[6] In an interesting contribution, Diebold and Chen (1996) provide simulation evidence (but no theory) that the parametric bootstrap works well for structural change tests applied to AR(1) processes.

methods are also inappropriate in conditional regression models (except in the special case in which x_{ni} is strictly exogenous). We wish to avoid methods which require the joint modelling of y_{ni} and x_{ni} which require the correct specification of the marginal distribution (including any structural changes). Such modelling violates the principle of the regression model, where the goal is to condition on the regressors, and hence ignore their marginal distribution.

Despite these concerns, we are able to successfully employ what we call the "Fixed Regressor Bootstrap," which treats the regressors x_{ni} as if they are fixed (exogenous) even when they contain lagged dependent variables. The discussion which follows is for the SupF test, yet the method applies as well for the other tests. There are two forms of the fixed regressor bootstrap introduced here, one appropriate if the error e_{ni} is homoskedastic (4) and the other appropriate under heteroskedasticity.

For the homoskedastic bootstrap, let $\{y_{ni}(b) : i = 1, ..., n\}$ be a random sample from the N(0,1) distribution. Regress $y_{ni}(b)$ on x_{ni} to get residual variance $\hat{\sigma}^2(b)$ and regress $y_{ni}(b)$ on x_{ni} and $x_{ni}I(i \leq t)$ to get the residual variance $\hat{\sigma}_t^2(b)$ and Wald sequence

$$F_t(b) = \frac{(n-m)\hat{\sigma}^2(b) - (n-2m)\hat{\sigma}_t^2(b)}{\hat{\sigma}_t^2(b)}.$$

The bootstrap test statistic is $\text{SupF}_n(b) = \sup_{t_1 \leq t \leq t_2} F_t(b)$. Let $G_n(x) = P\left(\text{SupF}_n(b) \leq x \mid \Im_n\right)$ denote the conditional distribution function of $\text{SupF}_n(b)$ (conditional on the data). The bootstrap p-value is $p_n = 1 - G_n\left(\text{SupF}_n\right)$. The bootstrap test rejects H_0 when p_n is small.

We can allow for heteroskedastic errors by making a small modification. Set $y_{ni}^h(b) = u_i(b)\tilde{e}_i$, where $\{u_i(b) : i = 1, ..., n\}$ is an iid $N(0,1)$ sample and $\tilde{e}_i$ are the regression residuals defined in section 2. Construct the bootstrap test statistic $\text{SupF}_n^n(b)$ as before, substituting $y_{ni}^h(b)$ for $y_{ni}(b)$ in the previous paragraph. This has the bootstrap distribution $G_n^h(x) = P\left(\text{SupF}_n^h(b) \leq x \mid \Im_n\right)$ from which we define the bootstrap p-value $p_n^h = 1 - G_n^h\left(\text{SupF}_n\right)$.

While $G_n(\cdot)$ is unknown, it may be calculated by simulation. Let $\text{SupF}_n(j)$, $j = 1, ..., J$ denote (conditionally) independent draws from the distribution $\text{SupF}_n(b)$. The simulated bootstrap p-value $p_n(J)$ is then calculated by counting the percentage of simulated bootstrap test statistics $\text{SupF}_n(j)$ which exceed the sample value SupF_n. As $J \to \infty$, $p_n(J) \to p_n$ almost surely, so the error due to simulation can be made arbitrarily small. For the case of heteroskedastic errors, a simulation estimate of p_n^h, denoted $p_n^h(J)$ may be constructed similarly. We follow the bootstrap literature (see Hall (1994)) and consider p_n and p_n^h to be the bootstrap statistics of interest. It is important to remember, however, that inference in practice is based on the randomized estimates $p_n(J)$ or $p_n^h(J)$, which may induce additional error unless J is large.

5.2 Asymptotic Theory

The following notation will be helpful. Let $T(\delta) = \sup_{\pi_1 \leq r \leq \pi_2} F(r \mid \delta)$ so that by Theorem 2, $\text{SupF}_n \rightarrow_d T(\delta)$ and $T(0)$ denotes the null distribution. Let $G(x) = P\left(T(0) \leq x\right)$ denote the null asymptotic distribution function, and define the random variable $p(\delta) = 1 - G\left(T(\delta)\right).$ Note that the distribution of $p(0)$ is $U[0,1]$.

Let " $\Rightarrow_p$ " denote weak convergence in probability as defined by Gine and Zine (1990). The concept "weak convergence in probability" generalizes convergence in distribution to allow for conditional (i.e. random) distribution functions. This is necessary for bootstrap theory as the empirical distribution used for re-sampling is data-dependent. We first state the results for the homoskedastic bootstrap.

Theorem 5 *Under Assumptions 1, 2, and (4),* $\text{SupF}_n(b) \Rightarrow_p T(0)$ *and* $p_n \rightarrow_d p(\delta)$*.*

Corollary 1 *Under Assumptions 1, 2, (4), and* H_0*,* $p_n \rightarrow_d U[0,1]$*.*

The first result of Theorem 5 states that the conditional distribution function $G_n(\cdot)$ is close to $G(\cdot)$ if n is sufficiently large. This means that p-value calculations based on G_n are asymptotically equivalent to those based on G. The second result of Theorem 5 gives the asymptotic distribution of the bootstrap p-value p_n. In particular, we find in Corollary 1 that under H_0, p_n is asymptotically distributed $U[0,1]$, which is pivotal, so the nuisance parameter problem has been solved (for large samples). We now state the result for the heteroskedastic bootstrap.

Theorem 6 *Under Assumptions 1 and 2,* $\text{SupF}_n^n(b) \Rightarrow_p T(0)$ *and* $p_n^h \rightarrow_d p(\delta)$*.*

Corollary 2 *Under Assumptions 1, 2, and* H_0*,* $p_n^h \rightarrow_d U[0,1]$*.*

Theorem 6 and Corollary 2 show that the heteroskedastic bootstrap achieves the correct asymptotic distribution under general forms of conditional heteroskedasticity. This holds even though SupF_n has not been constructed to allow for heteroskedasticity.

It is also possible to derive the asymptotic local power functions of the bootstrap tests from Theorems 5 and 6, which allows us to assess the behavior of the tests under (local) alternatives. From Theorem 5, we can calculate that

$$\begin{aligned}\pi_a(\delta) &= \lim_{n\to\infty} P\left(p_n \leq a\right) \\ &= P\left(p(\delta) \leq a\right) \\ &= P\left(T(\delta) > b\right)\end{aligned}$$

where $b = G^{-1}(a)$. The function $\pi_a(\delta)$ is the asymptotic local power function. As $|\delta| \to \infty$ the non-centrality effect in $F(r \mid \delta)$ causes $T(\delta) \to \infty$ almost surely. Hence for any a, $\pi_a(\delta) \to 1$ as $|\delta| \to \infty$. In other words, the asymptotic local power function is increasing in $|\delta|$, and the asymptotic probability of rejecting the null hypothesis can be made arbitrarily high by selecting a sufficiently large δ.

This argument does not prove that the test is consistent against fixed alternatives, however, as θ_n fixed and independent of n is not covered by our theory. A formal proof of consistency against fixed alternatives appears to be quite intricate, and will not be attempted in this paper.

5.3 Small Sample Distributions

To investigate the performance of our bootstrap tests in a small sample, we report a limited Monte Carlo experiment. The regression model is a single equation from a bi-variate VAR:

$$y_i = \alpha_0 + \alpha_1 y_{i-1} + \alpha_2 y_{i-2} + \alpha_3 y_{i-3} + \beta_1 x_{i-1} + \beta_2 x_{i-2} + \beta_3 x_{i-3} + e_i$$

with a sample size of $n = 50$. We set $\alpha_1 = .5$ and $\beta_1 = 1$ while the other regression parameters are set to zero.

Seven models for the regressors x_i are considered. Below, let η_i be iid student t with 5 degrees of freedom, let u_i be iid $N(0, 1)$, and let $c_i = 1 + 2i/n$.

- IID: $x_i = \eta_i$.
- Mean Break: $x_i = \eta_i$ for $i \leq 25$ and $x_i = 5 + \eta_i$ for $i > 25$.
- Variance Break: $x_i = \eta_i$ for $i \leq 25$ and $x_i = 5\eta_i$ for $i > 25$.
- Mean Trend: $x_i = c_i + u_i$.
- Variance Trend: $x_i = \sqrt{c_i} u_i$.
- Stochastic Trend: $x_i = x_{i-1} + u_i$.
- Stochastic Variance: $x_i = \left(\sum_{j=1}^{j} w_i\right) u_i$, where w_i is iid $N(0,1)$.

Two specifications for the regression error e_i are considered. The first is that e_i is iid student t with 5 degrees of freedom. The second is that e_i is conditionally heteroskedastic $e_i \sim N\left(0, 1 + .25x_{i-1}^2\right)$.

The random regressors x_i are independently generated for each Monte Carlo replication. In all experiments, the number of bootstrap replications is $J = 1000$ and the number of simulation replications is 5000. We report results for tests of nominal size 10%.

We restrict attention to the SupF statistic of Andrews (1993) for simplicity. Three distributional approximations are considered. The first uses the asymptotic approximation of Andrews (1993), the second uses the homoskedastic fixed regressor bootstrap, and the third the heteroskedastic fixed regressor bootstrap. In all experiments, the null hypothesis of coefficient stability holds, so the rejection rate should be ideally 10%.

The results are summarized in Table 1. First consider the case of homoskedastic errors, reported in the first half of the Table. In all cases, tests based on the conventional asymptotic approximation substantially over-reject. Tests based on the homoskedastic bootstrap have better size, and tests based on the heteroskedastic bootstrap are close to the correct rejection frequency.

The second half of the table reports results for the heteroskedastic error process. The performance of all test statistic deteriorates significantly. Tests based on the conventional asymptotic approximation are extremely poor. Tests based on the homoskedastic bootstrap are not much improved, which should not be surprising since the technique is not designed

Table 1: Small Sample Type I Error, 10% Nominal Size

	Regressor Process						
	IID	Mean Break	Variance Break	Mean Trend	Variance Trend	Stochastic Mean	Stochastic Variance
Homoskedastic Regression Error							
Asymptotic Distribution	16	21	21	19	18	22	21
Homoskedastic Bootstrap	12	14	14	13	12	15	15
Heteroskedastic Bootstrap	10	9	7	10	09	10	10
Heteroskedastic Regression Error							
Asymptotic Distribution	21	43	71	22	18	40	50
Homoskedastic Bootstrap	14	33	64	15	12	31	42
Heteroskedastic Bootstrap	10	19	34	11	8	20	23

Note: Rejection Frequencies (%) in 5000 Replications. Standard Error of Estimates = 0.4

to be robust to heteroskedasticity. The heteroskedastic bootstrap works much better than the other tests for most regressor processes, achieving correct size in three of the seven cases. In the most extreme cases, however, the heteroskedastic bootstrap also over-rejects relative to the nominal size.

The simulation results show that the fixed regressor bootstrap improves on asymptotic approximations, but does not completely solve the inference problem.

6 Conclusion

This paper has attempted a careful examination of modern tests for structural change and the associated asymptotic distribution theory. We argue that the assumption that the regressors are stationary (without structural change) is inappropriate in the context of testing for structural change in a regression, and that relaxing this assumption has consequences for the asymptotic theory. We found that the Nyblom L statistic is sensitive to this assumption, so is generally not recommended for empirical application in regression models. In a numerical study, the Andrews-Ploberger exponentially-weighted ExpF statistic appears to be the most stable with respect to structural change in the marginal equation.

We show that correct asymptotic inference may be obtained from a bootstrap distribution. We consider a simple fixed regressor bootstrap, which treats the right-hand-side regressors as fixed (even the lagged dependent variables). This produces the correct asymptotic distribution under a wide range of conditions, such as arbitrary structural change in the regressors including multiple structural breaks, time trends and certain stochastic trends. The regressors need not be strictly exogenous and the regression errors may be conditionally heteroskedastic. This procedure is computationally cheap and easy to program. In many contexts, bootstrap techniques can improve on the first-order asymptotic distribution when they approximate an Edgeworth correction (see Hall, 1994). This appears unlikely in the case of the fixed regressor bootstrap. Other bootstrap procedures are conceivable, but care must be taken to correctly mimic the distribution under the null hypothesis, and not distort the possible non-stationarity in the conditioning variables.

A GAUSS program which implements the empirical techniques discussed in this paper is available upon request from the author or his web homepage.

7 Appendix: Mathematical Proofs

Proof of Theorem 1. It is well known that

$$F_t = \frac{\hat{S}_t' \left(M_t - M_t M_n^{-1} M_t\right)^{-1} \hat{S}_t}{\hat{\sigma}_t^2/\sigma^2}, \tag{23}$$

where $M_t = n^{-1} \sum_{i=1}^t x_{ni} x_{ni}'$ and $\hat{S}_t = \left(\sigma^2 n\right)^{-1/2} \sum_{i=1}^t x_{ni}\hat{e}_i$. Under H_0, $\hat{S}_t = S_t - M_t M_n^{-1} S_n$, where $S_t = \left(\sigma^2 n\right)^{-1/2} \sum_{i=1}^t x_{ni} e_{ni}$. By the asymptotic mse-stationarity of x_{ni}, $M_{[nr]} \Rightarrow rM$ where $M = \lim_{n\to\infty} n^{-1} \sum_{i=1}^n E\left(x_{ni} x_{ni}'\right)$. We next show that for any $\alpha \neq 0$,

$$\alpha' \frac{1}{n} \sum_{i=1}^{[nr]} x_{ni} x_{ni}' e_{ni}^2 \alpha \Rightarrow r\alpha' M \alpha \sigma^2. \tag{24}$$

Let $\xi_{ni} = \alpha' x_{ni} x_{ni}' \alpha \left(e_{ni}^2 - \sigma^2\right)$. By (4), ξ_{ni} is a martingale difference array. Thus for any fixed r, by Burkholder's inequality (e.g. Hall and Heyde (1980), p. 23)), for some $C < \infty$,

$$E\left|\frac{1}{n}\sum_{i=1}^{[nr]} \xi_{ni}\right|^q \le C\frac{1}{n^q} E\left|\sum_{i=1}^{[nr]} |\xi_{ni}|^2\right|^{q/2} \le C\frac{1}{n^q}\sum_{i=1}^{[nr]} E\left|\xi_{ni}\right|^q \le 2(\alpha'\alpha)^q C n^{1-q} \sup_{n\ge 1, i\le n} E\left|x_{ni}e_{ni}\right|^{2q} \to 0$$

by (10). By Markov's inequality it follows that $\frac{1}{n}\sum_{i=1}^{[nr]}\xi_{ni}\to_p 0$. Thus pointwise in r,

$$\alpha'\frac{1}{n}\sum_{i=1}^{[nr]}x_{ni}x_{ni}'e_{ni}^2\alpha=\alpha' M_{[nr]}\sigma^2\alpha+\frac{1}{n}\sum_{i=1}^{[nr]}\xi_{ni}\to_p r\alpha' M\alpha\sigma^2.$$

This convergence is also uniform over $0\leq r\leq 1$, since the left-hand argument of (24) is monotonically increasing in r and the limit function is continuous in r (for a proof, see Lemma A.9 of Hansen (1998)). This establishes (24).

Take any $\eta>0$, set $K=\operatorname{tr} M\sigma^2/\eta$ and $r_k=k/K$ for $k=0,...,K$. By (24),

$$\max_{i\leq n}\frac{1}{n}x_{ni}'x_{ni}e_{ni}^2\leq\max_{k\leq K}\sum_{i=[nr_k]+1}^{[nr_{k+1}]}\frac{1}{n}x_{ni}'x_{ni}e_{ni}^2\to_p \operatorname{tr} M\sigma^2/K=\eta. \tag{25}$$

Since $x_{ni}e_{ni}$ is a martingale difference sequence, (24) and (25) imply that $S_{[nr]}\Rightarrow B(r)$, a vector Brownian motion with covariance matrix M, see Davidson (1994), Theorem 27.14. It is not hard to show that $\hat{\sigma}_{[nr]}^2\Rightarrow\sigma^2$. The stated results follow by standard manipulations. □

Proof of Theorem 2. Under H_1 (see (3) and (5)) we can calculate that

$$\begin{aligned}\hat{S}_t &= \frac{1}{\sigma\sqrt{n}}\sum_{i=1}^{t}x_{ni}\hat{e}_i\\ &= S_t-M_tM_n^{-1}S_n+\left(M_t-M_{t_0}\right)I(t\geq t_0)\delta-M_tM_n^{-1}\left(M_n-M_{t_0}\right)\delta\end{aligned}$$

Thus

$$\begin{aligned}\hat{S}_{[nr]} &\Rightarrow N(r)-M(r)M(1)^{-1}N(1)+\left(M(r)-M(\tau_0)\right)I(r\geq\tau_0)\delta\\ &\quad -\left(M(r)-M(r)M(1)^{-1}M(\tau_0)\right)\delta\\ &= N^*(r)-Q(r)\delta,\end{aligned}$$

and

$$\begin{aligned}F_{[nr]} &= \frac{\hat{S}_{[nr]}'\left(M_{[nr]}-M_{[nr]}M_n^{-1}M_{[nr]}\right)^{-1}\hat{S}_{[nr]}}{\hat{\sigma}_{[nr]}^2/\sigma^2}\\ &\Rightarrow \left(N^*(r)-Q(r)\delta\right)'\left(M(r)-M(r)M(1)^{-1}M(r)\right)^{-1}\left(N^*(r)-Q(r)\delta\right)\end{aligned} \tag{26}$$

by the continuous mapping theorem. The results for the test statistics follow from standard manipulations. □

Proof of Theorem 3.

Note that $M(r) = [1+\omega(1-\kappa)]\, v(r)$ where $v(r)$ is defined in (21). WLOG, rescaling allows us to set $M(r) = v(r)$. Let

$$\phi(s) = (1+\omega(1-\kappa))\, s - \frac{\omega}{1+\omega}\left((1+\omega(1-\kappa))\, s - \kappa\right)^{+}. \tag{27}$$

One can verify that $\phi(s)$ is monotonically increasing and $v(\phi(s)) = s$. Thus $M(\phi(s)) = s$, and $M^*(\phi(s)) = s(1-s)$. Further, $W(s) = N(\phi(s))$ is a standard Brownian motion and $W^*(s) = N^*(\phi(s))$ is a Brownian bridge. Making the change of variables $r = \phi(s)$, we conclude that

$$\sup_{\pi_1 \le r \le \pi_2} \frac{N^*(r)^2}{M^*(r)} = \sup_{\pi_1 \le \phi(s) \le \pi_2} \frac{N^*(\phi(s))^2}{M^*(\phi(s))} = \sup_{v(\pi_1) \le s \le v(\pi_2)} \frac{W^*(s)^2}{s(1-s)}.$$

□

Proof of Theorem 4. As in the proof of Theorem 3, rescale so that $M(r) = v(r)$. Note $v(1) = 1$. For the change-of-variables $r = \phi(s)$ defined in (27), the Jacobian is

$$J(s) = \begin{cases} 1+\omega(1-\kappa) & s < \kappa^* \\ \frac{1+\omega(1-\kappa)}{1+\omega} & s \ge \kappa^* \end{cases}$$

where $\kappa^* = \kappa / \left(1+\omega\left(1-\kappa\right)\right)$. As $\omega \to \infty$, $\kappa^* \to 0$ and

$$\begin{aligned} \frac{\int_0^1 N^*(r)^2 dr}{M(1)} &= \int_0^1 N^*(\phi(s))^2 J(s) ds \\ &= (1+\omega(1-\kappa)) \int_0^{\kappa^*} W^*(s)^2 ds + \frac{(1+\omega(1-\kappa))}{1+\omega} \int_{\kappa^*}^1 W^*(s)^2 ds \\ &\to_p (1-\kappa) \int_0^1 W^*(s)^2 ds. \end{aligned}$$

□

Proof of Theorem 5. Define

$$S_t(b) = \frac{1}{n} \sum_{i=1}^{t} x_{ni} u_i(b).$$

Since $u_i(b)$ is iid $N(0,1)$, the stochastic process $S^b_{[nr]}$,conditional on the sample, has an exact distribution as a mean-zero Gaussian process with covariance kernel $M_{[n \min(r,s)]} \Rightarrow$

$M(\min(r,s)) = V(\min(r,s))$ under (4). Hence $S_{[nr]}(b) \Rightarrow_p N(r)$. Thus

$$\begin{aligned} F_t(b) &= \left(S_t(b) - M_t M_n^{-1} S_n(b)\right)' \left(M_t - M_t M_n^{-1} M_t\right)^{-1} \left(S_t(b) - M_t M_n^{-1} S_n(b)\right) / \left(\hat{\sigma}_t^2(b)/\sigma^2\right) \\ &\Rightarrow_p \left(N(r) - M(r)M(1)^{-1}N(1)\right)' M^*(r)^{-1} \left(N(r) - M(r)M(1)^{-1}N(1)\right) \\ &\equiv F(r \mid 0) \end{aligned}$$

and

$$\text{SupF}_n(b) = \sup_{t_1 \le t \le t_2} F_t(b) \Rightarrow_p \sup_{\pi_1 \le r \le \pi_2} F(r \mid 0) \equiv T(0),$$

as stated. This means that $G_n(\cdot)$ converges uniformly in probability to $G(\cdot)$. By the continuous mapping theorem we conclude

$$p_n = 1 - G_n\left(\text{SupF}_n\right) \rightarrow_d 1 - G\left(T(\delta)\right) = p(\delta).$$

□

Proof of Theorem 6. Define

$$S_t(b) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{t} x_{ni} u_i(b) \tilde{e}_i.$$

The stochastic process $S_{[nr]}(b)$ has an exact distribution as a mean-zero Gaussian process with covariance kernel $\hat{V}_n(\min(r,s))$, where

$$\hat{V}_n(r) = \frac{1}{\sigma^2 n} \sum_{i=1}^{[nr]} x_{ni} x_{ni}' \hat{e}_i^2 \Rightarrow V(r).$$

Hence $S_{[nr]}^b \Rightarrow_p N(r)$. The remainder of the proof is identical to Theorem 5. □

References

[1] Andrews, D.W.K. (1993), "Tests for parameter instability and structural change with unknown change point," *Econometrica*, 61, 821-856.

[2] Andrews, D.W.K. and W. Ploberger (1994): "Optimal tests when a nuisance parameter is present only under the alternative," *Econometrica,* 62, 1383-1414.

[3] Basawa, I.V., A.K. Mallik, W.P. McCormick, J.H. Reeves, and R.L. Taylor (1991): "Bootstrapping unstable first-order autoregressive processes," *Annals of Statistics*, 19, 1098-1101.

[4] Bose, A. (1988): "Edgeworth correction by bootstrap in autoregressions," *Annals of Statistics*, 16, 1709-1722.

[5] Carlstein, E. (1986): "The use of subseries methods for estimating the variance of a general statistic from a stationary time series," *Annals of Statistics*, 14, 1171-1179.

[6] Davidson, J. (1994): *Stochastic Limit Theory: An Introduction for Econometricians*. Oxford: Oxford University Press.

[7] Efron, B. (1979): "Bootstrap methods: Another look at the jackknife," *Annals of Statistics*, 7, 1-26.

[8] Engle, R.F. and D.F. Hendry (1993): "Testing superexogeneity and invariance in regression models," *Journal of Econometrics*, 56, 119-139

[9] Engle, R.F., D.F. Hendry, and J.F. Richard (1983): "Exogeneity," *Econometrica* 51, 277-304.

[10] Gine, E. and J. Zinn (1990): "Bootstrapping general empirical measures," *The Annals of Probability*, 18, 851-869.

[11] Hall, P. (1994): "Methodology and theory for the bootstrap," *Handbook of Econometrics, Vol. IV*, eds. R.F. Engle and D.L. McFadden. New York: Elsevier Science.

[12] Hall, P. and C.C. Heyde (1980): *Martingale Limit Theory and Its Application*. New York: Academic Press.

[13] Hansen, B.E. (1997): "Approximate asymptotic p-values for structural change tests," *Journal of Business and Economic Statistics*, 15, 60-67.

[14] Hansen, B.E. (1998): "Sample splitting and threshold estimation," working paper, Boston College.

[15] Hendry, D.F. (1988): "The encompassing implications of feedback versus feedforward mechanisms in econometrics," *Oxford Economic Papers*, 40, 132-149.

[16] Künsch, H.R. (1989): "The jackknife and the bootstrap for general stationary observations," *Annals of Statistics*, 17, 1217-1241.

[17] Nyblom, J. (1989): "Testing the Constancy of Parameters Over Time," *Journal of the American Statistical Association*, 84, 223-230.

[18] Phillips, P.C.B. and M. Loretan (1991): "Estimating long run economic equilibria," *Review of Economic Studies*, 58, 407-436.

[19] Quandt, R. (1960): "Tests of the hypothesis that a linear regression obeys two separate regimes," *Journal of the American Statistical Association*, 55, 324-330.

[20] Saikkonen, P. (1991): “Asymptotically efficient estimation of cointegration regressions,” *Econometric Theory*, 7, 1-21.

[21] Stock, J.H. and M.W. Watson (1993): “A simple estimator of cointegrating vectors in higher order integrated systems,” *Econometrica,* 61, 783-821.

[22] Stock, J.H. and M.W. Watson (1996): “Evidence on structural instability in macroeconomic time series relations,” *Journal of Business and Economic Statistics*, 14, 11-30.

Figure 1: λ^* as a function of ω and κ

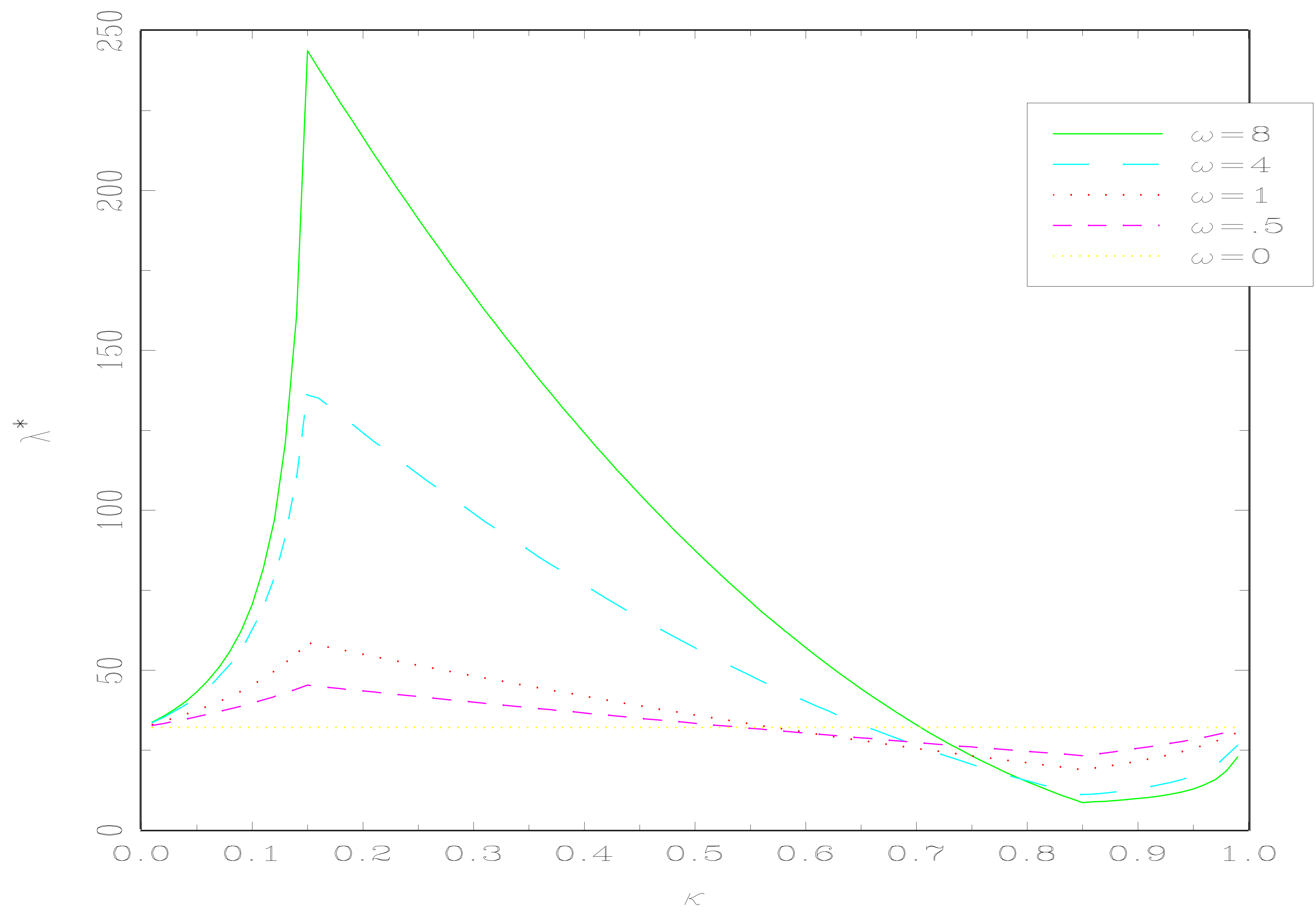

Figure 2: Asymptotic Type I Error as a Function o

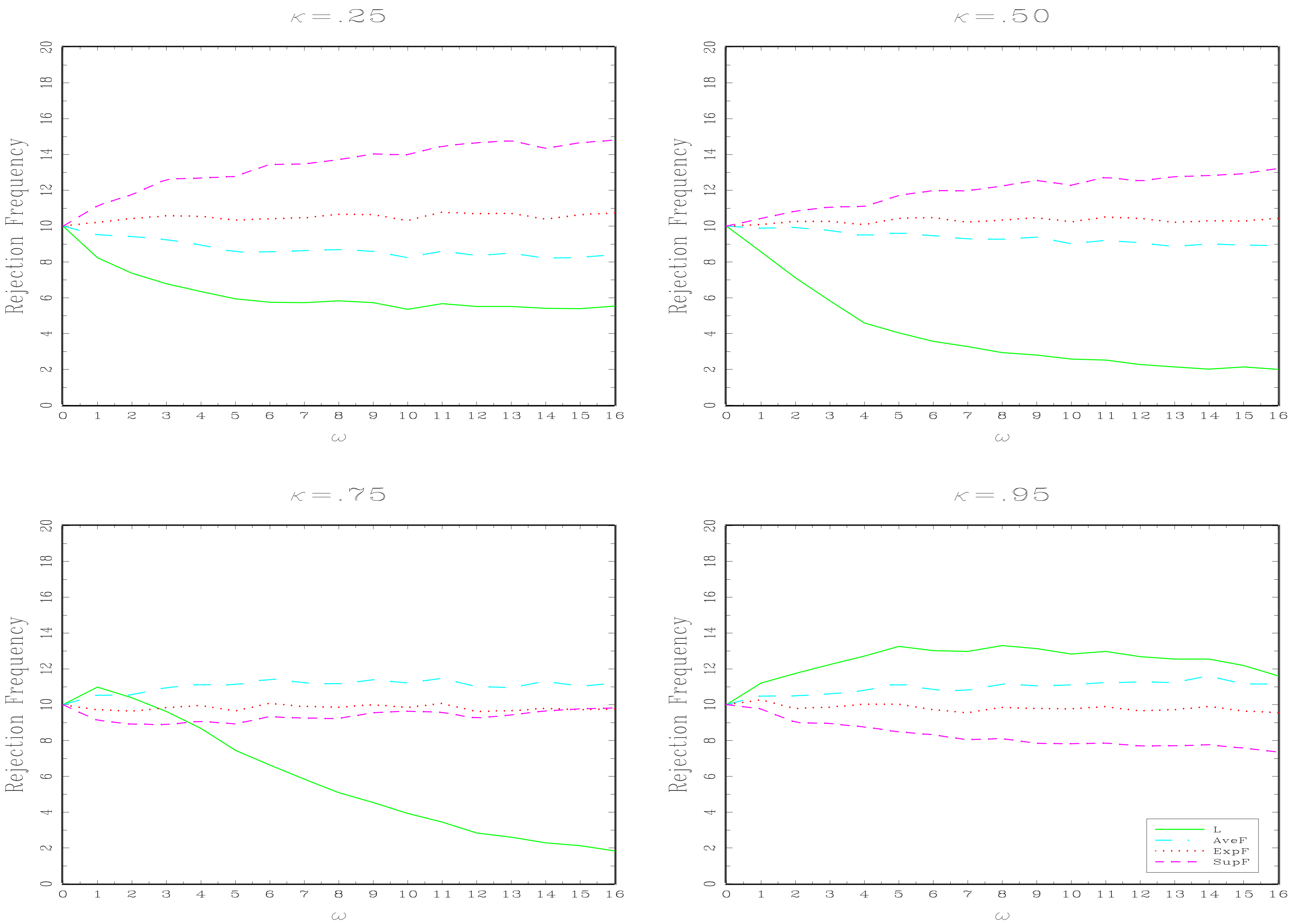

Figure 3: Asymptotic Local Power as a Function of