Implementing the Friedman Rule

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Abstract

In cash-in-advance models, necessary and sufficient conditions for the existence of an equilibrium with zero nominal interest rates and Pareto optimal allocations place restrictions only on the very long-run, or asymptotic, behavior of the money supply. When these asymptotic conditions are satisfied, they leave the central bank with a great deal of flexibility to manage the money supply over any finite horizon. But what happens when these asymptotic conditions fail to hold? This paper shows that the central bank can still implement the Friedman rule if its actions are appropriately constrained in the short run.

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1. Introduction

Milton Friedman (1969) presents his famous rule for optimal monetary policy-making. "Our final rule for the optimum quantity of money," he writes (p.34), "is that it will be attained by a rate of price deflation that makes the nominal rate of interest equal to zero." Friedman also suggests that this rule can be implemented by steadily contracting the money supply at the representative household's rate of time preference.

Wilson (1979) and Cole and Kocherlakota (1998) assess Friedman's proposals using fully-specified, general equilibrium models in which money is introduced through the imposition of a cash-in-advance constraint. These authors confirm the relevance of the Friedman rule by demonstrating that equilibrium allocations are efficient if and only if the nominal interest rate equals zero. But they also find that the Friedman rule can be implemented through any one from a broad class of monetary policies. Some of these policies call for the money supply to expand over an arbitrarily long, but finite, horizon; others call for the money supply to contract, but at a rate that is always slower than the representative household's rate of time preference. In fact, Wilson and Cole and Kocherlakota show that necessary and sufficient conditions for the existence of an equilibrium

with zero nominal interest rates and Pareto optimal allocations place restrictions only on the asymptotic behavior of the money supply: these restrictions simply require the money supply to eventually contract at a rate that is no faster than the representative household's rate of time preference.

For central bankers who wish to implement the Friedman rule, these asymptotic conditions are a double-edged sword. For when the conditions are satisfied, they leave the policymaker with considerable leeway in managing the money supply over any finite horizon. But what should the central banker do when, for some reason, these asymptotic conditions fail to hold? Must the Friedman rule be abandoned altogether? Or is there still a way to manage the money supply so that nominal interest rates are zero and equilibrium allocations are efficient, at least in the short run?

To answer these questions, section 2 sets up a cash-in-advance model like those used by Wilson (1979) and Cole and Kocherlakota (1998) and, for the sake of completeness, restates the asymptotic conditions that are both necessary and sufficient for implementing the Friedman rule over the infinite horizon. Section 3 then assumes that these asymptotic conditions do not hold and characterizes optimal monetary policies in this alternative case. Section 4 concludes by reinterpreting and extending the results of section 3 using a version of the model in

which private agents are boundedly rational in a very special way.

2. A Cash-in-Advance Model

An infinitely-lived representative household is endowed with one unit of productive time during each period $t = 0, 1, 2, ...$ Its preferences are described by

$$\sum_{t=0}^{\infty} \beta^t u(c_t, 1 - n_t),$$

where c_t denotes its consumption and $1 - n_t$ its leisure during period t. The discount factor satisfies $1 > \beta > 0$. The single-period utility function u is strictly increasing in both arguments, strictly concave, and twice continuously differentiable. Let u_i and u_{ij}, $i, j = 1, 2$, denote the first and second derivatives of u, and for $y \in (0, 1)$, define

$$V(y) = \frac{u_1(y, 1 - y)}{u_2(y, 1 - y)}.$$

It will be useful in what follows to assume that V is strictly decreasing with $\lim_{y \to 0} V(y) = \infty$ and $\lim_{y \to 1} V(y) = 0$. Since u is strictly increasing and concave, a sufficient condition for $V'(y) < 0$ is $u_{12} \geq 0$.

The household enters period t with money M_t and bonds B_t. The goods mar-

ket opens first; here, the description of production and trade draws on Lucas'
(1980) interpretation of the cash-in-advance model. Suppose that the representative household consists of two members: a shopper and a worker. The shopper
purchases consumption from workers from other households at the nominal price
P_t, subject to the cash-in-advance constraint

$$\frac{M_t}{P_t} \geq c_t.$$

Meanwhile, the worker produces output according to the linear technology $y_t = n_t$
and sells this output to shoppers from other households for $P_t n_t$ units of money.

The asset market opens last. In this end-of-period asset market, the representative household receives a lump-sum nominal transfer H_t from the central bank
and the household's bonds mature, providing B_t additional units of money. The
household spends $B_{t+1}/(1+r_t)$ on new bonds, where r_t is the net nominal interest
rate, and carries M_{t+1} units of money into period $t + 1$. The household's budget
constraint is therefore

$$\frac{M_t + H_t + B_t}{P_t} + n_t \geq c_t + \frac{B_{t+1}/(1 + r_t) + M_{t+1}}{P_t}.$$

In addition to the cash-in-advance and budget constraints, the household's choices must satisfy the nonnegativity constraints

$$c_t \geq 0, \ 1 \geq n_t \geq 0, \ M_{t+1} \geq 0.$$

And while the household is allowed to borrow by choosing negative values of B_{t+1}, it is not permitted to borrow more than it can ever repay. Let Q_t denote the present discounted value in the period-0 asset market of one unit of money received in the period-t asset market, so that $Q_0 = 1$ and

$$Q_t = \prod_{s=0}^{t-1} \left(\frac{1}{1+r_s} \right)$$

for $t = 1, 2, 3, ...$ Then the no-Ponzi-game constraint can be formalized as

$$W_{t+1} = Q_t \left(M_{t+1} + \frac{B_{t+1}}{1+r_t} \right) + \sum_{s=t+1}^{\infty} Q_s(H_s + P_s n_s) \geq 0.$$

Thus, the representative household chooses $\{c_t, n_t, M_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ to maximize its utility subject to the cash-in-advance constraint, the budget constraint, the nonnegativity constraints, and the no-Ponzi-game constraint, each of which must hold for all $t = 0, 1, 2, ...$ The appendix shows that when the market-clearing

conditions

$$y_t = c_t = n_t, \; M_{t+1} = M_t + H_t, \; B_{t+1} = 0$$

are imposed, necessary and sufficient conditions for a solution to the household's

problem can be written as

$$u_1(y_t, 1 - y_t) = \lambda_t + \mu_t, \tag{1}$$

$$u_2(y_t, 1 - y_t) = \lambda_t, \tag{2}$$

$$\frac{\lambda_t}{P_t} = \frac{\beta(\lambda_{t+1} + \mu_{t+1})}{P_{t+1}}, \tag{3}$$

$$\frac{\lambda_t}{(1 + r_t)P_t} = \frac{\beta\lambda_{t+1}}{P_{t+1}}, \tag{4}$$

and

$$\mu_t \geq 0, \; \frac{M_t}{P_t} \geq y_t, \; \mu_t \left(\frac{M_t}{P_t} - y_t\right) = 0 \tag{5}$$

for all $t = 0, 1, 2, \ldots$ and

$$\lim_{t \to \infty} \frac{\beta^t \lambda_t M_{t+1}}{P_t} = 0, \tag{6}$$

where λ_t and μ_t are Lagrange multipliers on the period-t budget and cash-in-

advance constraints. Accordingly, an equilibrium can be defined as a set of se-

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quences $\{y_t, \lambda_t, \mu_t, r_t, P_t, M_{t+1}\}_{t=0}^{\infty}$ that satisfy (1)-(6), with the initial condition M_0 pinned down by a choice of nominal units.

Under the maintained assumptions on the household's utility function, there is a unique symmetric Pareto optimal allocation for this economy. This allocation has $y_t = y^*$ for all $t = 0, 1, 2, ...$, where y^* is the unique value that satisfies the efficiency condition $V(y^*) = 1$: the marginal rate of substitution between leisure and consumption equals the corresponding marginal rate of transformation. What monetary policies, defined as sequences $\{M_{t+1}\}_{t=0}^{\infty}$, allow for the existence of an equilibrium in which allocations are Pareto optimal? To answer this question, Wilson (1979) and Cole and Kocherlakota (1998) obtain results like the following.

Proposition 1 An equilibrium with $y_t = y^*$ for all $t = 0, 1, 2, ...$ exists if and only if

$$\inf_t \beta^{-t} M_t > 0 \tag{7}$$

and

$$\lim_{t \to \infty} M_{t+1} = 0. \tag{8}$$

Proof To begin, suppose that (7) and (8) are satisfied, and set $y_t = y^*$, $\lambda_t = u_1(y^*, 1 - y^*) = u_2(y^*, 1 - y^*)$, $\mu_t = 0$, $r_t = 0$, and $P_t = \beta^t P_0$ for all $t = 0, 1, 2, ...$, where $P_0 > 0$ is chosen below. Clearly, these values satisfy

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(1)-(4). Since $\mu_t = 0$, (5) requires that

$$\beta^{-t} M_t \geq P_0 y^*$$

for all $t = 0, 1, 2, \ldots$. But (7) guarantees the existence of an $\varepsilon > 0$ such that $\beta^{-t} M_t \geq \varepsilon$ for all $t = 0, 1, 2, \ldots$, and thereby allows this last condition to be satisfied for any choice of $P_0 \leq \varepsilon/y^*$. Meanwhile, (8) guarantees that (6) will hold. Thus, (7) and (8) are sufficient conditions for the existence of an optimal equilibrium.

Next, suppose that an equilibrium with $y_t = y^*$ for all $t = 0, 1, 2, \ldots$ exists. By (1)-(4), $\lambda_t = u_1(y^*, 1-y^*) = u_2(y^*, 1-y^*)$, $\mu_t = 0$, $r_t = 0$, and $P_t = \beta^t P_0 > 0$ for all $t = 0, 1, 2, \ldots$ in any such equilibrium. Thus, (5) requires that

$$\beta^{-t} M_t \geq P_0 y^* > 0$$

for all $t = 0, 1, 2, \ldots$, which implies that (7) must be satisfied. Meanwhile (6) implies that (8) must hold. This establishes that (7) and (8) are also necessary conditions for the existence of an optimal equilibrium, completing the proof.

Proposition 1 and its proof support Friedman's (1969) assertion that Pareto optimal allocations are associated with price deflation and zero nominal interest rates. Friedman also suggests that his zero-nominal-interest-rate rule can be implemented by steadily contracting the money supply at the representative household's rate of time preference and, indeed, the policy that sets $M_t = \beta^t M_0$ for all $t = 0, 1, 2, ...$ satisfies both (7) and (8). As emphasized by Wilson (1979) and Cole and Kocherlakota (1998), however, many other monetary policies also satisfy (7) and (8), including ones that call for positive rates of money growth over arbitrarily long, but finite, horizons, and ones that set $M_t = \gamma^t M_0$, with $1 > \gamma \geq \beta$, for all $t = 0, 1, 2, ...$

In fact, (7) and (8) impose restrictions only on the very long-run behavior of the money supply. Condition (7) places a lower bound on the asymptotic money growth rate: since the gross inflation rate equals β under the Friedman rule, the money stock must eventually grow at a rate that is at least as large as β if the cash-in-advance constraint is not to become binding. Condition (8) places an upper bound on the asymptotic money growth rate: evidently, the money supply must eventually contract if the nominal interest rate is to remain at zero. Together, therefore, (7) and (8) simply require the money supply to asymptotically contract at a rate no faster than the household's rate of time preference.

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3. Implementing the Friedman Rule in the Short Run

When (7) and (8) hold, they leave the central bank with a great deal of flexibility; in fact, they allow the central bank to choose *any* time path for the money supply over *any* finite horizon while still implementing the Friedman rule. But what is a central banker to do when (7) or (8) fails to hold?

When (7) fails to hold, the money supply contracts asymptotically at a rate that exceeds the representative household's rate of time preference. A second result, also adapted from Cole and Kocherlakota (1998), is useful in considering this case.

Proposition 2 Let the single-period utility function take the form

$$u(c, 1 - n) = \ln(c) + v(1 - n),$$

where v is strictly increasing, strictly concave, and twice continuously differentiable with $\lim_{n \to 1} v'(1 - n) = \infty$. If $M_{t+1}/M_t < \beta$ for all $t = 0, 1, 2, ...,$ then no equilibrium exists.

Proof When $M_{t+1}/M_t < \beta$ for all $t = 0, 1, 2, ...,$ (7) fails to hold. Hence, by proposition 1, there is no equilibrium with $y_t = y^*$ for all $t = 0, 1, 2, ...$ So

suppose there is an equilibrium with $y_t = \tilde{y} \neq y^*$ for some t. Equations (1),

(2), and (5) then imply

$$\frac{1}{\tilde{y}} = \lambda_t + \mu_t \geq \lambda_t = v'(1 - \tilde{y}),$$

a condition that, along with the concavity of v, rules out the possibility that

$\tilde{y} > y^*$. Hence, any such equilibrium must have $y_t = \tilde{y} < y^*$ and $\mu_t > 0$.

But in this case, (5) also implies that $M_t = P_t\tilde{y}$. Using this result, together

with (1), (3), and (5) again,

$$\frac{1}{M_t} = \frac{1}{P_t\tilde{y}} \geq \frac{\beta}{P_{t+1}y_{t+1}} \geq \frac{\beta}{M_{t+1}}$$

or, more simply, $M_{t+1}/M_t \geq \beta$. But this contradicts the original assumption

that $M_{t+1}/M_t < \beta$ for all $t = 0, 1, 2, ...$; evidently, no equilibrium exists.

Proposition 2 suggests that when (7) fails to hold, the problem has to do with

the possible nonexistence of an equilibrium, rather than merely the suboptimality

of equilibrium allocations. What happens when a central bank adopts a policy

that is inconsistent with the existence of an equilibrium? Exploring the subtleties

of this issue is left for future research; instead, the remainder of this paper will

focus on the case in which the conditions of proposition 1 are violated because (8) fails to hold.

Suppose, for example, that a central banker is appointed at the beginning of period 0 and granted the authority to choose $\{H_t\}_{t=0}^{T-1}$, the monetary transfers for the first T periods. With the initial condition M_0 taken as given, this central banker's control over $\{H_t\}_{t=0}^{T-1}$ provides him or her with control over $\{M_{t+1}\}_{t=0}^{T-1}$, the time path for the money supply through the beginning of period T.

This central banker's term lasts only T periods, however: during period T, a new central banker takes over and arbitrarily decides that the money supply will grow at the constant gross rate $\pi \geq 1$, so that $M_{T+j} = \pi^j M_T$ for all $j = 0, 1, 2, \dots$. Under the maintained assumptions on the household's utility function, there is a unique steady-state equilibrium under the constant money growth rate π, in which output y_t and real balances $m_t = M_t/P_t$ are both constant and equal to $\bar{y}$, where $\bar{y} < y^*$ is the unique value that satisfies $V(\bar{y}) = \pi/\beta$. So suppose in addition that, independent of the first central banker's decisions, $y_{T+j} = m_{T+j} = \bar{y}$ for all $j = 0, 1, 2, \dots$.

The assumption that $\pi \geq 1$ implies that (8) will not hold when the first central banker takes office at the beginning of period 0. The question now becomes: can this first central banker, through an appropriate choice of $\{M_{t+1}\}_{t=0}^{T-1}$, nevertheless

guarantee the existence of an equilibrium in which nominal interest rates are zero and allocations are efficient, at least in the short run?

As a first step in answering this question, note that with $M_{T+j} = \pi^j M_T$ and $y_{T+j} = m_{T+j} = \bar{y}$ for all $j = 0, 1, 2, ...$, (1)-(5) are satisfied for all $t = T, T+1, T+2, ...$ and (6) is satisfied as well. Hence, the values of concern to the first central banker, $\{y_t, \lambda_t, \mu_t, r_t, P_t, M_{t+1}\}_{t=0}^{T-1}$, need only satisfy (1), (2), and (5) for all $t = 0, 1, ..., T-1$, (3) and (4) for all $t = 0, 1, ..., T-2$,

$$\frac{\lambda_{T-1}}{P_{T-1}} = \frac{\beta u_1(\bar{y}, 1 - \bar{y})}{M_T / \bar{y}}, \tag{9}$$

and

$$\frac{\lambda_{T-1}}{(1 + r_{T-1}) P_{T-1}} = \frac{\beta u_2(\bar{y}, 1 - \bar{y})}{M_T / \bar{y}}, \tag{10}$$

where these last two conditions correspond to (3) and (4) for $t = T - 1$ and make use of the fact that in the inefficient steady state, $\lambda_T = u_2(\bar{y}, 1 - \bar{y})$, $\mu_T = u_1(\bar{y}, 1 - \bar{y}) - u_2(\bar{y}, 1 - \bar{y}) > 0$, and $m_T = M_T / P_T = \bar{y}$. These observations are useful in establishing the following.

Proposition 3 Suppose that $M_{T+j} = \pi^j M_T$, with $\pi \geq 1$, for all $j = 0, 1, 2, ...$ and that from period T forward, the economy is in its unique steady state, with $y_{T+j} = m_{T+j} = \bar{y}$ for all $j = 0, 1, 2, ...$. Then an equilibrium with

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$y_t = y^*$ for all $t = 0, 1, ..., T - 1$ exists if and only if

$$M_T > 0 \tag{11}$$

and

$$M_t \geq \beta^t \left[\frac{u_1(y^*, 1 - y^*)y^*}{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}} \right] M_T \tag{12}$$

for all $t = 0, 1, ..., T - 1$.

Proof To begin, suppose that (11) and (12) are satisfied, and set $y_t = y^*$, $\lambda_t = u_1(y^*, 1 - y^*) = u_2(y^*, 1 - y^*)$, $\mu_t = 0$, and

$$P_t = \beta^t \left[\frac{u_1(y^*, 1 - y^*)}{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}} \right] M_T$$

for all $t = 0, 1, ..., T - 1$. In addition, set $r_t = 0$ for all $t = 0, 1, ..., T - 2$, and set $r_{T-1} = V(\bar{y}) - 1$. Equation (11) guarantees that $P_t > 0$ for all $t = 0, 1, ..., T - 1$, as required for the existence of this equilibrium. Clearly, (1) and (2) hold for all $t = 0, 1, ..., T - 1$ and, since $P_{t+1} = \beta P_t$, (3) and (4) hold for all $t = 0, 1, ..., T - 2$. Equations (9) and (10) hold as well. Since

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$\mu_t = 0$, (5) requires that

$$M_t \geq \beta^t \left[\frac{u_1(y^*, 1 - y^*)y^*}{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}} \right] M_T$$

for all $t = 0, 1, ..., T - 1$, but this last condition coincides with (12) and is

therefore guaranteed to hold. Thus, (11) and (12) are sufficient conditions

for the existence of an equilibrium with $y_t = y^*$ for all $t = 0, 1, ..., T - 1$.

Next, suppose that an equilibrium with $y_t = y^*$ for all $t = 0, 1, ..., T - 1$ exists.

By (1)-(3) and (9), $\lambda_t = u_1(y^*, 1 - y^*) = u_2(y^*, 1 - y^*)$, $\mu_t = 0$, and

$$P_t = \beta^t \left[\frac{u_1(y^*, 1 - y^*)}{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}} \right] M_T > 0$$

for all $t = 0, 1, ..., T - 1$ in any such equilibrium; this last condition implies

that (11) must hold. In addition, (5) requires that

$$M_t \geq \beta^t \left[\frac{u_1(y^*, 1 - y^*)y^*}{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}} \right] M_T$$

for all $t = 0, 1, ..., T - 1$, but this condition simply says that (12) must hold.

This establishes that (11) and (12) are also necessary conditions for the

existence of an equilibrium with $y_t = y^*$ for all $t = 0, 1, ..., T - 1$, completing

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the proof.

Before going on to interpret conditions (11) and (12), it is useful to note that proposition 3 holds much more generally. In particular, the assumption that the economy is in a steady state from period T forward is not essential. All that is required is that the monetary policy adopted from period T forward give rise to an equilibrium in which the cash-in-advance constraint binds in period T, so that $M_T/P_T = y_T$ for some $y_T < y^*$. In the more general case, the proof goes through unchanged, with y_T in place of $\bar{y}$. As stated, however, the proposition makes clear that optimal allocations can be achieved in periods $t = 0, 1, ..., T - 1$ even when the rate of money growth is positive for all $t = T, T + 1, T + 2, ...$, even when the cash-in-advance constraint binds for all $t = T, T + 1, T + 2, ...$, and even when allocations are suboptimal for all $t = T, T + 1, T + 2, ...$

Proposition 3 implies that the Friedman rule need not be abandoned when (8) fails to hold: the central bank can still select $\{M_{t+1}\}_{t=0}^{T-1}$ in a way that guarantees the existence of an equilibrium in which the nominal interest rate is zero for all $t = 0, 1, ..., T - 2$ and allocations are efficient for all $t = 0, 1, ..., T - 1$. Condition (11) simply insures that money is always in positive supply, given that (12) must hold for all $t = 0, 1, ..., T - 1$ and that $M_{t+1} = \pi M_t$ for all $t = T, T + 1, T + 2, ...$ Condition (12), in turn, places upper and lower bounds on the growth rate of the

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money supply and thereby provides finite-horizon analogs to (7) and (8).

Consider (12) for $t = 0$. Since the initial condition M_0 is given, this constraint places an upper bound on M_T:

$$\left[\frac{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}}{u_1(y^*, 1 - y^*)y^*} \right] M_0 \geq M_T. \tag{13}$$

Thus, like (8), (12) implies that money growth must be sufficiently slow if the nominal interest rate is to remain at zero. Given a choice of M_T that satisfies (13), (12) also places lower bounds on M_t, $t = 1, 2, ..., T - 1$. Thus, like (7), (12) implies that money growth must be sufficiently large if the cash-in-advance constraint is not to become binding.

Conditions (11) and (12) still leave the central bank with a great deal of flexibility in choosing its policy: the money supply can expand for the first $T - 1$ periods, for instance, so long as it eventually contracts so that (13) holds. Unlike (7) and (8), however, (11) and (12) do constrain the money supply over a finite horizon. Thus, proposition 3 implies that the central bank must act in the short run in order to implement the Friedman rule in the short run.

4. Interpretation as Limited Forecast Equilibria

Extending the example from the previous section, suppose that the central banker at period 0 announces a policy $\{M_{t+1}\}_{t=0}^{T-1}$ that satisfies the conditions of proposition 3 and thereby succeeds in giving rise to an equilibrium with $r_0 = 0$ and $y_0 = y^*$. But now, suppose that at the beginning of period 1, the representative household discovers that this first central banker will also be permitted to choose M_{T+1} and thereby delay by one period the economy's convergence to its inefficient steady state. And suppose further that this scenario repeats itself over the infinite horizon: at the beginning of each period $t = 0, 1, 2, ...$, the household believes that the money supply $\{M_{t+j}\}_{j=1}^{T}$ over the next T periods will be chosen by the benevolent central banker and that the money supply will grow at the constant gross rate $\pi \geq 1$ thereafter. In this case, the representative household behaves like the boundedly rational players in Jehiel's (1998) game-theoretic framework, having perfect foresight over the first T periods but having what might be considered vague, and in this case incorrect, beliefs about what will happen beyond this limited forecast horizon.

The logic used to prove proposition 3 now implies that the central bank can guarantee the existence of an equilibrium with $r_t = 0$ and $y_t = y^*$ for all $t =$

$0, 1, 2, ...$ by choosing $\{M_{t+1}\}_{t=0}^{T-1}$ at the beginning of period 0 to satisfy (11) and (12) and by choosing $M_{t+T} > 0$ at the beginning of each period $t = 1, 2, 3, ...$ to satisfy the appropriately generalized version of (12):

$$M_{t+j} \geq \beta^j \left[\frac{u_1(y^*, 1 - y^*)y^*}{\beta^T u_1(\bar{y}, 1 - \bar{y})\bar{y}} \right] M_{t+T} \tag{14}$$

for all $j = 0, 1, ..., T - 1$.

Thus, when the representative household is boundedly rational in this very special way, the Friedman rule can be implemented over the infinite horizon by any policy $\{M_{t+1}\}_{t=0}^{\infty}$ that satisfies $M_{t+1} > 0$ and (14) for all $t = 0, 1, 2, ...$. Like (12), but unlike (7) and (8), (14) places constraints on the short-run behavior of the money supply. Once again, therefore, the central bank must act in the short run to implement the Friedman rule.

It should be noted, however, that the boundedly rational household in this example does not learn: it continues to believe that the economy will eventually move to its inefficient steady state, even though this steady state is never reached. What monetary policies will implement the Friedman rule in environments where private agents' expectations gradually change in response to the observed actions of the central bank? This is another question for future research.

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5. Appendix

This appendix shows how (1)-(6) in the text can be derived from conditions that are both necessary and sufficient for a solution to the representative household's optimization problem. Since the household's utility function is strictly concave, the necessary conditions for optimality include the usual first-order conditions, which are given by

$$u_1(c_t, 1 - n_t) = \lambda_t + \mu_t, \tag{A.1}$$

$$u_2(c_t, 1 - n_t) = \lambda_t, \tag{A.2}$$

$$\frac{\lambda_t}{P_t} = \frac{\beta(\lambda_{t+1} + \mu_{t+1})}{P_{t+1}}, \tag{A.3}$$

$$\frac{\lambda_t}{(1 + r_t)P_t} = \frac{\beta\lambda_{t+1}}{P_{t+1}}, \tag{A.4}$$

$$\frac{M_t + H_t + B_t}{P_t} + n_t = c_t + \frac{B_{t+1}/(1 + r_t) + M_{t+1}}{P_t}, \tag{A.5}$$

and

$$\mu_t \geq 0, \frac{M_t}{P_t} \geq c_t, \mu_t\left(\frac{M_t}{P_t} - y_t\right) = 0 \tag{A.6}$$

for all $t = 0, 1, 2, ...$, where λ_t and μ_t are Lagrange multipliers on the period-t budget and cash-in-advance constraints.

The necessary conditions also include the transversality condition

$$\lim_{t \to \infty} W_{t+1} = \lim_{t \to \infty} Q_t \left(M_{t+1} + \frac{B_{t+1}}{1 + r_t} \right) = 0. \tag{A.7}$$

To derive (A.7), note first that the sequence $\{W_{t+1}\}_{t=0}^{\infty}$ is nonincreasing since, using the period-t budget constraint,

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\begin{aligned}
W_{t+1} - W_t &= Q_t \left(M_{t+1} + \frac{B_{t+1}}{1 + r_t} \right) - Q_{t-1} \left(M_t + \frac{B_t}{1 + r_{t-1}} \right) - Q_t (H_t + P_t n_t) \\
&\leq -Q_t P_t c_t - (Q_{t-1} - Q_t) M_t.
\end{aligned}
$$

In any equilibrium, $P_t > 0$, $c_t \geq 0$, $M_t \geq 0$, and $Q_{t-1} \geq Q_t > 0$ must hold; it therefore follows from the expression above that $W_{t+1} \leq W_t$ must also hold.

Next, note that if $\{c_t, n_t, M_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ solve the household's problem, the implied sequence $\{W_{t+1}\}_{t=0}^{\infty}$ must satisfy $\inf_t W_{t+1} = 0$. To see this, suppose to the contrary that there exists an $\varepsilon > 0$ such that $W_{t+1} \geq \varepsilon$ for all $t = 0, 1, 2, \ldots$ and construct alternative sequences $\{\tilde{c}_t, \tilde{n}_t, \tilde{M}_{t+1}, \tilde{B}_{t+1}\}_{t=0}^{\infty}$ that coincide with $\{c_t, n_t, M_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ except for

$$\tilde{c}_1 = c_1 + \frac{\varepsilon}{P_1},$$

$$\tilde{M}_1 = M_1 + \varepsilon,$$

and

$$\tilde{B}_{t+1} = B_{t+1} - \frac{\varepsilon}{Q_{t+1}}$$

for all $t = 0, 1, 2, ...$ These alternative sequences satisfy all of the cash-in-advance, budget, nonnegativity, and no-Ponzi-game constraints and provide the household with a higher level of utility than the original sequences, which contradicts the assumption that the original sequences are optimal. Thus, $\inf_t W_{t+1} = 0$ must hold.

Together, $\{W_{t+1}\}_{t=0}^{\infty}$ nonincreasing and $\inf_t W_{t+1} = 0$ imply that (A.7) must hold and that, more generally, the first-order and transversality conditions are necessary for optimality.

It is also possible to show that the first-order and transversality conditions are sufficient for optimality. Suppose that $\{c_t, n_t, M_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ satisfy (A.1)-(A.7) but that $\{\tilde{c}_t, \tilde{n}_t, \tilde{M}_{t+1}, \tilde{B}_{t+1}\}_{t=0}^{\infty}$ satisfy all of the constraints and yield a higher level of utility. Then

$$\lim_{T \to \infty} \sum_{t=0}^{T} \beta^t [u(\tilde{c}_t, 1 - \tilde{n}_t) - u(c_t, 1 - n_t)]$$
$$< \lim_{T \to \infty} \sum_{t=0}^{T} \beta^t [u_1(c_t, 1 - n_t)(\tilde{c}_t - c_t) - u_2(c_t, 1 - n_t)(\tilde{n}_t - n_t)]$$

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$$
\begin{aligned}
=\ & \lim_{T\to\infty}\sum_{t=0}^{T}\beta^{t}[\lambda_t(\tilde{c}_t - c_t) - \lambda_t(\tilde{n}_t - n_t) + \mu_t(\tilde{c}_t - c_t)] \\
\leq\ & \lim_{T\to\infty}\sum_{t=0}^{T}\beta^{t}\lambda_t\left[\frac{\tilde{M}_t - M_t}{P_t} + \frac{\tilde{B}_t - B_t}{P_t} - \frac{\tilde{M}_{t+1} - M_{t+1}}{P_t} - \frac{\tilde{B}_{t+1} - B_{t+1}}{(1 + r_t)P_t}\right] \\
& + \sum_{t=0}^{T}\beta^{t}\mu_t\left(\frac{\tilde{M}_t - M_t}{P_t}\right) \\
=\ & \lim_{T\to\infty}\frac{\beta^{T}\lambda_T(M_{T+1} - \tilde{M}_{T+1})}{P_T} + \frac{\beta^{T}\lambda_T(B_{T+1} - \tilde{B}_{T+1})}{(1 + r_T)P_T} \\
=\ & \left(\frac{\lambda_0}{P_0}\right)\lim_{T\to\infty}\left[Q_T\left(M_{T+1} + \frac{B_{T+1}}{1 + r_T}\right) - Q_T\left(\tilde{M}_{T+1} + \frac{\tilde{B}_{T+1}}{1 + r_T}\right)\right] \\
=\ & -\left(\frac{\lambda_0}{P_0}\right)\lim_{T\to\infty}Q_T\left(\tilde{M}_{T+1} + \frac{\tilde{B}_{T+1}}{1 + r_T}\right) \\
\leq\ & 0
\end{aligned}
$$

by concavity, by the first-order conditions for c_t and n_t, by the budget constraint, the cash-in-advance constraint, and the complementary slackness condition, by the first-order conditions for M_{t+1} and B_{t+1}, by the first-order conditions for M_{t+1} and B_{t+1} again, by the transversality condition, and by the no-Ponzi-game constraint. But all of this contradicts the assumption that $\{\tilde{c}_t, \tilde{n}_t, \tilde{M}_{t+1}, \tilde{B}_{t+1}\}_{t=0}^{\infty}$ yield a higher level of utility than $\{c_t, n_t, M_{t+1}, B_{t+1}\}_{t=0}^{\infty}$.

Thus, (A.1)-(A.7) are both necessary and sufficient conditions for a solution to the household's problem. After the market-clearing conditions are imposed, these equations can be rewritten as (1)-(6) in the text.

6. References

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