On the Shapley-Scarf Economy: The Case of Multiple Types of Indivisible Goods*

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Abstract

We study a generalization of Shapley-Scarf's (1974) economy in which multiple types of indivisible goods are traded. We show that many of the distinctive results from the Shapley-Scarf economy do not carry over to this model, even if agents' preferences are strict and can be represented by additively separable utility functions. The core may be empty. The strict core, if nonempty, may be multi-valued, and might not coincide with the set of competitive allocations. Furthermore, there is no Pareto efficient, individually rational, and strategy-proof social choice rule. We also show that the core may be empty in the class of economies with a single type of indivisible good and agents consuming multiple units, even if no complementarity exists among the goods.

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1. Introduction

One of the most important models in cooperative game theory is that of Shapley and Scarf (1974). In this model there are n agents (or **players** or **traders**), each of whom is initially endowed with one indivisible good (say a house). Each trader also has a preference ordering over the n houses in the market, with different traders in general having different preferences. They then swap houses with one another, each in an effort to obtain the highest possible house on his own preference list.[1]

Shapley and Scarf's fundamental result was that the core of this economy is non-empty, regardless of individual preferences. Moreover, they showed that a competitive equilibrium always exists, using the "top trading cycles" algorithm of David Gale.

In the time since Shapley-Scarf's paper first appeared, other authors have shown that this economy has some remarkable properties. Roth and Postlewaite (1977) proved that if no agent is indifferent between any two houses, then the economy has a unique competitive allocation which is also the unique strict core allocation.[2] Roth (1982) first studied a strategic implication of the Shapley-Scarf economy, and presented a strategy-proof social choice rule for reaching a competitive allocation. Ma (1994) showed that the strict core mechanism is the only rule which is Pareto efficient, individually rational and strategy-proof. Finally, Sönmez (1996, 1999) and Quint (1997) formulated more general matching problems which include the Shapley-Scarf economy as a special case. Quint gives conditions for core nonemptiness in his model, which the Shapley-Scarf economy satisfies. Sönmez proves a general result concerning strategic issues, which also has implication for the Shapley-Scarf economy.

>From the above, we might conclude that the Shapley-Scarf economy is indeed a very "well-behaved" economic model. A natural question is then, "How robust these results are for a perturbation of the specification of the model?" Can we preserve any of these results if we generalize the model in some way?

One possible generalization is to allow the agents to trade Q types of good, where $Q \geq 1$. The $Q = 1$ case is just the Shapley-Scarf model. The $Q = 2$ case was specifically brought up by Hervé Moulin in his recent book (1995, p. 110). Here there are two kinds of goods, say, houses and cars. Assume that each agent initially owns one house and one car. Suppose, moreover, that individual preferences are **separable** (i.e., which car I own does not affect my ranking of various houses and vice versa). Moulin then posed the question: is the core nonempty in this generalized barter economy? Do others of the aforementioned properties of the Shapley-Scarf economy carry over to this setting?

In this paper we provide a comprehensive answer to these questions, by studying a

[1] Moulin (1995) has a nice discussion on this economy.

[2] Some general results are known for the cases where indifferences are allowed. Shapley and Scarf gave an example with an empty strict core. Wako (1984) gave an example in which the strict core is a nonempty proper subset of the set of competitive allocations, and proved that the strict core is in general a subset of the set of competitive allocations. Wako (1991) and Ma (1994) showed that the strict core allocations, if they exist, are unique in terms of utility payoffs for the players.

model in which there are two kinds[3] of good and in which no indifference is allowed in the agents' preferences. First, if n (the number of agents) is three, then the core is always nonempty, but the strict core may be empty. If n is more than three, even the core may be empty. In either case, it is also possible for the strict core to be multi-valued. The set of competitive allocations is still a subset of the strict core, but, if $n > 2$, there is no longer necessarily an equivalence between these sets. Furthermore, there is no Pareto efficient, individually rational, and strategy-proof social choice rule for this economy. We also show that another natural generalization of the Shapley-Scarf economy does not preserve the nonemptiness of the core – a housing market where agents are allowed to consume multiple houses, and their utility functions are additively separable. In conclusion, our results show that the "good behavior" of the Shapley-Scarf economy is heavily dependent on its special setting.

In Section 2, we present the model and analyze the core and the strict core. In Section 3, we show the relationship between the set of competitive allocations and the strict core. In Section 4, we consider a strategic implication of the model. Finally, in the last section, we consider the model in which there is only one type of indivisible good, but in which traders may consume more than one unit of that good.

2. The Model, the Core, and the Strict Core

Consider an economy with agent set $N = \{1, ..., n\}$ and in which there are Q types of indivisible goods. Each agent $i \in N$ owns exactly one good of each type initially, and her endowment is denoted by the Q-vector $(i, ..., i)$. Here the "i" in the qth component represents the type-q good initially owned by i. Note that the sets of existing goods of type-q ($q = 1, ..., Q$) can each also be denoted by N. Finally, let 2^N represent the set of subsets of N.

We assume that each agent desires to consume exactly one unit of each type. Agent i's preferences over possible such bundles are modeled in the standard way via a complete, reflexive and transitive **preference ordering** $\succsim_i$. Hence, $(j_1, ..., j_Q) \succ_i (k_1, ..., k_Q)$ means that i strictly prefers bundle $(j_1, ..., j_Q)$ to $(k_1, ..., k_Q)$; $(j_1, ..., j_Q) \sim_i (k_1, ..., k_Q)$ means he is indifferent; and $(j_1, ..., j_Q) \succsim_i (k_1, ..., k_Q)$ means either of these two. We say that i has **strict preferences** if he is never indifferent between distinct bundles, i.e., $(j_1, ..., j_Q) \sim_i (k_1, ..., k_Q) \Longleftrightarrow j_q = k_q$ for $q = 1, ..., Q$. In this case we say he has a **strict preference ordering**, which we denote by $\succ_i$.

Because of the assumption that each agent wants only one good of each type, there are a limited number of interesting feasible outcomes, which we call **allocations**. An allocation is a function $x : N \to N^Q \equiv N \times ... \times N$ with $x_1(N) = x_2(N) = ... = x_Q(N) = N$, where $x_q(i)$ denotes the type-q good that agent i obtains.[4] Hence, all of

[3] So we consider the case where $Q = 2$. All of our "negative" results for this case imply the same negative results for higher values of Q.

[4] Hence, $x(i) = (x_1(i), ..., x_Q(i))$. In addition, the notation $x_q(S)$ means $\cup_{i \in S} x_q(i)$, etc.

the indivisible goods in the game are distributed amongst the agents, with each agent obtaining precisely one good of each type.

An allocation x is said to be **blocked** by coalition S if there is another allocation y with (a) $y_1(S) = ... = y_Q(S) = S$, and (b) $y(i) \succ_i x(i)$ for all $i \in S$. It is **weakly blocked** by S if condition (b) is replaced by $y(i) \succsim_i x(i)$ for all $i \in S$ and $y(j) \succ_j x(j)$ for at least one $j \in S$. The **core** of the economy is the set of allocations that are not blocked by any coalition $S \subseteq N$, while the **strict core** is the set of allocations that are not weakly blocked by any coalition $S \subseteq N$. Note that the strict core is a subset of the core. An allocation x is said to be **individually rational** if $x(i) \succsim_i (i, ..., i)$ for all $i \in N$. Both the core and the strict core are subsets of the individually rational allocations.

As Moulin (1995) reports, if $Q = 2$ and complementarity between the two types of goods is allowed, then it is easy to find three-player examples with empty cores. Thus, throughout the paper, we assume that each agent i has an **additively separable strict preference ordering**, i.e., a strict preference ordering $\succ_i$ for which there exist real valued functions $u_i^1 : N \to \Re$, ..., $u_i^Q : N \to \Re$ such that for any $(j_1, ..., j_Q)$, $(k_1, ..., k_Q) \in N^Q$ with $(j_1, ..., j_Q) \neq (k_1, ..., k_Q)$,

$$(j_1, ..., j_Q) \succ_i (k_1, ..., k_Q) \Leftrightarrow \sum_{q=1}^{Q} u_i^q(j_q) > \sum_{q=1}^{Q} u_i^q(k_q).$$

We denote by $\mathcal{M}$ this economy with additively separable strict preference orderings.

At this point, we remark that in the case with $Q = 1$, $\mathcal{M}$ is the familiar Shapley-Scarf (1974) houseswapping economy (with strict preferences). The main idea of this paper, though, is to show that many of the properties of the Shapley-Scarf economy do not necessarily carry over to the cases with $Q \geq 2$. To accomplish this, it will be sufficient to consider the case with $Q = 2$; we feel the (negative) inferences for the cases with $Q > 2$ will be obvious.

Hence, for the remainder of Sections 2,3, and 4, unless otherwise specified, let us assume that $Q = 2$.

Our first result is actually a "positive" one. The proof is in the appendix:

Proposition 2.1. *Suppose that there are three agents in economy $\mathcal{M}$ (with $Q = 2$). Then the core is nonempty.*

We comment that if the number of agents is less than three, then it is obvious that the core and the strict core are both nonempty.

To "complete the picture" for the case $n = 3$, we present an example with additively separable strict preferences and an empty strict core. This means that Roth and Postlewaite's (1977) theorem for the Shapley-Scarf case $Q = 1$ does not carry over to economy $\mathcal{M}$ with $Q \geq 2$.

Example 2.2. *Consider the economy which is described by* $N = \{1, 2, 3\}$ *with the following additively separable strict preferences:*[5]

$(1,3) \succ_1 (3,3) \succ_1 (1,2) \succ_1 (1,1) \succ_1$ *anything,*
$(2,3) \succ_2 (2,1) \succ_2 (3,3) \succ_2 (3,1) \succ_2 (2,2) \succ_2$ *anything,*
$(2,1) \succ_3 (2,2) \succ_3 (3,1) \succ_3 (1,1) \succ_3 (3,2) \succ_3 (1,2) \succ_3 (2,3) \succ_3 (3,3) \succ_3 (1,3)$.

Claim: *The strict core is empty in the economy above.*

Proof. It is easy to see that these strict preferences can be represented by additively separable utility functions. We find all the individually rational allocations, and show that each of them is blocked or weakly blocked by some coalition. There are twelve individually rational allocations in the economy (for convenience allocations are denoted as lists of agents' consumption bundles):

$$\begin{aligned}
\text{Allocation } x^1 &= \{(1,1),(2,2),(3,3)\},\\
\text{Allocation } x^2 &= \{(1,1),(3,3),(2,2)\},\\
\text{Allocation } x^3 &= \{(1,2),(2,1),(3,3)\},\\
\text{Allocation } x^4 &= \{(1,3),(2,2),(3,1)\},\\
\text{Allocation } x^5 &= \{(1,3),(3,1),(2,2)\},\\
\text{Allocation } x^6 &= \{(1,2),(3,3),(2,1)\},\\
\text{Allocation } x^7 &= \{(1,2),(2,3),(3,1)\},\\
\text{Allocation } x^8 &= \{(1,3),(2,1),(3,2)\},\\
\text{Allocation } x^9 &= \{(1,2),(3,1),(2,3)\},\\
\text{Allocation } x^{10} &= \{(3,3),(2,1),(1,2)\},\\
\text{Allocation } x^{11} &= \{(3,3),(2,2),(1,1)\},\\
\text{Allocation } x^{12} &= \{(1,1),(2,3),(3,2)\}.
\end{aligned}$$

It is then easy to see that allocations x^1, x^4, and x^5 are each weakly blocked by coalition $\{2,3\}$ via allocation x^2; allocations x^2 and x^6 are each weakly blocked by coalition $\{1,2\}$ via allocation x^3; and allocations $x^3, x^7, x^8, x^9, x^{10}, x^{11}$, and x^{12} are each weakly blocked by $\{1,3\}$ via allocation x^4. Hence, the strict core is empty in this economy.[6] ■

If we consider a three-agent economy $\mathcal{M}$ with not $Q = 2$ but $Q = 3$, then we can find an example with an empty core. See Example 6.2 in the Appendix. Thus, for the case

[5] Both here and in future examples, by "anything" we really mean "any strict preference ordering of the remaining bundles that preserves additive separability". Hence, in this example Trader 1's complete preference ordering over the bundles might be $(1,3) \succ_1 (3,3) \succ_1 (1,2) \succ_1 (1,1) \succ_1 (3,2) \succ_1 (3,1) \succ_1 (2,3) \succ_1 (2,2) \succ_1 (2,1)$, but it could not be $(1,3) \succ_1 (3,3) \succ_1 (1,2) \succ_1 (1,1) \succ_1 (2,2) \succ_1 (2,3) \succ_1 ...$, because the latter is clearly not additively separable.

[6] Allocations x^5, x^6, x^7 and x^8 form the core in this example.

$n = 3$, Proposition 2.1 and Examples 2.2 and 6.2 give a complete answer to the question about core nonemptiness.

For $n > 3$, it turns out that even having preferences which are both additively separable and strict is not enough to guarantee core nonemptiness. The next example is of a four-agent economy with $Q = 2$ and additively separable strict preferences which has an empty core. The strict core is then also empty, since the strict core is a subset of the core.

Example 2.3. *Consider the economy which is described by* $N = \{1, 2, 3, 4\}$ *with the following additively separable strict preferences:*

$$\begin{array}{l}(4,1) \succ_1 (4,2) \succ_1 (1,1) \succ_1 \textit{anything},\\ (2,1) \succ_2 (1,1) \succ_2 (2,3) \succ_2 (2,2) \succ_2 \textit{anything},\\ (3,4) \succ_3 (3,2) \succ_3 (4,4) \succ_3 (3,3) \succ_3 \textit{anything},\\ (3,4) \succ_4 (3,3) \succ_4 (2,4) \succ_4 (4,4) \succ_4 \textit{anything}.\end{array}$$

Claim: *The core is empty in the economy above.*

Proof. Again it is easy to find additively separable utility representations of the preferences. This economy has only four individually rational allocations as below. We show that each of them is blocked by some coalition.

$$\begin{array}{rcl}\text{Allocation } x^1 &=& \{(1,1),(2,2),(3,3),(4,4)\},\\ \text{Allocation } x^2 &=& \{(1,1),(2,3),(3,2),(4,4)\},\\ \text{Allocation } x^3 &=& \{(4,2),(1,1),(3,3),(2,4)\},\\ \text{Allocation } x^4 &=& \{(1,1),(2,2),(4,4),(3,3)\}.\end{array}$$

It is clear that the initial allocation x^1 is blocked by coalition $\{2, 3\}$ via allocation x^2; allocation x^2, which shows trade between $\{2, 3\}$, is blocked by coalition $\{1, 2, 4\}$ via allocation x^3; allocation x^3, which shows trade among $\{1, 2, 4\}$, is blocked by coalition $\{3, 4\}$ via allocation x^4; and allocation x^4, which shows trade between $\{3, 4\}$, is blocked by coalition $\{2, 3\}$ via allocation x^2. Hence, there is no core allocation in this economy. ■

If we consider the strict core of each sub-market independently, then we obtain allocation x^3, which we call a *commodity-wise strict core* allocation. A commodity-wise strict core allocation exists uniquely in economy $\mathcal{M}$. However, as we have shown in the proof, it is not immune to a coalitional deviation. The main difference from the original Shapley-Scarf economy is that having two types of goods, an agent can be better off by taking a less preferable commodity in one type of good, as long as she can get a much more preferable commodity in the other category.

It should be noted that the deviation cycle in the example above is essentially made by agents 2, 3, and 4, and agent 1 is a dummy player who makes the deviation cycle

work by her endowment.[7]

The results mentioned so far are summarized as follows:

Proposition 2.4. *Economy $\mathcal{M}$ has a nonempty core if (1) $n \leq 2$, or (2) $Q = 1$, or (3) $Q = 2$ and $n = 3$. However, it may have an empty core if (4) $Q = 2$ and $n \geq 4$, or (5) $Q \geq 3$ and $n \geq 3$. The strict core may be empty if $Q \geq 2$ and $n \geq 3$.*

Proof. Core nonemptiness follows from the definition of the core for case (1), Shapley and Scarf (1974) for case (2), and Proposition 2.1 for case (3). On the other hand, the core may be empty by Examples 2.3 and 6.2 for cases (4) and (5), respectively. The strict core may be empty by Example 2.2. ■

The last example of the section shows that multiplicity of strict core allocations may appear, even in a case with two agents:

Example 2.5. *Consider the economy which is described by $N = \{1, 2\}$ with the following additively separable strict preferences:*

$$(2,1) \succ_1 (2,2) \succ_1 (1,1) \succ_1 (1,2),$$
$$(1,1) \succ_2 (2,1) \succ_2 (1,2) \succ_2 (2,2).$$

Claim: *The economy above has multiple strict core allocations.*

Proof. It is easy to see that the preferences have additively separable representations. Apparently, there are two strict core allocations: $x^1 = \{(2,2),(1,1)\}$ and $x^2 = \{(2,1),(1,2)\}$. ■

3. The Competitive Equilibrium

In this section, we analyze the relationship between the (strict) core and the set of competitive allocations. A **competitive equilibrium** of economy $\mathcal{M}$ (with $Q = 2$) is a triple (x, p_1, p_2) consisting of an allocation x and a pair of price functions $p_1 : N \to \Re_+$ and $p_2 : N \to \Re_+$ such that for any $i \in N$, (i) $\sum_{q=1}^{2} p_q(x_q(i)) \leq \sum_{q=1}^{2} p_q(i)$, and (ii) if $(j,k) \succ_i x(i)$, then $p_1(j) + p_2(k) > \sum_{q=1}^{2} p_q(i)$. The first condition is that prices are such that allocation x is "affordable" for each agent i; condition (ii) implies that x gives each agent his favorite two-good bundle out of the ones he can afford. Note that by adding over all $i \in N$, we can substitute an "=" for the "$\leq$" in (i).

If x is an allocation for which there exists a comptetive equilibrium (x, p_1, p_2), we call x a **competitive allocation**. We now show that any competitive allocation is in the strict core.

[7] Wako (1997) showed that the commodity-wise strict core allocation of economy $\mathcal{M}$ is the unique Coalition-Proof Nash outcome of a simple strategic form game. Furthermore, Miyagawa (1997) characterized it as the unique solution that satisfies individual rationality, unanimity and a strong version of strategy proofness.

Proposition 3.1. *Every competitive allocation is a strict core allocation in economy* $\mathcal{M}$.

Proof. Suppose that a competitive allocation x is weakly blocked by a coalition S via an allocation y with $y_1(S) = S$ and $y_2(S) = S$. Since each agent has a strict preference ordering, if agent $i \in S$ is not strictly better off by joining S, then $y(i) = x(i)$ needs to hold. Thus, there must be a nonempty subset $T \subseteq S$ such that $y(i) \succ_i x(i)$ for any $i \in T$, and $y(i) = x(i)$ for any $i \in S \backslash T$. Since x is a competitive allocation, we have $\sum_{q=1}^{2} p_q(y_q(i)) > \sum_{q=1}^{2} p_q(i)$ for any $i \in T$, and $\sum_{q=1}^{2} p_q(y_q(i)) = \sum_{q=1}^{2} p_q(i)$ for any $i \in S \backslash T$. It follows that $\sum_{i \in S} \left(\sum_{q=1}^{2} p_q(y_q(i)) \right) > \sum_{i \in S} \left(\sum_{q=1}^{2} p_q(i) \right)$. However, since $y_1(S) = S$ and $y_2(S) = S$, we must have $\sum_{i \in S} \left(\sum_{q=1}^{2} p_q(y_q(i)) \right) = \sum_{i \in S} \left(\sum_{q=1}^{2} p_q(i) \right)$. This is a contradiction. ■

It is easy to see that the proof of Proposition 3.1 can be generalized to the case with $Q > 2$. Note also that separable preferences do not play any role in the proof. However, the 'no indifference' assumption is crucial. In fact, it is a substitute for the standard "local nonsatiation assumption".

Because of Propositions 2.4 and 3.1, we immediately have the following:

Corollary 3.2. *In economy* $\mathcal{M}$*, it is possible for there to be no competitive equilibria.*

The following example shows that the converse of Proposition 3.1 does not hold.

Example 3.3. *Consider the economy which is described by* $N = \{1, 2, 3\}$ *with the following additively separable strict preferences:*

$$(2,3) \succ_1 (1,3) \succ_1 (2,2) \succ_1 (1,2) \succ_1 (2,1) \succ_1 (1,1) \succ_1 (3,3) \succ_1 (3,2) \succ_1 (3,1),$$
$$(1,3) \succ_2 (1,2) \succ_2 (3,3) \succ_2 (3,2) \succ_2 (2,3) \succ_2 (2,2) \succ_2 (1,1) \succ_2 (3,1) \succ_2 (2,1),$$
$$(2,2) \succ_3 (3,2) \succ_3 (2,1) \succ_3 (3,1) \succ_3 (2,3) \succ_3 (3,3) \succ_3 (1,2) \succ_3 (1,1) \succ_3 (1,3).$$

Claim: *The economy above has two strict core allocations* $x^1 = \{(2,1), (1,3), (3,2)\}$ *and* $x^2 = \{(2,3), (1,2), (3,1)\}$. *Allocation* x^1 *is a competitive allocation, but allocation* x^2 *is not.*

Proof. It is easy to see that the preferences have additively separable representations. It is also easy to check that the strict core consists of allocations $x^1 = \{(2,1), (1,3), (3,2)\}$ and $x^2 = \{(2,3), (1,2), (3,1)\}$. Allocation x^1 is competitive. Indeed, any prices with $p_1(1) = p_1(2) > p_1(3)$ and $p_2(1) < p_2(2) = p_2(3)$ support allocation x^1. However, allocation x^2 is not competitive. This can be seen as follows. Suppose that $(p_1(1), p_1(2), p_1(3); p_2(1), p_2(2), p_2(3))$ supports allocation x^2. Then, we have the following inequalities:

$$\begin{aligned} p_1(1) + p_2(1) &= p_1(2) + p_2(3), \\ p_1(1) + p_2(3) &> p_1(2) + p_2(2) = p_1(1) + p_2(2), \\ p_1(3) + p_2(2) &> p_1(3) + p_2(3) = p_1(3) + p_2(1). \end{aligned}$$

From the third equality, $p_2(3) = p_2(1)$ holds. Then, from the first equation, $p_1(1) = p_1(2)$ follows. Thus, the second inequality implies $p_2(3) > p_2(2)$. However, the third inequality says $p_2(2) > p_2(3)$. This is a contradiction. Hence, there is no price support for allocation x^2. ■

The economy described in this example has two more core allocations: $\{(1,2),(3,3),(2,1)\}$ and $\{(2,2),(1,3),(3,1)\}$. However, the former is weakly blocked by coalition $\{2,3\}$ via the allocation $\{(1,1),(3,3),(2,2)\}$, and the latter is weakly blocked by coalition $\{1,3\}$ via the allocation $\{(1,3),(2,2),(3,1)\}$. Thus, this economy has four core allocations, two of which are strict core allocations, and one of the strict core allocations, is a competitive allocation.

In economy $\mathcal{M}$ with $Q \geq 2$, the set of competitive allocations is a subset, sometimes a proper subset, of the strict core and the core. This relationship is the same as we observe in many market models with divisible commodities. However, in economy $\mathcal{M}$ with $Q = 1$, i.e., the Shaply-Scarf economy, the set of competitive allocations cannot be a *proper* subset of the strict core (see Roth and Postlewaite (1977) and Wako (1984)).

4. A Strategic Implication of the Economy

Let $\mathcal{A}$ be the set of allocations in economy $\mathcal{M}$, and $\mathcal{P}$ the set of additively separable strict preference orderings. A **social choice rule** is a function $\varphi : \mathcal{P}^N \to \mathcal{A}$. A social choice rule φ is said to be **individually rational** if it gives an individually rational allocation for any $\succ \equiv (\succ_1, ..., \succ_n) \in \mathcal{P}^N$. A social choice rule φ is said to be **Pareto efficient** if it gives a Pareto efficient allocation for any $\succ \in \mathcal{P}^N$, where a Pareto efficient allocation is an allocation that is not weakly blocked by the grand coalition N. We say that φ is **strategy-proof** if, for any $\succ \in \mathcal{P}^N$, $i \in N$, and $\succ_i' \in \mathcal{P}$, we have $\varphi_i(\succ) \succsim_i \varphi_i(\succ_i', \succ_{-i})$. Here $\varphi_i(\succ)$ is the ith component of $\varphi(\succ)$ and $(\succ_i', \succ_{-i}) \in \mathcal{P}^N$ is a list of preference orderings which is made by replacing $\succ_i$ by $\succ_i'$. Although the Shapley-Scarf economy admits the strict core as the unique social choice rule which is Pareto efficient, individually rational, and strategy-proof, we cannot obtain such a nice result for the economy with multiple types of indivisible goods.

Proposition 4.1. *For economy $\mathcal{M}$ with $n \geq 2$ (and $Q \geq 2$), there is no Pareto efficient, individually rational, and strategy-proof social choice rule.*

Proof. We start with the case of $N = \{1,2\}$. Suppose that the two players' true preference orderings, $\succ_1$ and $\succ_2$, respectively, satisfy:

$$(2,1) \succ_1 (2,2) \succ_1 (1,1) \succ_1 (1,2),$$
$$(1,1) \succ_2 (2,1) \succ_2 (1,2) \succ_2 (2,2).$$

Then there are two Pareto efficient and individually rational allocations: $\{(2,1),(1,2)\}$ and $\{(2,2),(1,1)\}$. Let $\varphi : \mathcal{P}^N \to \mathcal{A}$ be a social choice function that is Pareto efficient,

individually rational, and strategy-proof. By definition, φ needs to select one of the two allocations above. First suppose $\varphi(\succ_1, \succ_2) = \{(2,1),(1,2)\}$. Then, player 2 can manipulate outcomes by reporting $\succ_2'$ with

$$(2,1) \succ_2' (1,1) \succ_2' (2,2) \succ_2' (1,2).$$

Under preference profile $(\succ_1, \succ_2')$, allocation $\{(2,2),(1,1)\}$ is the only Pareto efficient and individually rational allocation. Thus it holds that $\varphi(\succ_1, \succ_2') = \{(2,2),(1,1)\}$. Next, suppose $\varphi(\succ_1, \succ_2) = \{(2,2),(1,1)\}$. Then, player 1 can manipulate outcomes by reporting $\succ_1'$with

$$(2,1) \succ_1' (1,1) \succ_1' (2,2) \succ_1' (1,2).$$

This time we must have $\varphi(\succ_1', \succ_2) = \{(2,1),(1,2)\}$. Thus, in the two player case, any individually rational and Pareto efficient mechanism is manipulable. For the case with more than two players, we let the preferences satisfy:

$$\begin{array}{l}(2,1) \succ_1 (2,2) \succ_1 (1,1) \succ_1 (1,2) \succ_1 \text{anything},\\ (1,1) \succ_2 (2,1) \succ_2 (1,2) \succ_2 (2,2) \succ_2 \text{anything},\\ (i,i) \succ_i \text{anything} \quad \text{for any } i \in N\backslash\{1,2\}.\end{array}$$

Using the same logic, we see that there is no such φ in this case, too. ■

Note that the two allocations in the proof above are the strict core allocations in the two agent economy. Ma (1994) showed that the strict core mechanism is the unique rule which is Pareto efficient, individually ratioonal and strategy-proof in the Shapley-Scarf economy. Sönmez (1999) proved the same result in a "generalized matching problem" that contains the Shapley-Scarf economy as a special case. However, Sönmez's result does not cover our case. It is an open question if such a result applies in our economy with additively separable preference domain.

5. Another Related Model

In the previous sections we have investigated an extension of the Shapley-Scarf economy, in which multiple indivisible goods are traded and each agent consumes exactly one unit of each type. Another variation is where there is only one type of indivisible good, but where agents can consume any amount of these goods. Shapley and Scarf (1974, section 8) showed that such an economy may have an empty core if there are complementarities among the goods traded. We show, by the following example, that even if there is no complementarity among the goods, the core may be empty.

The model considered here is as follows: The set of agents is $N = \{1, ..., n\}$. The set of indivisible goods to be traded is $G = \{g_1, ..., g_m\}$. Let 2^G be the set of subsets of G. An allocation is a partition $x = \{(x_i)_{i\in N}\}$ of G, i.e., $\cup_{i\in N} x_i = G$ and $x_i \cap x_j = \emptyset$ for all $i \neq j$.[8]

[8] We allow some x_j to be empty.

There is a special allocation w which represents the agents' initial endowments. Each agent i has a sub-utility function $u_i : G \cup \{\emptyset\} \to \Re_+$ with $u_i(\emptyset) = 0$, and her preferences over 2^G are represented by a utility function $U_i : 2^G \to \Re_+$ with $U_i(x) = \sum_{g \in x} u_i(g)$ for all $x \in 2^G$. Here, the additive separability of utility function U_i implies that there is no complementarity among the goods traded.

Example 5.1. *Let* $N = \{1, 2, 3, 4\}$, $G = \{a, b, c, d, e\}$, *and* $w = \{(w_1), (w_2), (w_3), (w_4)\} = \{(a), (b), (c), (d, e)\}$. *Suppose that a sub-utility function of each agent is defined as follows:*

agent i	$u_i(a)$	$u_i(b)$	$u_i(c)$	$u_i(d)$	$u_i(e)$
1	10	6	0.1	0.01	5
2	0.1	1	3	2	0.01
3	0.1	2	1	0.01	3
4	20	0.1	6	10	5

Claim: *The core is empty in the economy above.*

Proof. It is easy to see that the economy has four individually rational allocations: $w = \{(a), (b), (c), (d, e)\}$, $x = \{(a), (c), (b), (d, e)\}$, $y = \{(a), (b), (e), (c, d)\}$, $z = \{(b, e), (d), (c), (a)\}$. However, allocation w is blocked by coalition $\{2, 3\}$ via allocation x; allocation x is blocked by coalition $\{3, 4\}$ via allocation y; allocation y is blocked by coalition $\{1, 2, 4\}$ via allocation z; allocation z is blocked by coalition $\{2, 3\}$ via allocation x. Therefore, the core is empty. ■

The example above suggests that the "each agent consumes one unit" assumption is crucial for an exchange market with indivisible goods to function well without money.

6. Appendix

In this appendix we first prove Proposition 2.1, which shows core nonemptiness of economy $\mathcal{M}$ with $n = 3$ and $Q = 2$. We then show by example that economy $\mathcal{M}$ may have an empty core if $n = 3$ and $Q > 2$.

Proposition2.1. *Suppose that there are three agents in economy* $\mathcal{M}$ *(with* $Q = 2$*). Then the core is nonempty.*

Consider an economy $\mathcal{M}$ with three agents swapping houses and cars, i.e., $N = \{1, 2, 3\}$ and $Q = 2$. Let (u_i^1, u_i^2) be functions representing agent i's additively separable preference ordering (see the main body of Section 2). Let Z be the set of subsets of $\Re^N$ and $\mathcal{A}$ be the set of allocations. For this economy, define the **associated NTU-game** $G = (N, V)$ with player set N and characteristic function $V : 2^N \to Z$ given by $V(\emptyset) = \emptyset$ and

$$V(S) = \{z \in \Re^N \mid \exists x \in \mathcal{A} \text{ with } \quad x_1(S) = x_2(S) = S \text{ and } \\ u_i^1(x_1(i)) + u_i^2(x_2(i)) \geq z_i \text{ for all } i \in S\}$$

for $S \in 2^N \backslash \{\emptyset\}$. Clearly, the core of $\mathcal{M}$ will be nonempty iff the core of the associated NTU-game G is nonempty, i.e., $V(N) - \bigcup_{\emptyset \neq S \subseteq N} \text{int}V(S) \neq \emptyset$.

It is easy to see that G is a superadditive three-person NTU game. So, as a corollary to Scarf's Theorem[9] (1967), we have the following proposition, by which we prove Proposition 2.1:

Proposition 6.1. *If $V(\{1,2\}) \cap V(\{1,3\}) \cap V(\{2,3\}) \subseteq V(N)$, then G has a nonempty core.*

Proof of Proposition 2.1. Suppose that there exists a vector $z \in \Re^3$for which $(z_1,\ z_2, *) \in V(\{1,2\})$, $(*,\ z_2, z_3) \in V(\{2,3\})$ and $(z_1, *, z_3) \in V(\{1,3\})$, where symbol $*$ represents any real number. It follows from Proposition 6.1 that Proposition 2.1 is proved if we show that $(z_1,\ z_2, z_3) \in V(N)$. In the following, for each S of $\{1,2\}$, $\{2,3\}$ and $\{1,3\}$, we denote by $\mathcal{A}_S$ the set of allocations with $x_1(S) = x_2(S) = S$, and let a, b and c be the generic symbols for allocations in $\mathcal{A}_{\{1,2\}}$, $\mathcal{A}_{\{2,3\}}$ and $\mathcal{A}_{\{1,3\}}$, respectively.

Provided that $(z_1,\ z_2, *) \in V(\{1,2\})$, there must be a *supporting* allocation $a \in \mathcal{A}_{\{1,2\}}$ under which agents 1 and 2 gain a payoff of at least z_1 and z_2, respectively. There are precisely four possibilities for this supporting allocation a: (1) allocation a^{NN}, in which both agents 1 and 2 keep their own houses and cars; (2) allocation a^{NT}, in which they keep their own houses but swap cars; (3) allocation a^{TN}, in which they keep their own cars but swap houses; and (4) allocation a^{TT}, in which they swap both cars and houses. The superscripts here describe what happens in the two sub-markets: hence, for example, " a^{TN} " means that houses are Traded but cars are Not traded between agents 1 and 2; and that agent 3 does not trade with anyone.

Similarly, there must be a supporting allocation $b \in \mathcal{A}_{\{2,3\}}$ ($c \in \mathcal{A}_{\{1,3\}}$) under which agents 2 and 3 (1 and 3) gain a payoff of at least z_2 and z_3 (z_1 and z_3) respectively. We apply the same notational convention as above to these allocations. We then have a total of $4 \times 4 \times 4 = 64$ cases, in each of which we must check that (z_1, z_2, z_3) is indeed an element of $V(N)$. However, by considering symmetry, those cases can be reduced to the 20 cases in Table 1 (see the last page).

Table1 is here

Let us start by checking, say, Case 5. Here, since the supporting allocation a is a^{NN}, agent 1 must value house-car bundle $(1,1)$ at no less than z_1. Next, since the supporting allocation b is b^{NT}, agent 2 values house-car bundle $(2,3)$ at no less than z_2. Finally, since $b = b^{NT}$ again, we have that agent 3 values house-car bundle $(3,2)$ at no less than z_3. Now, when we consider the grand coalition $N = \{1,2,3\}$, we see that it could

[9]We note the following conditions hold for our model: (a) $V(S)$ is closed for all S; (b) $V(S)$ is comprehensive for all S; and (c) the set of elements in $V(S)$ for which each player in S receives no less than the maximum that he can obtain by himself is a nonempty bounded set. These are necessary in order to use Scarf's Theorem.

institute allocation $\{(1,1),(2,3),(3,2)\}$. By the above, this would pay the agents at least (z_1, z_2, z_3), and so indeed $(z_1, z_2, z_3) \in V(N)$.

This type of reasoning can be used to verify that $(z_1, z_2, z_3) \in V(N)$ in cases 1-8, 11, 14, and 17-19 on Table 1. Note that no use is made of the "additive separability of preferences" hypothesis. For the other cases (marked with an * in Table 1), we must use a more complicated argument, which uses additive separability.

For instance, let us consider Case 9. In Case 9, (a) $a = a^{NT}$, (b) $b = b^{TN}$ and (c) $c = c^{TN}$. From (a), we know that agent 1 values bundle $(1,2)$ at no less than z_1, while from (c) we know that he values $(3,1)$ at no less than z_1. By additive separability, this means that $u_1^1(1) + u_1^2(2) \geq z_1$ and $u_1^1(3) + u_1^2(1) \geq z_1$. But this in turn implies that either (d) $u_1^1(1) + u_1^2(1) \geq z_1$ or (e) $u_1^1(3) + u_1^2(2) \geq z_1$. Assume that (d) holds. Then we have by (d) that agent 1 values $(1,1)$ at z_1 or more; by (b) that agent 2 values $(3,2)$ at z_2 or more; and again by (b) that agent 3 values $(2,3)$ at z_3 or more. If (e) holds, then (e), (a), and (c) imply that agent 1 values $(3,2)$ at z_1 or more; agent 2 values $(2,1)$ at z_2 or more; and agent 3 values $(1,3)$ at z_3 or more, respectively. Either way, the implication is that $(z_1, z_2, z_3) \in V(N)$.

In the same manner, we can show that $z \in V(N)$ in all of the other cases. Hence we have proven that economy $\mathcal{M}$ with three agents has a nonempty core.[10] ■

Finally we give an example of a three-agent economy with $Q = 3$ and an empty core.

Example 6.2. *Consider the economy which is described by* $N = \{1,2,3\}$, $Q = 3$ *and additively separable strict preferences* $\succ_i$, $i \in N$, *such that each* $\succ_i$ *is represented by the sum* $\sum_{q=1}^{3} u_i^q(j_q)$ *of functions* $u_i^q : N \to \Re$, $q = 1,2,3$, *defined as below:*

$u_1^q(j)$	$q=1$	$q=2$	$q=3$
$j=1$	0	0	0
$j=2$	−1.2	1.5	−10
$j=3$	−20	1.1	−1

$u_2^q(j)$	$q=1$	$q=2$	$q=3$
$j=1$	1.1	−1	−10
$j=2$	0	0	0
$j=3$	−1.2	−20	1.5

$u_3^q(j)$	$q=1$	$q=2$	$q=3$
$j=1$	−10	−1.2	1.5
$j=2$	1.1	−20	−1
$j=3$	0	0	0

where the (j,q) *component of each matrix shows value* $u_i^q(j)$.

Claim: *The core is empty in the economy above.*

Proof. First we show that there are only four individually rational allocations

[10] Note that we do not need strictness of the preferences for the proof.

$x^k : N \to N^3$, $k = 0, 1, 2, 3$, as follows:

$$\begin{aligned}
\text{Allocation } x^0 &= \{(1,1,1),(2,2,2),(3,3,3)\},\\
\text{Allocation } x^1 &= \{(2,2,1),(1,1,2),(3,3,3)\},\\
\text{Allocation } x^2 &= \{(1,3,3),(2,2,2),(3,1,1)\},\\
\text{Allocation } x^3 &= \{(1,1,1),(3,2,3),(2,3,2)\},
\end{aligned}$$

where the i th component of each allocation x^k gives player i's commodity bundle $(x_1^k(i), x_2^k(i), x_3^k(i))$. Since the initial allocation x^0 is clearly individually rational, it suffices to show that the other three allocations are individually rational. To prove this, we focus on player 1's preferences over allocations that are better than her endowments, which is described as follows:

$$(1,2,1) \succ_1 (1,3,1) \succ_1 (1,2,3) \succ_1 (2,2,1) \succ_1 (1,3,3) \succ_1 (1,1,1).$$

Next we check whether an individually rational allocation exists for each case of player 1 consuming one of the above bundles. Denote by x an individually rational allocation.

(i) Suppose that $x(1) = (1,2,1)$. Then $x_2(2) = 1$ (otherwise, individually rationality for player 2 will not be satisfied: $x_2(2) = 3$ generates -20). Thus, for player 2 to have a positive utility, $x_3(2) = 3$ is needed. But, then $x_3(3) = 2$, and $x_1(3) = 2$ needs to follow in turn. Hence, $x_1(2) = 3$. However, $x(2) = (3,1,3)$ does not attain a non-negative utility. A contradiction.

(ii) Suppose that $x(1) = (1,3,1)$. If $x_2(3) = 2$, then player 3 cannot get nonnegative utility anyway. Thus, $x_2(3) = 1$. To attain a nonnegative utility for player 3, there is only one way: $x_3(3) = 1$. But this contradicts that $x_3(1) = 1$.

(iii) Suppose that $x(1) = (1,2,3)$. If $x_2(2) = 3$ or $x_3(2) = 1$, then player 2 cannot get nonnegative utility anyway. Thus, $x_2(2) = 1$ and $x_3(2) = 2$. To attain a nonnegative utility for player 2, there is only one way: $x_1(2) = 1$. But this contradicts that $x_1(1) = 1$.

(iv) Suppose that $x(1) = (2,2,1)$. This implies $x_1(3) = 3$ and $x_3(3) \neq 1$. Thus, to give player 3 a nonnegative utility, we must have $x(3) = (3,3,3)$. Thus, this allocation is x^1.

(v) Suppose that $x(1) = (1,3,3)$. This implies $x_3(2) = 2$ and $x_2(3) = 1$. Then, $x_2(2) = 2$. To give player 2 a nonnegative utility, $x_1(2) = 2$. Thus, this allocation is x^2.

(vi) Suppose that $x(1) = (1,1,1)$. This implies $x_2(2) = 2$ and $x_2(3) = 3$ right away. Obviously, there are two individually rational allocations: x^0 and x^3.

Thus, there are only four individually rational allocations. Obviously, the initial allocation is not a core allocation. Allocation x^1 is blocked by coalition $\{2,3\}$ via x^3; allocation x^3 is blocked by coalition $\{1,3\}$ via x^2; and allocation x^2 is blocked by coalition $\{1,2\}$ via x^1. Thus this economy has an empty core. ■

Table 1

Combinations of supporting allocation types

No.	{1,2}	{2,3}	{1,3}	No.	{1,2}	{2,3}	{1,3}
1	$(a^{NN},$	$b^{NN},$	$c^{NN})$	11	$(a^{TN},$	$b^{NN},$	$c^{NN})$
2	$(a^{NT},$	$b^{NT},$	$c^{NT})$	12*	$(a^{TN},$	$b^{NT},$	$c^{NT})$
3	$(a^{TN},$	$b^{TN},$	$c^{TN})$	13*	$(a^{TN},$	$b^{TT},$	$c^{TT})$
4	$(a^{TT},$	$b^{TT},$	$c^{TT})$	14	$(a^{TT},$	$b^{NN},$	$c^{NN})$
5	$(a^{NN},$	$b^{NT},$	$c^{NT})$	15*	$(a^{TT},$	$b^{NT},$	$c^{NT})$
6	$(a^{NN},$	$b^{TN},$	$c^{TN})$	16*	$(a^{TT},$	$b^{TN},$	$c^{TN})$
7	$(a^{NN},$	$b^{TT},$	$c^{TT})$	17	$(a^{NN},$	$b^{NT},$	$c^{TN})$
8	$(a^{NT},$	$b^{NN},$	$c^{NN})$	18	$(a^{NN},$	$b^{NT},$	$c^{TT})$
9*	$(a^{NT},$	$b^{TN},$	$c^{TN})$	19	$(a^{NN},$	$b^{TN},$	$c^{TT})$
10*	$(a^{NT},$	$b^{TT},$	$c^{TT})$	20*	$(a^{NT},$	$b^{TN},$	$c^{TT})$

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