

Min, Max, and Sum*

Uzi Segal[†] and Joel Sobel[‡]

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Abstract

This paper provides characterization theorems for preferences that can be represented by $U(x_1, \dots, x_n) = \min\{x_k\}$, $U(x_1, \dots, x_n) = \max\{x_k\}$, $U(x_1, \dots, x_n) = \sum u(x_k)$, or combinations of these functionals. The main assumption is partial separability, where changing a common component of two vectors does not reverse strict preferences, but may turn strict preferences into indifference. We discuss applications of our results to social choice.

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[†]Department of Economics, Boston College, Chestnut Hill, MA 02467, U.S.A. E-mail: segalu@bc.edu

[‡]Department of Economics, University of California, San Diego, La Jolla, CA 92093, U.S.A. E-mail: jsobel@weber.ucsd.edu

1 Introduction

This paper provides characterization theorems for preferences over \mathfrak{R}^n that can be represented by $U(x_1, \dots, x_n) = \min\{x_k\}$, $U(x_1, \dots, x_n) = \max\{x_k\}$, $U(x_1, \dots, x_n) = \sum u(x_k)$, or combinations of these functional forms.

The problem of finding characterization theorems for the additive and minimum functional forms has been studied from the perspective of social-choice theory, decision making under ignorance, and consumer theory. In social choice theory, the arguments of the utility function U are (von Neumann & Morgenstern) utilities of the members of the society. In decision making under ignorance, the arguments are state-contingent utilities, while in consumer theory, the arguments are commodities.

The standard assumptions on preferences made in this literature are some form of continuity, monotonicity, symmetry, linearity, and separability. The interpretation and importance of continuity and monotonicity¹ are familiar and well understood. The linearity assumption² requires that rankings do not change when one applies a common affine transformation to all components. In the social-choice context it defines the extent to which individual utilities are comparable. The assumption of complete separability states that changing a common component of two vectors does not reverse a weak preference. The main technical contribution of this paper is to study the implications of a weaker version of the separability assumption. Under this weaker assumption, called partial separability, changing a common component of two vectors does not reverse strict preferences, but may turn strict preferences into indifference (see Blackorby, Primont, and Russell [4] and Mak [16]). Complete separability suggests that changes in one component should have no affect over the desirability of the rest of the components. Partial separability replaces this with a limited influence, namely that a change in one component may not reverse preferences, but may make the rest of the component immaterial.

This paper provides novel characterizations of the min and max criteria and conditions under which preferences are represented by the sum of x_i over a subset of the domain and either the max or min of x_i over the rest

¹The social choice literature often refers to monotonicity assumptions as the Pareto principle.

²In the decision theoretic literature, this assumption is known as constant (absolute and relative) risk aversion.

of the domain. To put these results in perspective, we briefly review related literature. Luce and Raiffa [15] survey the research of several authors on the problem of decision making under ignorance done in the 1950s. Perhaps the most elegant contribution in this group is due to Milnor [19]. He presents axiomatic treatments of the maxmin and sum criteria. His characterization of the Hurwicz (see [15]) α criterion (weighted average of maxmax and maxmin) is based on a “column duplication” assumption that stipulates that the ranking between a pair of vectors does not change if a component is added to each vector that duplicates an existing component. This assumption is plausible for problems of choice under ignorance (where the duplicate component could be the result of an arbitrary redefinition of the states of the world), but is harder to justify in other applications, especially to the theory of social choice. His characterization of the sum criterion (in the context of decision making under ignorance, this is the principal of insufficient reason) depends on monotonicity and complete separability assumptions.

Maskin [18] provides a characterization of the sum and maxmin criteria similar to Milnor’s. He relates his results to the social choice literature. Maskin relaxes the continuity assumption to provide characterizations of lexicographic maxmin and lexicographic maxmax criteria as well.

We assume throughout that preferences are symmetric, continuous, weakly increasing, and satisfy partial separability. We supplement these assumptions with different conditions to obtain the representation functions described above. We introduce the assumptions formally in Section 2. By focusing on partial separability rather than the stronger complete separability axiom, this paper provides axioms that imply not only the sum representation, but also max, min, and combination of these two and sum.

If preferences are strictly monotonic, then partial separability implies complete separability (see Färe and Primont [9]). In order to get more than just additively separable representations we must permit some flatness. In Section 3 we analyze the implications of local flatness with respect to one variable at a point along the main diagonal. We show that if symmetry is assumed, then the indifference curve through this point is either min or max. Section 4 shows how an additional indifference monotonicity axiom implies that if one indifference curve is described by the min function, then all lower indifference curves are min as well, and that higher indifference curves are either min or additively separable. Similarly, if one indifference curve is described by the max function, then all higher indifference curves are max as

well, and all lower indifference curves are either max or additively separable.

The third result, presented in Section 5, shows that a linearity axiom combines with the partial separability axiom to guarantee that preferences must be represented throughout their domain by min, max, or $\sum x_i$. In this section we also point out that the linearity axiom is stronger than necessary to obtain this trichotomy. Under a weaker condition, preferences must be represented throughout their domain by min, max, or $\sum u(x_i)$.

Section 6 discusses the connection between our results and a social choice problem, where the question is how should society allocate indivisible goods when it decides to use a lottery.

2 Axioms and Preliminary Results

Consider a preference relation \succeq on \mathfrak{R}^n . We denote by \succ and \sim the strict and indifference relations, respectively. The preferences \succeq are assumed to be complete, transitive, and continuous. Denote $\mathbf{e} = (1, \dots, 1) \in \mathfrak{R}^n$ and let $\mathbf{e}^i = (0, \dots, 1, \dots, 0)$. Assume:

(M) Monotonicity $x \gg y$ implies $x \succ y$.³

(S) Symmetry For every permutation π of $\{1, \dots, n\}$ and for every x ,

$$(x_1, \dots, x_n) \sim (x_{\pi(1)}, \dots, x_{\pi(n)})$$

Define x_{-k} to be the vector in \mathfrak{R}^{n-1} that is obtained from x by eliminating component k , and let (x_{-k}, y_k) be the vector obtained from x by replacing x_k with y_k .

All of the results of the paper depend on the following axiom.

(PS) Partial Separability For all $x, y \in \mathfrak{R}^n$ and every k , if $x = (x_{-k}, x_k) \succ (y_{-k}, x_k)$, then $(x_{-k}, y_k) \succeq (y_{-k}, y_k) = y$.

The partial separability axiom is weaker than the common separability axiom:

(CS) Complete Separability For all x and y and for every k , $(x_{-k}, x_k) \succeq (y_{-k}, x_k)$ iff $(x_{-k}, y_k) \succeq (y_{-k}, y_k)$.

³For $x, y \in \mathfrak{R}^n$, $(x_1, \dots, x_n) \gg (y_1, \dots, y_n)$ iff, for all i , $x_i > y_i$.

It follows immediately that complete separability implies partial separability. The converse is of course not true (see for example Theorem 1). However, if we make a stronger monotonicity assumption, such an equivalence will follow.

(SM) Strict Monotonicity $x \succeq y$ implies $x \succ y$.

Färe and Primont [9] show that if the preferences \succeq satisfy strict monotonicity and partial separability, then they satisfy complete separability.

It is useful to note equivalent forms of the partial separability assumption. For $x \in \mathfrak{R}^n$, let $\gamma^k(x) = \{\bar{x}_{-k} : (\bar{x}_{-k}, x_k) \succeq (x_{-k}, x_k) = x\}$. That is, $\gamma^k(x)$ is the intersection of the upper set of x with the hyperplane where the k -th component equals x_k . Let N denote the set $\{1, \dots, n\}$ and N_{-k} denote $N \setminus \{k\}$.

Proposition 1 *Let U be a continuous utility function that represents \succeq . The following three conditions are equivalent.*

1. *For all $x, y \in \mathfrak{R}^n$, if $x = (x_{-k}, x_k) \succ (y_{-k}, x_k)$, then $(x_{-k}, y_k) \succeq (y_{-k}, y_k) = y$.*
2. *There exist continuous $f : \mathfrak{R}^{n-1} \rightarrow \mathfrak{R}$ and $g : \mathfrak{R}^2 \rightarrow \mathfrak{R}$ such that $U(x_{-k}, x_k) = g(f(x_{-k}), x_k)$ where g is nondecreasing in its first argument.*
3. *For each $x, x' \in \mathfrak{R}^n$ either $\gamma^k(x') \subseteq \gamma^k(x)$ or $\gamma^k(x) \subseteq \gamma^k(x')$.*

Proof Bliss [5] and Blackorby, Primont, and Russell [4, Theorem 3.2b, p. 57 and Theorem 3.3b, p. 65] prove that the last two conditions are equivalent. Mak [16, Proposition (2.11)] proves the equivalence of the first and the third conditions. ■

If $x_k = x'_k$, then $\gamma^k(x)$ and $\gamma^k(x')$ are upper sets of the same induced preferences on \mathfrak{R}^{n-1} and are of course nested. Condition 3 of Proposition 1 is therefore restrictive only when $x_k \neq x'_k$. This condition does not require that indifference curves of the induced preferences on \mathfrak{R}^{n-1} at the level where the k -th component is x_k or x'_k will be the same. Blackorby, Primont, and

Russell [4, pp. 43–46] and Mak [16] define the set of variables N_{-k} to be separable from $\{k\}$ if and only if the third condition in Proposition 1 holds.⁴

Some of our arguments derive local implications of our assumptions. In order to extend these properties, we make other assumptions. The following one, which we do not use until Section 5, is the strongest.

(L) Linearity For all $x, y \in \mathfrak{R}^n$,

1. For all α , $x \succeq y$ iff $x + \alpha \mathbf{e} \succeq y + \alpha \mathbf{e}$.
2. For all $\beta > 0$, $x \succeq y$ iff $\beta x \succeq \beta y$.

In the decision theoretic literature this axiom is called constant risk aversion, where together with the independence axiom it is known to imply expected value maximization.⁵ It is also widely used in the literature concerning income distribution. In the social choice literature, where preferences are defined over individual utilities, the axiom states that the ordering is invariant with respect to a common positive affine transformation of utilities (see Maskin [17]). In Section 6 we provide an example where this axiom may seem acceptable.

To investigate the implications of the partial separability axiom, we need the following definition.

Definition 1 *The two vectors $x, y \in \mathfrak{R}^n$ are comonotonic if, for all k and k' , $x_k \geq x_{k'}$ iff $y_k \geq y_{k'}$. For $x \in \mathfrak{R}^n$, the comonotonic sector $M(x)$ is the set of all points y such that x and y are comonotonic.*

Two vectors are comonotonic if they have the same ranking of their components. Observe that for α and β as in the definition of axiom **L**, the three vectors x , $x + \alpha \mathbf{e}$, and βx are comonotonic.

Definition 2 *Preferences \succeq satisfy strict monotonicity at x if and only if for all $x', x'' \in \mathfrak{R}^n$, $x' \not\geq x \not\geq x''$ implies $x' \succ x \succ x''$.*

⁴It is possible to amend the conditions of Proposition 1 and the partial separability axiom to obtain the stronger notion of partial set separability, where a set of variables I is partially separable from the set $N \setminus I$. We do not make this assumption, but discuss it briefly in Section 6.

⁵For an analysis of constant risk aversion without the independence axiom, see Safra and Segal [23].

If the preferences \succsim satisfy strict monotonicity at x , then the linearity axiom implies that \succsim satisfy strict monotonicity for all $x' \in M(x)$ and hence symmetry implies strict monotonicity. In combination with the other axioms, therefore, the linearity axiom implies that preferences will violate strict monotonicity everywhere, or satisfy the complete separability axiom. Weaker assumptions suffice for most of our analysis.

We say that the preferences \succeq are strictly monotonic with respect to the k -th component at the point x if for all $x'_k > x_k > x''_k$, $(x_{-k}, x'_k) \succ x \succ (x_{-k}, x''_k)$.

(CF) Comonotonic Flatness For every x , if the preferences are strictly monotonic with respect to the k -th component at the point x , then they are strictly monotonic with respect to this component for all $y \in M(x)$.

Axioms relating to comonotonic vectors are popular in the literature concerning income distribution and decision making under uncertainty. The most popular alternative to expected utility theory, called rank dependent (Quiggin [20]), assumes that the utility from an outcome x_i , which is to be obtained with probability p_i , is multiplied by a function of p_i , and the adjusted value of p_i depends on x_i 's rank. Formally, denote $p_0 = 0$ and let $x_1 \leq \dots \leq x_n$. The value of the lottery $(x_1, p_1; \dots; x_n, p_n)$ is given by

$$\sum_{i=1}^n u(x_i) \left[f \left(\sum_{j=0}^i p_j \right) - f \left(\sum_{j=0}^{i-1} p_j \right) \right]$$

where $f(0) = 0$, $f(1) = 1$, and f is increasing and continuous. Consider now the set of lotteries $(x_1, \frac{1}{n}; \dots; x_n, \frac{1}{n})$, and suppose that at a point where all outcomes are distinct, $(\dots; x_i + \varepsilon, \frac{1}{n}; \dots) \sim (\dots; x_i, \frac{1}{n}; \dots)$. In expected utility theory this indifference implies $u(x_i + \varepsilon) = u(x_i)$, therefore the indifference holds regardless of the rest of the outcomes. In the rank dependent model, on the other hand, this indifference may hold because $f(\frac{i}{n}) = f(\frac{i-1}{n})$ (assume that the outcomes are ordered from lowest to highest). In that case, indifference need not hold at x_i if the outcome x_j changes for $j \neq i$. On the other hand, it does hold for all outcomes that are ranked i -th from below. The comonotonic indifference axiom complies with this model.

The linearity axiom implies comonotonic flatness, but of course not conversely. In combination with the symmetry axiom, however, comonotonic flatness implies that either preferences satisfy strict monotonicity globally,

or they violate this axiom globally. For Theorem 1 we need an even weaker assumption.

(IM) Indifference Monotonicity Let x and y be comonotonic, and suppose that $x \sim y$. The preferences \succeq satisfy strict monotonicity at x if, and only if, they satisfy strict monotonicity at y .

This axiom is weaker than comonotonic flatness in two respects. Let x and y be comonotonic vectors. If at x the preferences are strictly monotonic with respect to the k -th component and flat with respect to the ℓ -th component, then axiom **CF** requires the same at y , while indifference monotonicity is satisfied even if preferences at y are strictly monotonic with respect to the ℓ -th component and flat with respect to the k -th one. Secondly, unlike the comonotonic flatness axiom, where uniform behavior is required over the whole comonotonic sector $M(x)$, axiom **IM** only restricts behavior along the comonotonic part of the indifference curve through x . Together with symmetry, indifference monotonicity implies that if $x \sim y$, then the preferences \succeq satisfy strict monotonicity at x if, and only if, they satisfy strict monotonicity at y . In the sequel, we use this version of axiom **IM**.

Under the partial separability axiom, weak monotonicity of preferences (that is, $x \gg y$ implies $x \succeq y$) guarantees that the functions f and g in the representation in Proposition 1 can be taken to be weakly monotonic. Under the stronger monotonicity condition **M** and under symmetry, we can guarantee that both of the functions in the representation of Proposition 1 are increasing.⁶ Formally:

Proposition 2 *Assume that the preferences \succeq satisfy monotonicity, symmetry, and partial separability, and let U be a continuous utility function that represents \succeq . Then there exist increasing and continuous functions $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ and $g : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that $U(x_{-k}, x_k) = g(f(x_{-k}), x_k)$.*

Proof Take $x_{-k} \gg y_{-k}$ and assume $f(x_{-k}) \leq f(y_{-k})$. To simplify notation, assume $k > 1$ and $a = y_1 = \min_{j \neq k} y_j$. Let z be obtained by permuting the first and k -th components of (x_{-k}, a) . It follows that $x_{-k} \geq z_{-k} \geq y_{-k}$. Weak monotonicity and Prop. 1(2) imply that f is nondecreasing, hence

$$f(x_{-k}) = f(z_{-k}) = f(y_{-k}) \tag{1}$$

⁶The function $h : \mathbb{R}^m \rightarrow \mathbb{R}$ is weakly monotonic if $x \geq y$ implies that $h(x) \geq h(y)$. It is increasing if $x \gg y$ implies that $h(x) > h(y)$.

and therefore

$$\begin{aligned}
 U(y_{-k}, a) &= g(f(y_{-k}), a) = g(f(x_{-k}), a) = U(x_{-k}, a) = U(z) = \\
 & \\
 U(z_{-k}, x_1) &= g(f(z_{-k}), x_1) = g(f(x_{-k}), x_1) = U(x_{-k}, x_1)
 \end{aligned}
 \tag{2}$$

where the second and seventh equations follow from eq. (1), the first, third, sixth, and eighth equations follow from the definition of g and f , the fourth equation follows from the symmetry axiom, and the fifth equation because the k -th component of z is x_1 . It follows from eq. (2) that $U(y_{-k}, a) = U(x_{-k}, x_1)$, which contradicts **M** since $a < x_1$. This establishes that f is increasing. That g is increasing (when its first argument is in the range of f) now follows directly from axiom **M**. ■

3 Monotonicity Along the Main Diagonal

Although partial separability restricts upper sets, and its verification therefore needs the analysis of the preferences \succeq at many points, it turns out that when monotonicity and symmetry are also assumed, much information is contained in the preferences' behavior along the main diagonal $\{\lambda \mathbf{e} : \lambda \in \mathfrak{R}\}$. The present section is devoted to this analysis.

Fix a point $\lambda^0 \mathbf{e}$. The next proposition, which does not require linearity, comonotonic flatness, or indifference monotonicity, shows that preferences are either strictly monotonic at this point, or the indifference curve through the point is either min or max. As before, $\mathbf{e}^i = (0, \dots, 1, \dots, 0)$.

Proposition 3 *Let $n \geq 3$. Assume monotonicity, symmetry, and partial separability, and suppose that there exist λ^0 , $\varepsilon > 0$, and m such that $\lambda^0 \mathbf{e} + \varepsilon \mathbf{e}^m \sim \lambda^0 \mathbf{e}$. Then $x \sim \lambda^0 \mathbf{e}$ if, and only if, $\min\{x_i\} = \lambda^0$.*

Axiom **PS** places no restriction on monotonic preferences when $n = 2$, and simple examples show that Proposition 3 does not hold when $n = 2$ (see, e.g., the preferences that are represented by eq. (20) below). We will prove Proposition 3 using Lemmas 1 and 2. A symmetric argument establishes the next result.

Proposition 4 *Let $n \geq 3$. Assume monotonicity, symmetry, and partial separability, and suppose that there exist $\lambda^0, \varepsilon > 0$, and m such that $\lambda^0 \mathbf{e} - \varepsilon \mathbf{e}^m \sim \lambda^0 \mathbf{e}$. Then $x \sim \lambda^0 \mathbf{e}$ if, and only if, $\max\{x_i\} = \lambda^0$.*

Lemma 1 *Assume monotonicity, symmetry, and partial separability, and suppose that there exist $\lambda^0, a > 0$, and $1 \leq m < n - 1$ such that $\lambda^0 \mathbf{e} + \sum_{i=1}^m a \mathbf{e}^i \sim \lambda^0 \mathbf{e}$. Then there exists $\theta > 0$ such that $\lambda^0 \mathbf{e} + \sum_{i=1}^{m-1} a \mathbf{e}^i + \sum_{i=m}^{n-1} \theta \mathbf{e}^i \sim \lambda^0 \mathbf{e}$.*

This lemma implies, in particular, that if it is possible to increase the value of one component of a point $\lambda^0 \mathbf{e}$ on the main diagonal without moving to a strictly better point, then in an open neighborhood of $\lambda^0 \mathbf{e}$, the indifference curve through $\lambda^0 \mathbf{e}$ is derived from the min representation function.

Proof Suppose that there exist $b > 0$ and k such that

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{m-1} a \mathbf{e}^i + \sum_{i=m}^{k-1} b \mathbf{e}^i \sim \lambda^0 \mathbf{e} \quad (3)$$

but, for all $\varepsilon > 0$,

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{m-1} a \mathbf{e}^i + \sum_{i=m}^{k-1} b \mathbf{e}^i + \varepsilon \mathbf{e}^k \succ \lambda^0 \mathbf{e}. \quad (4)$$

We want to show that $k = n$. Assume that $k < n$. Let $x_{-k} = (x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_n)$ where $x_i = \lambda^0 + a$ for $i < m$, $x_i = \lambda^0 + b$ for $m \leq i < k$, and $x_i = \lambda^0$ for $i > k$. It follows from (3) that

$$(x_{-k}, \lambda^0) \sim \lambda^0 \mathbf{e} \quad (5)$$

and for all $0 < \varepsilon < a$,

$$(x_{-k}, \lambda^0 + \varepsilon) \succ \lambda^0 \mathbf{e} \sim \lambda^0 \mathbf{e} + \sum_{i=1}^{m-1} a \mathbf{e}^i + \varepsilon \mathbf{e}^k \quad (6)$$

where the strict preference follows from (4) and the indifference follows from the assumption of the lemma and axiom **S**. By continuity, (6) implies that there exists $\delta > 0$ such that if $y = (\lambda^0 + \delta) \mathbf{e} + \sum_{i=1}^{m-1} a \mathbf{e}^i$, then

$$(x_{-k}, \lambda^0 + \varepsilon) \succ (y_{-k}, \lambda^0 + \varepsilon). \quad (7)$$

Furthermore, unless the statement of the lemma holds true, it must be the case that $(y_{-k}, \lambda^0) \succ \lambda^0 \mathbf{e}$. It follows from (5) that

$$(y_{-k}, \lambda^0) \succ (x_{-k}, \lambda^0). \quad (8)$$

Equations (7) and (8) violate axiom **PS**. Hence we have a contradiction, proving that $k = n$. \blacksquare

Lemma 2 *Assume monotonicity, symmetry, and partial separability, and suppose that there exist λ^0 , $a > 0$, and $1 \leq m < n - 1$ such that $\lambda^0 \mathbf{e} + \sum_{i=1}^m a \mathbf{e}^i \sim \lambda^0 \mathbf{e}$. Then $\lambda^0 \mathbf{e} + \sum_{i=1}^{n-1} a \mathbf{e}^i \sim \lambda^0 \mathbf{e}$.*

In other words, the value of θ in Lemma 1 is at least a .

Proof Suppose that for some $k \geq 2$,

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i \sim \lambda^0 \mathbf{e} \quad (9)$$

but $\lambda^0 \mathbf{e} + \sum_{i=1}^k a \mathbf{e}^i \succ \lambda^0 \mathbf{e}$. We want to show that $k = n$. Assume that $k < n$ and argue to a contradiction. First, we claim that there exists $b > 0$ such that

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + b \mathbf{e}^k \sim \lambda^0 \mathbf{e}. \quad (10)$$

If (10) did not hold, then it follows from (9) and continuity that for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + \varepsilon \mathbf{e}^k \succ \lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + \sum_{i=k}^n \delta \mathbf{e}^i. \quad (11)$$

Applying axioms **PS** and **M**, we can conclude from (11) that

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \delta \mathbf{e}^{k-1} + \varepsilon \mathbf{e}^k \succeq \lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \sum_{i=k-1}^n \delta \mathbf{e}^i \succ \lambda^0 \mathbf{e}. \quad (12)$$

But if $k < n$ and δ and ε are sufficiently small, then by Lemma 1,

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \delta \mathbf{e}^{k-1} + \varepsilon \mathbf{e}^k \sim \lambda^0 \mathbf{e}. \quad (13)$$

Since (13) contradicts (12), (10) must hold.

Now let $\bar{b} \equiv \sup\{b : \lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + b \mathbf{e}^k \sim \lambda^0 \mathbf{e}\}$. We know that $\bar{b} > 0$ and we want to show that $\bar{b} \geq a$. Assume $\bar{b} < a$. Again by continuity it follows that for all $\varepsilon > 0$, there exists $\delta \in (0, \bar{b})$ such that

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + (\bar{b} + \varepsilon) \mathbf{e}^k \succ \lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + \sum_{i=k}^n \delta \mathbf{e}^i.$$

By **PS**

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \delta \mathbf{e}^{k-1} + (\bar{b} + \varepsilon) \mathbf{e}^k \succeq \lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \sum_{i=k-1}^n \delta \mathbf{e}^i. \quad (14)$$

However, by axiom **S**,

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \delta \mathbf{e}^{k-1} + (\bar{b} + \varepsilon) \mathbf{e}^k \sim \lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + (\bar{b} + \varepsilon) \mathbf{e}^{k-1} + \delta \mathbf{e}^k \quad (15)$$

and, provided that $\bar{b} + \varepsilon \leq a$, axiom **M** implies that

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + (\bar{b} + \varepsilon) \mathbf{e}^{k-1} + \delta \mathbf{e}^k \preceq \lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + \delta \mathbf{e}^k \preceq \lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + \bar{b} \mathbf{e}^k. \quad (16)$$

Since

$$\lambda^0 \mathbf{e} + \sum_{i=1}^{k-1} a \mathbf{e}^i + \bar{b} \mathbf{e}^k \sim \lambda^0 \mathbf{e}$$

by the definition of \bar{b} , (14), (15), and (16) combine to imply that

$$\lambda^0 \mathbf{e} \succeq \lambda^0 \mathbf{e} + \sum_{i=1}^{k-2} a \mathbf{e}^i + \sum_{i=k-1}^n \delta \mathbf{e}^i$$

a violation of axiom **M**. This contradiction establishes the lemma. \blacksquare

Proof of Proposition 3 From Lemma 2 it is sufficient to show that $\sup\{x_1 : \lambda^0 \mathbf{e} + x_1 \mathbf{e}^1 \sim \lambda^0 \mathbf{e}\} = \infty$. Denote the supremum by a and assume that a is finite. By assumption $a > \lambda^0$. By Lemma 2 and $n > 2$, there exists $\delta > 0$

such that $\lambda^0 \mathbf{e} + a \mathbf{e}^1 + \delta \mathbf{e}^n \sim \lambda^0 \mathbf{e}$. Therefore, by continuity and monotonicity, there exists $\varepsilon' > 0$ such that

$$(\lambda^0 + \delta) \mathbf{e} \succ \lambda^0 \mathbf{e} + (a + \varepsilon') \mathbf{e}^1 + \delta \mathbf{e}^n. \quad (17)$$

On the other hand, for sufficiently small δ ,

$$\lambda^0 \mathbf{e} + (a + \varepsilon') \mathbf{e}^1 \succ \lambda^0 \mathbf{e} \sim \lambda^0 \mathbf{e} + \delta \sum_{i=1}^{n-1} \mathbf{e}^i \quad (18)$$

where the indifference follows from Lemma 1 and the strict preference follows from the definition of a . Eqs. (17) and (18) violate axiom **PS**. This contradiction establishes the proposition. \blacksquare

4 Indifference Monotonicity

Consider the function $U : \mathfrak{R}^n \rightarrow \mathfrak{R}$, given by

$$U(x) = \begin{cases} \prod x_i & x \in \mathfrak{R}_{++}^n \\ \min\{x_i\} & \text{otherwise} \end{cases} \quad (19)$$

(See Fig. 1 for the case $n = 2$). This function satisfies monotonicity, symmetry, partial separability, and indifference monotonicity. As we show in this section, to a certain extent, it is typical of the functions satisfying these axioms.

Theorem 1 *Let $n \geq 3$. The following two conditions on \succeq are equivalent.*

1. \succeq satisfy monotonicity, symmetry, partial separability, and indifference monotonicity.
2. \succeq satisfy one of the following conditions.
 - (a) There exists $\lambda^* \in [-\infty, \infty]$ and a function $u : (\lambda^*, \infty) \rightarrow \mathfrak{R}$ with $\lim_{x \downarrow \lambda^*} u(x) = -\infty$, such that for $\lambda \leq \lambda^*$, $x \sim \lambda \mathbf{e}$ iff $\min\{x_i\} = \lambda$, and for $\lambda > \lambda^*$, $x \sim \lambda \mathbf{e}$ iff $\sum u(x_i) = nu(\lambda)$.
 - (b) There exists $\lambda^* \in [-\infty, \infty]$ and a function $u : (-\infty, \lambda^*) \rightarrow \mathfrak{R}$ with $\lim_{x \uparrow \lambda^*} u(x) = \infty$, such that for $\lambda \geq \lambda^*$, $x \sim \lambda \mathbf{e}$ iff $\max\{x_i\} = \lambda$, and for $\lambda < \lambda^*$, $x \sim \lambda \mathbf{e}$ iff $\sum u(x_i) = nu(\lambda)$.

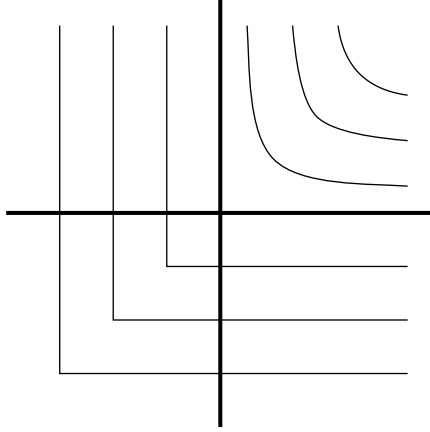


Figure 1: The function U for $n = 2$

Adding a quasi concavity assumption in Condition 1 eliminates possibility 2(b). Likewise, assuming quasi convexity eliminates 2(a). As noted above, axiom **PS** is not restrictive when $n = 2$. Consequently the theorem requires that $n > 2$. Even if we invoke the stronger linearity axiom instead of indifference monotonicity, when $n = 2$, indifference curves in a comonotonic sector must be parallel straight lines, but they are otherwise not restricted (see Roberts [22, p. 430]). For example, the preferences over \mathfrak{R}^2 that are represented by the utility function

$$U(x_1, x_2) = \begin{cases} x_1 + 2x_2 & x_1 \leq x_2 \\ 2x_1 + x_2 & x_1 \geq x_2 \end{cases} \quad (20)$$

satisfy axioms **M**, **S**, **L**, and **PS** but cannot be represented by any of the utility functions in the theorem.

By Propositions 3 and 4 we know that lack of strict monotonicity at a point along the main diagonal implies that the indifference curve through this point is either max or min. The next lemma utilizes the indifference monotonicity axiom to obtain restrictions on upper and lower sets of such indifference curves.

Lemma 3 *Assume that the preferences \succeq satisfy monotonicity, symmetry, partial separability, and indifference monotonicity. If for some point $\lambda^0 \mathbf{e}$ and for some $\varepsilon > 0$, $\lambda^0 \mathbf{e} + \varepsilon \mathbf{e}^i \sim \lambda^0 \mathbf{e}$, then for all $\lambda \leq \lambda^0$, $x \sim \lambda \mathbf{e}$ if and only if $\min x_i = \lambda$.*

Proof Let V be a continuous representation of \succeq . Suppose that for some λ^0 and $\varepsilon > 0$, $\lambda^0 \mathbf{e} + \varepsilon \mathbf{e}^i \sim \lambda^0 \mathbf{e}$. By Proposition 3, $x \sim \lambda^0 \mathbf{e}$ iff $\min x_i = \lambda^0$. Suppose that for some $\lambda < \lambda^0$, j , and $\varepsilon' > 0$, $\lambda \mathbf{e} + \varepsilon' \mathbf{e}^j \succ \lambda \mathbf{e}$. By symmetry, these preferences hold for every j . We show that such preferences contradict our assumptions. Suppose first that for some $\varepsilon'' > 0$ and j , and for the same λ , $\lambda \mathbf{e} - \varepsilon'' \mathbf{e}^j \sim \lambda \mathbf{e}$. Then by Proposition 4, $x \sim \lambda \mathbf{e}$ iff $\max x_i = \lambda$. Let $\tilde{\lambda} = \sup\{\lambda \leq \lambda^0 : x \sim \lambda \mathbf{e} \text{ iff } \max x_i = \lambda\}$. The sup is attained by continuity. If $\tilde{\lambda} = \lambda^0$, then $\lambda^0 \mathbf{e} + \sum_{i=1}^{n-1} \mathbf{e}^i \sim \lambda^0 \mathbf{e} - \mathbf{e}^n$, a violation of monotonicity. Likewise, we assume, wlg, that $\lambda^0 = \min\{\lambda \geq \tilde{\lambda} : x \sim \lambda \mathbf{e} \text{ iff } \min x_i = \lambda\}$. It follows that for every $\lambda \in (\tilde{\lambda}, \lambda^0)$, the preferences \succeq are strictly monotonic at $\lambda \mathbf{e}$, and by axiom **IM**, the preferences are strictly monotonic along the indifference curve through $\lambda \mathbf{e}$.

By axiom **IM** and continuity it is possible to find an open box $B = (\lambda^0 - \delta, \lambda^0 + \delta)^n$ around $\lambda^0 \mathbf{e}$ such that the preferences \succeq are strictly monotonic on the set $C = B \cap \{x : \min x_i < \lambda^0\}$. Since, by monotonicity, indifference curves are connected on C , and since strict monotonicity and axiom **PS** imply axiom **CS**, it follows from Segal [24] that \succeq on C can be represented by a transformation of an additively separable function. Using symmetry we obtain that on C , $V(x) = h(\sum_{i=1}^n v(x_i))$ for continuous, strictly increasing functions h and v . Choose $0 < a' < \delta$, and for $a \in (0, a']$ construct a sequence $x^m(a) \in C$ of the form $x_j^m(a) = \lambda^0 + a$ for $j > 1$ and $x_1^m(a) \uparrow \lambda^0$. Let $x(a) = \lim_{m \rightarrow \infty} x^m(a) = \lambda^0 \mathbf{e} + a \sum_{i=2}^n \mathbf{e}^i$. By assumption, V is continuous at $x(a')$, and since $\lambda^0 + a \in (\lambda^0 - \delta, \lambda^0 + \delta)$, it follows that v is continuous at $\lambda^0 + a$ for all $a \in (0, a']$. Moreover, for all $a \in (0, a']$, h is continuous at $v(\lambda^0) + (n-1)v(\lambda^0 + a)$. To see this, note that by the strict monotonicity of v , $v(a) < v(a')$ for all $a \in (0, a')$. Hence $v(\lambda^0) + (n-1)v(\lambda^0 + a) = v(x_1) + (n-1)v(\lambda^0 + a'')$ for some $x_1 < \lambda^0$ sufficiently close to λ^0 and $a'' \in (a, a')$. Since $x_1 \mathbf{e}^1 + a'' \sum_{i=2}^n \mathbf{e}^i \in C$, h is continuous at $v(\lambda^0) + (n-1)v(\lambda^0 + a)$. Since $V(x^m(a)) = h(v(x_1^m(a)) + (n-1)v(\lambda^0 + a))$, we have, by the continuity of V , h , and v ,

$$V(x(a)) = h((n-1)v(\lambda^0 + a) + v(\lambda^0)).$$

Since $V(x(a))$ is constant for $a > 0$, this equation contradicts the strict monotonicity of h and v . \blacksquare

Proof of Theorem 1 (2) \implies (1): Monotonicity, symmetry, and indifference monotonicity are obviously satisfied. We will use the second part of Proposition 1 to obtain the **PS** axiom. For part 2(a), let

$$f(x_1, \dots, x_{n-1}) = \begin{cases} \min\{x_k\} & \min\{x_k\} \leq \lambda^* \\ \exp\left(\sum_{k=1}^{n-1} u(x_k)\right) + \lambda^* & \min\{x_k\} > \lambda^* \end{cases}$$

and

$$g(y_1, y_2) = \begin{cases} \min\{y_1, y_2\} & \min\{y_1, y_2\} \leq \lambda^* \\ \exp\left(\ln(y_1 - \lambda^*) + u(y_2)\right) + \lambda^* & \min\{y_1, y_2\} > \lambda^* \end{cases}$$

For Part 2(b) of the theorem, let

$$f(x_1, \dots, x_{n-1}) = \begin{cases} \max\{x_k\} & \max\{x_k\} > \lambda^* \\ -\exp\left(-\sum_{k=1}^{n-1} u(x_k)\right) + \lambda^* & \max\{x_k\} \leq \lambda^* \end{cases}$$

and

$$g(y_1, y_2) = \begin{cases} \max\{y_1, y_2\} & \max\{y_1, y_2\} > \lambda^* \\ -\exp\left(\ln(-y_1 + \lambda^*) - u(y_2)\right) + \lambda^* & \max\{y_1, y_2\} \leq \lambda^* \end{cases}$$

(1) \implies (2): If, for all λ , \succeq are strictly monotonic at $\lambda \mathbf{e}$, then by **IM** the preferences \succeq satisfy **SM**. By symmetry it follows that such preferences can be represented by $\sum u(x_i)$ for a strictly increasing, continuous function u (see Debreu [6] and Gorman [10]).

If there exists λ such that the preferences \succeq are not strictly monotonic at $\lambda \mathbf{e}$, then, by symmetry, there are three possibilities.

1. There exists $\varepsilon > 0$ such that for all $\varepsilon' < \varepsilon$, $\lambda \mathbf{e} + \varepsilon' \mathbf{e}^1 \sim \lambda \mathbf{e} \succ \lambda \mathbf{e} - \varepsilon' \mathbf{e}^1$.
2. There exists $\varepsilon > 0$ such that for all $\varepsilon' < \varepsilon$, $\lambda \mathbf{e} + \varepsilon' \mathbf{e}^1 \succ \lambda \mathbf{e} \sim \lambda \mathbf{e} - \varepsilon' \mathbf{e}^1$.
3. There exists $\varepsilon > 0$ such that for all $\varepsilon' < \varepsilon$, $\lambda \mathbf{e} + \varepsilon' \mathbf{e}^1 \sim \lambda \mathbf{e} \sim \lambda \mathbf{e} - \varepsilon' \mathbf{e}^1$.

As in the proof of Lemma 3, it follows by Propositions 3 and 4 that the third case violates monotonicity. We show here that the first case implies the representation of part 2(a) of the theorem. The proof that the second case implies 2(b) is similar.

Suppose the first case holds, and let $\lambda^* = \sup\{\lambda : \exists \varepsilon > 0 \text{ such that } \lambda \mathbf{e} + \varepsilon \mathbf{e}^1 \sim \lambda \mathbf{e}\}$. If $\lambda^* = \infty$, we are through. Otherwise, it follows by Lemma 3 that for all $\lambda \leq \lambda^*$, $x \sim \lambda \mathbf{e}$ iff $\min\{x_i\} = \lambda$.

By definition, for all $\lambda > \lambda^*$ the preferences \succeq are strictly monotonic at $\lambda \mathbf{e}$. By **IM**, they are strictly monotonic on $(\lambda, \infty)^n$. Again by Debreu [6] and Gorman [10], there exists a function $u : (\lambda^*, \infty) \rightarrow \mathfrak{R}$ such that these preferences satisfy $x \sim \lambda \mathbf{e}$ iff $\sum u(x_i) = nu(\lambda)$. Finally, if $\lim_{x \downarrow \lambda} u(x) > -\infty$, then there are two points y, y' such that $\min\{y_i\} = \min\{y'_i\} = \lambda$, but $\lim_{x \downarrow y} \sum u(x_i) \neq \lim_{x \downarrow y'} \sum u(x_i)$, while $y \sim y'$, a contradiction. ■

Note the role that axiom **IM** plays in the proof.

Example 1 Define $U : \mathfrak{R}^n \rightarrow \mathfrak{R}$ by

$$U(x) = \begin{cases} \min\{x_i\} & \min\{x_i\} > 1 \\ \prod_{i=1}^n \min\{x_i, 1\} & \min\{x_i\} \in [0, 1] \\ \min\{x_i\} & \min\{x_i\} < 0 \end{cases}$$

The preferences that are represented by this function are monotonic, symmetric, quasi-concave, and satisfy axiom **PS**. They fail to satisfy the conclusion of the theorem, as an interval over which preferences are strictly monotonic along the diagonal is sandwiched between two non-empty sets in which preferences are $\min x_i$. The example fails to satisfy the assumptions of the theorem. Although preferences are strictly monotonic at $\lambda \mathbf{e}$ for $\lambda \in (0, 1)$, preferences are not strictly monotonic for all points on such an indifference curve.

5 Linearity and Comonotonic Flatness

In this section we discuss the implications of axiom **L** on our results.

Theorem 2 *Let $n \geq 3$. The following two conditions on the preferences \succeq over \mathfrak{R}^n are equivalent.*

1. \succeq satisfy monotonicity, symmetry, linearity, and partial separability.
2. \succeq can be represented by one of the following functions.

$$(a) \ U(x_1, \dots, x_n) = \max\{x_k\}.$$

$$(b) \ U(x_1, \dots, x_n) = \min\{x_k\}.$$

$$(c) \ U(x_1, \dots, x_n) = \sum x_k.$$

When partial separability is replaced with complete separability, Maskin [17] proves the equivalence of the first condition and the third possible representation in a social choice framework.

Proof (2) \implies (1): Since on comonotonic sectors, the three functions suggested by the theorem are linear, and since the changes that are permitted by axiom **L** do not take a point to a new comonotonic sector, it follows that all three functions satisfy axiom **L**. Monotonicity and symmetry are obviously satisfied. Using Proposition 1, the **PS** axiom follows easily. For case (a), let $f(x_1, \dots, x_{n-1}) = \max\{x_k\}$ and $g(x_1, x_2) = \max\{x_1, x_2\}$; for case (b), let $f(x_1, \dots, x_{n-1}) = \min\{x_k\}$ and $g(x_1, x_2) = \min\{x_1, x_2\}$; and for case (c), let $f(x_1, \dots, x_{n-1}) = \sum x_k$ and $g(x_1, x_2) = x_1 + x_2$.

(1) \implies (2): Since axiom **L** is a property of preferences, rather than of representation functions, we can choose U such that $U(\lambda \mathbf{e}) = \lambda$. Then for every x ,

$$U(\beta x) = \beta U(x) \text{ and } U(x + \alpha \mathbf{e}) = U(x) + \alpha \tag{21}$$

We first analyze preferences that satisfy the complete separability axiom.

Lemma 4 *Let M be a comonotonic section of \mathfrak{R}^n . If the preferences \succeq on M satisfy monotonicity, linearity and complete separability, then they can be represented by a function of the form $\sum a_k x_k$.*

Maskin [17] proved this lemma (with $a_1 = \dots = a_n = 1$) under the additional assumption of symmetry.

Proof By an extension of theorems of Debreu [6] and Gorman [10] (see Wakker [26]), we know that \succeq on M can be represented by $\sum u_k(x_k)$, so on M , $U(x) = h(\sum u_k(x_k))$ for some increasingly monotonic function h . By eq. (21) we obtain for all $x \in M$ and sufficiently small α

$$h\left(\sum u_k(x_k + \alpha)\right) = h\left(\sum u_k(x_k)\right) + \alpha \quad (22)$$

and for β sufficiently close to 1

$$h\left(\sum u_k(\beta x_k)\right) = \beta h\left(\sum u_k(x_k)\right). \quad (23)$$

The functions h and u_k , $k = 1, \dots, n$ are monotonic, therefore almost everywhere differentiable. The rhs of eq. (22) is differentiable with respect to α , therefore h and u_k are differentiable functions. Differentiate this equation with respect to α to obtain

$$h'\left(\sum u_k(x_k + \alpha)\right) \cdot \sum u'_k(x_k + \alpha) = 1. \quad (24)$$

In particular, for $\alpha = 0$ we obtain

$$h'\left(\sum u_k(x_k)\right) \cdot \sum u'_k(x_k) = 1. \quad (25)$$

Differentiate eq. (22) with respect to x_ℓ and obtain

$$h'\left(\sum u_k(x_k + \alpha)\right) u'_\ell(x_\ell + \alpha) = h'\left(\sum u_k(x_k)\right) u'_\ell(x_\ell). \quad (26)$$

From eq. (24) and eq. (25) it follows that $h'(\sum u_k(x_k + \alpha))$ and $h'(\sum u_k(x_k))$ are not zero. Therefore, if $u'_\ell(x_\ell) = 0$, then by eq. (26), for all α , $u'_\ell(x_\ell + \alpha) = 0$, and $u_\ell(x_\ell) \equiv a_\ell$. If, for all ℓ , $u'_\ell(x_\ell) = 0$, then by the above argument the claim is satisfied with $a_1 = \dots = a_n = 0$. Otherwise, suppose wlg that $u'_1(x_1) \neq 0$. If, for all other ℓ , $u'_\ell(x_\ell) = 0$, then \succeq are represented by $v(x_1) := h(u_1(x_1) + \sum_{k=2}^n a_k)$, where for every α ,

$$\begin{aligned} v(x_1 + \alpha) &= h\left(u_1(x_1 + \alpha) + \sum_{k=2}^n a_k\right) = \\ &= h\left(u_1(x_1) + \sum_{k=2}^n a_k\right) + \alpha = v(x_1) + \alpha. \end{aligned}$$

(The second equation follows by eq. (22)). Similarly, by eq. (23), $v(\beta x_1) = \beta v(x_1)$. Hence v is linear. So suppose $u'_1(x_1) \neq 0$ and $u'_2(x_2) \neq 0$. From eq. (26) it follows that

$$\frac{u'_1(x_1 + \alpha)}{u'_1(x_1)} = \frac{h'(\sum u_k(x_k))}{h'(\sum u_k(x_k + \alpha))} = \frac{u'_2(x_2 + \alpha)}{u'_2(x_2)}.$$

Fix x_2 and consider x_1 and α as variables to obtain that

$$\frac{u'_1(x_1 + \alpha)}{u'_1(x_1)} = g(\alpha) \implies u'_1(x_1 + \alpha) = g(\alpha)u'_1(x_1). \quad (27)$$

The solution of this functional equation is $u'_1(x_1) = \eta e^{\zeta x_1}$ and $g(\alpha) = e^{b\alpha}$ (see Aczél [1, p. 143, Theorem 2]). Hence $u_1(x_1) = a_1 e^{b_1 x_1} + c_1$ if $\zeta \neq 0$, and $u_1(x_1) = a_1 x_1 + c_1$ if $\zeta = 0$. Similarly, for every k , if $u'_k(x_k) \neq 0$, then either $u_k(x_k) = a_k e^{b_k x_k} + c_k$, $b_k \neq 0$, or $u_k(x_k) = a_k x_k + c_k$. Since we can define $h^*(z) = h(z + \sum c_k)$, we can assume wlg that $c_k = 0$ for all k .

If for some k , $u_k(x_k) = a_k x_k$, then from eq. (27) it follows that $g(\alpha) = 1$. On the other hand, if $u_k(x_k) = a_k e^{b_k x_k}$, then $g(\alpha) = e^{b_k \alpha}$. In other words, either for every k such that $u'_k \neq 0$ we have $u_k(x_k) = a_k e^{b_k x_k}$, or for all such k , $u_k(x_k) = a_k x_k$.

Suppose the first case. By eq. (25), if $\sum u_k(x_k)$ does not change, then neither should $\sum u'_k(x_k)$. Moreover, since by monotonicity h' is almost everywhere non-zero, $\sum u_k(x_k)$ and $\sum u'_k(x_k)$ have the same indifference curves. In particular, at each point $\sum u_k$ and $\sum u'$ should have the same MRS (marginal rate of substitution). The MRS between k and ℓ for $\sum u_k$ is given by $a_k b_k e^{b_k x_k} / a_\ell b_\ell e^{b_\ell x_\ell}$, while the corresponding MRS for $\sum u'_k$ is given by $a_k b_k^2 e^{b_k x_k} / a_\ell b_\ell^2 e^{b_\ell x_\ell}$, hence $b_k = b_\ell = b$. Differentiate eq. (23) with respect to β , set $\beta = 1$, and obtain

$$h'(\sum u_k(x_k)) \cdot \sum x_k u'_k(x_k) = h(\sum u_k(x_k)).$$

Therefore, the MRS between k and ℓ for the function $\sum x_k u'_k(x_k)$ must be the same as the corresponding MRS for $\sum u_k(x_k)$. It follows that

$$\frac{a_k b e^{b x_k} + a_k b^2 x_k e^{b x_k}}{a_\ell b e^{b x_\ell} + a_\ell b^2 x_\ell e^{b x_\ell}} = \frac{a_k b e^{b x_k}}{a_\ell b e^{b x_\ell}} \implies b = 0$$

hence the claim of the lemma. ■

Conclusion 1 If we add symmetry to the assumptions of Lemma 4 we obtain that \succeq can be represented by $\sum x_k$ (see Maskin [17]).

Suppose now that there exists a point x where the preferences \succeq satisfy strict monotonicity. As noted above (see discussion after Definition 2), the linearity axiom then implies that the preferences \succeq are strictly monotonic. Lemma 4 then implies the third possible representation of the theorem. On the other hand, if for some x , i , and $\varepsilon > 0$, $x + \varepsilon \mathbf{e}^i \sim x$, then linearity implies that for all λ , $\lambda \mathbf{e} + \varepsilon \mathbf{e}^i \sim \lambda \mathbf{e}$. Proposition 3 then implies the second possible representation of the theorem. Finally, if for some x , i , and $\varepsilon > 0$, $x - \varepsilon \mathbf{e}^i \sim x$, then the first possible representation of the theorem likewise follows by Proposition 4. ■

The linearity axiom plays two roles in the proof of Theorem 2. First, it enables us to show that when preferences are completely separable, then they can be represented by a linear function. Second, it guarantees that a failure of monotonicity will hold throughout a comonotonic sector. Theorem 1 provides one characterization of preferences when the linearity axiom does not hold, while Example 1 demonstrates that some assumption is needed to have control over the way in which local violations of monotonicity influence the global behavior of preferences. To isolate the second role of the linearity axiom, we discuss the implications of replacing linearity with the weaker condition of comonotonic flatness.

Theorem 3 *Let $n \geq 3$. The following two conditions on the preferences \succeq are equivalent.*

1. \succeq satisfy monotonicity, symmetry, comonotonic flatness, and partial separability.
2. \succeq can be represented by one of the following functions.
 - (a) $U(x_1, \dots, x_n) = \max\{x_k\}$.
 - (b) $U(x_1, \dots, x_n) = \min\{x_k\}$.
 - (c) $U(x_1, \dots, x_n) = \sum u(x_k)$ for some strictly increasing u .

Proof Axiom **CF** implies **IM**, therefore Propositions 3 and 4 guarantee that when preferences fail to be strictly monotonic, they can be represented by

either the min or the max function. By Proposition 2, if preferences satisfy axiom **SM**, then they satisfy axiom **CS**. By Debreu [6] and Gorman [10] the theorem follows. ■

Figure 2 summarizes Theorems 1–3. In all three cases we assume that the preferences satisfy monotonicity, symmetry, and partial separability.

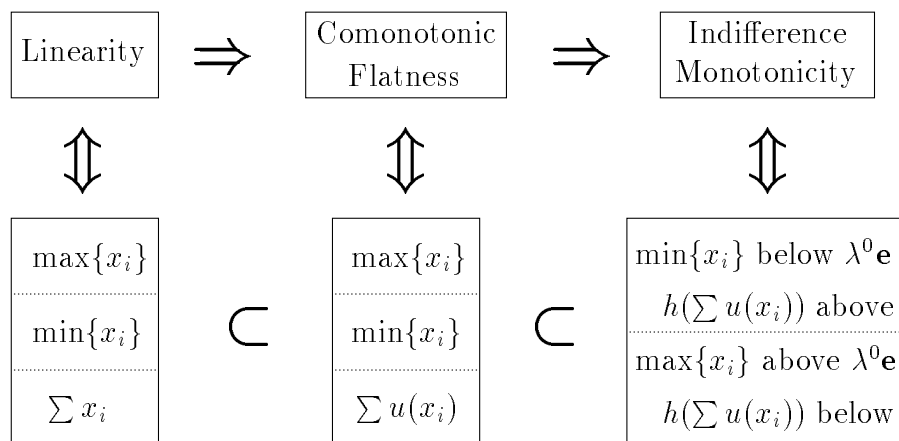


Figure 2: Theorems 1–3

Remark All the results of the paper can be obtained for a symmetric box $(a, b)^n \subset \mathbb{R}^n$. The only place where a more detailed (but trivial) argument is needed is when the linearity axiom is invoked.⁷ We omit this discussion. Note however that the box needs to be symmetric, as our results strongly depend on the symmetry axiom. This is in contrast with papers dealing with complete separability, where the main diagonal usually plays no important role.⁸

⁷In the context of social choice, Blackorby and Donaldson [3] assume complete separability and split linearity into ratio-scale comparability and translation-scale comparability. If they assume only the multiplicative part of linearity (ratio-scale comparability) their assumptions characterize a form of generalized mean. Their arguments are complicated by the sign of x_i , hence domain restrictions could be important when one assumes only the second condition in the linearity assumption.

⁸An exception is Wakker [26], where preferences are defined over a comonotonic sector.

6 Harsanyi and Rawls

Social welfare functions are usually functions of individual (von Neumann & Morgenstern) utilities. This approach and its axiomatic foundations have a serious flaw. vN&M utility functions are unique up to increasing affine transformations, but only one version of each utility function appears in the social welfare function. It is not clear which representation of individual preferences should enter the social welfare function. (For a rigorous discussion of this problem, see Weymark [27].) Recently, three similar (axiomatic) choices of affine transformations were offered by Dhillon and Mertens [7], Karni [14], and Segal [25]. Here we suggest another possible choice of the vN&M utility functions and illustrate some of the paper's results. The trick is to have social preferences defined directly over allocations, without making references to individual preferences. This is possible because of the specific set-up of the social choice problem we discuss.

An n -person society has to allocate, with probability p , $m < n$ units of an indivisible good. For example, there is a p -probability that the country will have to go to war, in which case the army will need to draft m extra soldiers. If each person can receive at most one unit of this good,⁹ then there are $\binom{n}{m}$ pure social policies, none of which are egalitarian. Indeed, by definition, it is impossible to obtain an egalitarian ex post distribution of the m units. But society is not bound, ex ante, to choose such a pure policy. As argued by Harsanyi [11, 12, 13], society can randomize, and may have preferences over such randomizations. In our case, society may assign each member i a probability p_i , denoting the probability this person will be drafted. The constraints are

1. $\sum p_i = pm$; and
2. $p_i \leq p$, $i = 1, \dots, n$.

Society can randomize over pure social policies (that is, over m person selections) such that exactly m people will be drafted, and the probability person i will be selected is p_i (see the Appendix for such a procedure). We assume that society has preferences \succeq over distributions of the form $(p_1,$

⁹If individuals could receive more than one unit (while others received none), then the assumption that society is indifferent regarding the actual receiver of each unit (see below) is much less appealing.

\dots, p_n).¹⁰ Assuming that society’s preferences depend only on (p_1, \dots, p_n) is a strong assumption. It requires that society is indifferent between two random social policies that lead to the same distribution of prizes. Such preferences may, but need not, favor egalitarian distributions over uneven ones. These preferences are ex ante, and they compare probability allocations that cannot be compared ex post. Specifically, once society knows whether it is in war, the probability p is no longer relevant, and the new social choice problem becomes to draft m people into the army. One may indeed suggest deferring such decisions till later, moreover, since m too may change, to wait until we know exactly how many soldiers need to be drafted.

The separation of the actual social choice problem from other situations that could have, but did not happen, is far from being obvious. Harsanyi [12] is usually understood to assume such separation. Yaari [28] points out that such a separation may make (weighted) utilitarian and Rawlsian allocations identical. The issue is formally explored in Segal [25], where it is assumed that a solution must apply simultaneously to all possible sets of resources. We offer a similar approach, and would like social preferences to hold at the point where the eventuality of war is still uncertain, and the number of recruits m may change. Next we try to justify the assumptions of Theorem 1 in the context of this social choice problem.

Monotonicity and partial separability need some explanation. If p and m are fixed, then the first constraint makes the “if” part of the monotonicity axiom empty. We therefore assume that p and m can vary, and that social preferences are over the set $(0, 1)^n$. The monotonicity axiom asserts that if p or m are increasing, and each member of society receives at least some of the added probability, then society is better off.¹¹

Partial separability too is reasonable in the present context. Changing the probability for one person should not strictly reverse the induced order on the rest of the probability distribution. This assumption makes sense even if there are special links between individuals. Suppose, for example, that 15%

¹⁰Observe that although these are probabilities, the vector (p_1, \dots, p_n) is not a probability distribution, as $\sum p_i = pm$. Further, since precisely m people are drafted, the probabilities are not stochastically independent.

¹¹Of course, if the items to be allocated are considered “bads” (for example, draft service), then either the monotonicity axiom should be reversed, or one should redefine the commodity to be allocated. In the draft example, the good should be “not serving in the army,” and society will have $n - m$ units of this good.

or the population are of race r_1 , while the remaining 85% are of race r_2 . For simplicity, let $n = 100$. Let ε be close to zero, and consider the following four distributions.

	$i = 1, \dots, 14$	$i = 15$	$i = 16, \dots, 100$
A	$p_i = \frac{1}{2}$	$p_i = \frac{1}{2}$	$p_i = \frac{1}{2}$
B	$p_i = \frac{1}{2}$	$p_i = 1$	$p_i = \frac{1}{2} - \frac{1}{170}$
C	$p_i = \varepsilon$	$p_i = \frac{1}{2}$	$p_i = \frac{1}{2}$
D	$p_i = \varepsilon$	$p_i = 1$	$p_i = \frac{1}{2} - \frac{1}{170}$

If the allocation is not racially biased (as is the case with allocations A and B), then society will probably (strictly) prefer the more egalitarian distribution A to B . However, if the allocation favors group r_2 , society may prefer to compensate at least some of the members of group r_1 , hence $D \succ C$. Partial separability alone does not rule out such preferences, since changing person 1's outcome from $\frac{1}{2}$ to ε may make society indifferent, and then changing person 2's outcome from $\frac{1}{2}$ to ε may reverse these preferences. Note, however, that such a reversal of preferences is ruled out by partial set separability (see footnote 4).¹²

The linearity axiom has two parts, homogeneity and additivity. In the present context, homogeneity suggests that social preferences for probability distributions conditional on p are always the same. In other words, society has preferences for distributions of the probabilities needed to select m individuals out of n . These preferences do not depend on the probability that society will actually need to select these people. The additivity part suggests that if m increases, and the *added* probability is equally distributed, the preferences between two distributions do not change.

The most controversial of our assumptions is symmetry. This axiom is often used in the social choice literature in reference to utilities.¹³ Our model so far has no utilities (in fact, we did not even introduce individual preferences), so the symmetry axiom needs a fresh defense. In the absence of information about individual well being, it is plausible for the social planner to treat individual members of society as having equal rights to allocations. That

¹²A strict reversal of preferences is not consistent with the assumptions used in any of our theorems. For all of the functional forms identified in these theorems, it is impossible for society to strictly prefer A to B and to strictly prefer D to C .

¹³See, for example, Diamond [8], or Ben-Porath, Gilboa, and Schmeidler [2]

is, the preferences over allocations should be independent of the identities of the agents. The symmetry axiom requires precisely this level of anonymity, namely that probability distributions are ranked with no reference to the identity of the individuals receiving these probabilities.¹⁴

Given our assumptions, Theorem 2 implies¹⁵ that society ranks probability distributions of the form (p_1, \dots, p_n) by one of the following three social welfare functions: 1. $\sum p_i$; 2. $\min\{p_i\}$; and 3. $\max\{p_i\}$. Note that given the constraint $\sum p_i = pm$, rule 1 effectively says that society is indifferent over all feasible probability distributions.

Suppose that all members of society are expected utility maximizers. Choose a normalization of the utilities from the indivisible good such that $u_1(0) = \dots = u_n(0) = 0$, and $u_1(1) = \dots = u_n(1) = 1$. Then p_i stands not only for the probability that person i will receive a unit of the good, but also for his expected utility from the lottery this probability generates. Rule 1 above is therefore the same as Harsanyi's [11, 12] social welfare function, while 2 is a Rawlsian-like [21] function. Note that we do not claim that 1 yields a utilitarian social ranking, because nothing in our model enables us to compare individual utilities. Indeed, as argued by Weymark [27], utilitarianism is inconsistent with the above normalization unless initially $u_1(1) = \dots = u_n(1)$.

If instead of linearity we assume indifference monotonicity we may get a combination of Harsanyi and Rawls. For example, define

$$U_{a,\tau}(p_1, \dots, p_n) = \begin{cases} \min p_i & \min p_i \leq a \\ \exp(\sum u_{\tau,a}(p_i)) + a & \min p_i > a \end{cases}$$

Where

$$u_{a,\tau}(p) = \begin{cases} \log(p - a) & a < p \leq \tau \\ \frac{\log(\tau - a)}{\tau} p & p > \tau \end{cases}$$

As $\tau \rightarrow a$, the area where indifference curves of $U_{a,\tau}$ are linear becomes almost the whole upper set of the indifference curve $\min\{p_i\} = a$. In other words, these social preferences are Rawlsian up to a certain threshold, beyond which they are Harsanyian.

¹⁴For a similar intuition, but with a different notion of symmetry, see Segal [25].

¹⁵See the remark after Theorem 3.

Appendix: Choosing m out of n People

To simplify the analysis, we assume throughout that $p = 1$, that is, society knows for sure that it will need to draft m people.

For a set $A_n = \{1, \dots, n\}$, let $\mathcal{S}_{m,n}$ be the set of all m -elements subsets of A_n . Let $p = (p_1, \dots, p_n) \in \mathfrak{R}_+^n$ such that $\sum p_i = m$. We want to show that there are $q_S, S \in \mathcal{S}_{m,n}$ such that

1. For every S , $q_S \in [0, 1]$.
2. $\sum_S q_S = 1$.
3. For every i , $\sum_{S:i \in S} q_S = p_i$.

The task is obvious either when $m = 1$ ($q_{\{i\}} = p_i$), or when $m = n$ (for every i , p_i must be 1, and $q_{A_n} = 1$). Let $m < n$, and suppose we found appropriate q -vectors for all $(m', n') \neq (m, n)$ such that $m' \leq m$ and $n' \leq n$. We now show how to construct the vector q for the pair m, n .

If for some i , $p_i = 1$, then assume, wlg, that $p_n = 1$, get the vector q' for the pair $m, n - 1$, and define $q_S = 0$ if $n \notin S$ and $q_S = q'_{S \setminus \{n\}}$ if $n \in S$. If for some i , $p_i = 0$, then assume, wlg, that $p_n = 0$, get the vector q' for the pair $m, n - 1$, and define $q_S = 0$ if $n \in S$ and $q_S = q'_{S \setminus \{n\}}$ if $n \notin S$. Otherwise, assume, wlg, that for all i , $0 < p_1 \leq p_2 \leq \dots \leq p_n < 1$. Define $q^* = q_{\{1, \dots, m\}} = \min\{p_1, 1 - p_n\}$, and set $r_i = \frac{p_i - q^*}{1 - q^*}$ for $i = 1, \dots, m$ and $r_i = \frac{p_i}{1 - q^*}$ for $i = m + 1, \dots, n$. Clearly, for all i , $r_i \in [0, 1]$, $\sum r_i = m$, and either $r_1 = 0$ or $r_n = 1$. In the first case, define for $i = 2, \dots, n$, $r'_i = r_{i+1}$, find a vector q' that solves the problem for the pair $m, n - 1$ with the vector r' , define $q_S = p_1$ for $S = \{1, \dots, m\}$, $q_S = 0$ for all other sets S containing person 1, and $q_S = (1 - p_1)q'_{S'}$ otherwise, where $j \in S'$ iff $j + 1 \in S$. Since $\sum_{S \subset \{2, \dots, n\}} q'_S = 1$, it follows that $\sum_S q_S = 1$. By definition, $\sum_{S:1 \in S} q_S = p_1$. For all other $i = 2, \dots, m$, $\sum_{S:i \in S} q_S = p_1 + (1 - p_1) \sum_{S \subset \{2, \dots, n\}: i \in S} q'_{S'} = p_1 + (1 - p_1)r_i = p_i$. Finally, for $i = m + 1, \dots, n$, $\sum_{S:i \in S} q_S = (1 - p_1) \sum_{S \subset \{2, \dots, n\}: i \in S} q'_{S'} = (1 - p_1)r_i = p_i$.

If $r_n = 1$, then solve the problem for $m - 1, n - 1$ for r_1, \dots, r_{n-1} , and get the vector q' . Define $q_S = p_n q'_{S \setminus \{n\}}$ if $n \in S$, and $q_S = 0$ otherwise.

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