Referrals in Search Markets*

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Abstract

This paper compares the equilibrium outcomes in search markets with and without referrals. Although it seems clear that consumers would benefit from honest referrals, it is not at all clear whether firms would unilaterally provide information about competing offers since such information could encourage the consumer to purchase the product elsewhere. In a model of a horizontally differentiated product and sequential consumer search, we show that valuable referrals can arise in the equilibrium: a firm will give referrals to consumers whose ideal product is sufficiently far from the firm's offering. It is found that prices in the equilibrium are higher in markets with referrals. Although referrals can make consumers worse off, referrals lead to a Pareto improvement as long as the search cost is not too low relative to product heterogeneity. The effects of referral fees and third-party referrals are examined, and policy implications are drawn.

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1 Introduction

In a number of industries, consumers have to incur substantial costs to learn about available products and their characteristics. Examples include high-tech products such as digital cameras, specialized services such as picture framing, and professional services in areas of law, accounting, real estate, and health care. These products and services are typically purchased infrequently, or else their characteristics change often and are thus difficult to assess. As a result, a typical consumer needs to conduct a costly search in order to find the products and services matching her tastes.

In contrast, firms know the industry well and can gather information on products and services provided by their competitors without much effort. They obtain useful information simply by engaging in their everyday business. Through conversations with their customers, firms could also ascertain the customers' preferences and inform them about where they could find the product or service they are looking for. Such referrals can reduce consumers' search costs tremendously. However, referrals made by competing firms or intermediaries are usually subject to incentive problems. A referring firm faces an incentive problem in choosing between serving a consumer itself or referring the consumer to another seller. Referrals can also be influenced by kickbacks – side payments in the form of flat referral fees or commissions.

To illustrate the type of referrals we have in mind, consider the following scenarios from retail markets. A consumer asking for a light-weight camper's sleeping mat at *Sears* is referred to *Sports Authority*. A *Garden Ridge* customer seeking to custom frame an oil painting is referred to a competing framing store that can provide the exact type of service the customer is looking for. A shopper visiting *HHGregg* finds that the particular TV model she is searching for is unavailable, and she is referred to *Sears*. At *Circuit City*, a shopper cannot find a small digital camera with a wide angle lens, and he is advised to look for the camera at *Amazon.com*. A jewelry store customer in need of a battery replacement for her watch is referred to another jewelry store that performs the task. In each of these situations experienced by the authors first-hand, the store's sales clerk chose to provide a valuable referral to the consumer who did not find the what he or she was looking for in the store.

Referrals play an important role in markets for services as well. Professional associations in law, accounting, and real estate have established codes of honor that regulate referral activity to avoid conflicts of interest. For instance, the Code of Professional Responsibility of the American Bar Association prescribes the division of fees by lawyers in proportion to the actual services performed or responsibility assumed. In particular, the division of fees for "pure" referrals is not allowed. Similarly, the conduct rules of the American Institute of Certified Public Accountants prohibited referral fees until FTC found the ban to have anti-competitive effects on potential competition.

In this paper, we focus on the effects of referrals on the equilibrium prices, profits, and consumer welfare. In particular, we ask the following questions. Which referral policies can exist in the equilibrium? Are referral practices beneficial for buyers? Are they beneficial for sellers? Do referrals increase or decrease the equilibrium market prices? Do the sizes of flat referral fees or commissions influence the equilibrium price? Are third-party referrals good for consumers and/or for the industry?

To analyze the role of referrals in search markets we introduce referrals into a model of competition between firms producing a horizontally differentiated product. The product space is a unit circle, with firms evenly and consumers uniformly distributed over it. Consumer willingness-to-pay for a product decreases as a mismatch between a firm's product and the consumer's ideal product increases. A consumer can engage in sequential search by randomly sampling products at a constant marginal search cost. Upon a visit to a firm, the consumer learns the features of the product offered by the firm. Wolinsky (1983) analyzed symmetric oligopolistic price equilibrium in such a model.

In this paper, we allow firms to refer consumers who visit them to other sellers. A firm does not have to provide consumers with referrals if the firm would rather sell its own commodity to them. However, we assume that when a firm knows that a consumer would not purchase its product, it would refer her to the best-matching firm. We first derive and compare equilibria with and without referrals in the case of no referral fees. Then, we introduce referral fees and third party referrals into the model and observe how they affect the equilibrium.

The results are somewhat surprising. The norm of referring consumers to the best-matching competitors tends to increase prices and is generally preferred by sellers. Although referrals provide consumers with valuable information that decreases search costs and improves the product match, consumers may be worse off in the presence of referrals. This happens when the benefits from referrals (i.e. lower search costs and better product match) are outweighed by the loss to consumers from the equilibrium price increase. We show that for sufficiently low search costs relative to degree of product heterogeneity, consumers would prefer markets without referrals. At the same time, referrals lead to a Pareto improvement in markets with relatively high search costs (Proposition 4). If referral fees are introduced, the equilibrium price increases with the referral fees. This is not surprising. Referral fees encourage a firm to raise its price because it can obtain referral fees when it cannot sell its product (Propositions 6 and 6'). Third-party brokers do not face the incentive problem of firms selling their own products, but the presence of third-party referrals further increases the equilibrium price. The reason is that a firm's demand curve becomes more inelastic because consumers who visit the third-party brokers obtain the information on which firms sell their ideal products, and they would not search for a better-matching product. Such a negative externality on consumers from improved information was first demonstrated by Anderson and Renault (2000) in an economy with horizontally differentiated search goods.[1] As a result, even though the incentive problem is improved by third-party referrals, consumers can be worse off due to an increase in the equilibrium prices (Propositions 7 and 7').

The rest of the paper is organized as follows. Section 2 briefly discusses related literature. Section 3 presents a model of price competition between firms selling a horizontally differentiated product under imperfect consumer information. We consider search markets with and without referrals and compare the referral and random search equilibria that arise in such markets. In Section 4, we extend the basic model by examining markets with referral fees and third-party brokers. Section 5 offers concluding comments. Appendix contains all

[1] In contrast, this negative externality from better-informed consumers is not present in a homogeneous commodity search model. In that setting, Diamond (1971) shows that all firms charge the monopoly price when all consumers face the same positive search cost. Having informed consumers (with search costs in the neighborhood of zero) generates price dispersion and lower prices (Rob, 1985, and Stahl, 1989). One of the results in Burdett and Judd (1983) is that price dispersion and lower prices can also arise due to an introduction of an information-improving noise in the sequential search technology of the Diamond model such that a single random search gives at least one quote but may give multiple quotes. Thus, improvements in price information create a positive externality in the form of a price reduction for consumers in homogeneous product search markets. In the context of dynamic search in the labor market, Rogerson, Shimer and Wright (2005) discuss two different ways to generate price (wage) dispersion in Diamond-type models.

the proofs.

2 A Brief Literature Review

Given differences in opinion about referral policies and their prominence in many industries, it is perhaps surprising that there is not much economic theory on the topic. One exception is a study by Garicano and Santos (2004), which examines referrals between vertically differentiated firms (vertical referrals). Due to complementarity between the values of opportunities and firms' skills, efficient matching involves assigning more valuable opportunities to high-quality firms. The authors show that flat referral fees can support efficient referrals from high-quality to low-quality firms but not vice versa. The moral hazard problem arises because a low-quality firm has an incentive to keep the best opportunities to itself rather than refer them to a high-quality firm.[2] However, the model does not allow the firms' clients to conduct searches on their own. In contrast, this paper focuses on consumer search versus referrals in a market with horizontally differentiated firms (horizontal referrals). Our model of sequential consumer search follows Wolinsky (1983, 1986). Consumers randomly visit a firm and at a cost learn about the firm's product and price.[3]

Although it is not directly related to referrals, there is a nice literature on the role of the middleman in buyer-seller search markets: see Rubinstein and Wolinsky (1987), Gehrig (1993), Yavas (1994), Bloch and Ryder (2000), Rust and Hall (2003) and Miao (2006), among others. Sellers and buyers engage in searches for each other, and middlemen help match the two sides or set up markets for them. Gehrig (1993), Rust and Hall (2003) and Miao (2006) consider static and dynamic models in which agents have the choice between an intermediated market and a decentralized search market accompanied by some bargaining procedure. These papers explore which types of sellers and buyers would use middlemen instead of their own search and how the social welfare would be affected by their presence. In particular, Miao (2006) has a result related to ours that social welfare can be reduced when a search market is intermediated. We also show that social welfare can be lower in

[2] For an early study on the effects of fee splitting on referrals in health care see Pauly (1979). An empirical study by Spurr (1990) on referral practices among lawyers examines the proportion of cases referred between lawyers, as dependent on the value and nature of a claim, advertising activity, and other factors.

[3] An interesting alternative model of consumer search that decomposes the cost of search for product and price information was proposed by Bakos (1997) and further studied by Harrington (2001).

the presence of referrals, although the mechanisms for getting the results are very different. There are a number of differences between our paper and the existing literature on search with information intermediaries. First of all, in our paper, referral services are provided by sellers (except for the third party referrals), and thus referral services are subject to the incentive problem. Second, unlike these papers, we assume that only consumers engage in searches, while sellers have perfect information about the industry and provide referral services to consumers. Finally, we assume a horizontally differentiated market, which is unique in the literature.

3 The Model

We model competition between firms that produce a horizontally differentiated product following Wolinsky (1983, 1986). We assume that there is a *large* number of firms, n, located symmetrically on a circle of unit circumference and producing the product of the location at a zero marginal cost. Our analysis will focus on the monopolistically competitive market with an arbitrary large number of firms, where for every product there is a firm located arbitrarily close to it. A unit mass of consumers with unit demand is characterized by consumers' valuation of the product (willingness-to-pay), and their preferences over the horizontally differentiated products. Consumers' ideal positions are distributed over the unit circle. Independently of their spatial preferences, each consumer has a value v for her ideal product (the product that is a perfect match with her tastes). We assume that v is distributed over an interval $[0, 1]$, and thus the equilibrium consumer participation would be endogenously determined in the model. Each consumer's ideal position and valuation are independently uniformly distributed.

Firms know positions of all other firms in the industry, whereas consumers do not know which firms offer which products. When a consumer visits a firm, she learns of the firm's position and its price, while the firm learns of the consumer's ideal product (tastes). The firm remains uncertain about the consumer's willingness-to-pay even after her visit. This assumption captures the idea that firms may find it easier to extract information about a consumer's ideal product than about the consumer's willingness-to-pay for the ideal product. Suppose a consumer whose product valuation is v learns that the firm's product is located at distance x from her ideal position. Let the consumer's utility for the firm's product (gross of price and search costs) be $u(x, v) = v - hx$, where $h > 0$ is the taste parameter that denotes the degree of product heterogeneity and hx represents the disutility a consumer receives from consuming a product located at a distance x in the product space from the consumer's ideal product. For example, a high h corresponds to markets where consumers feel strongly about product characteristics, e.g. color, weight, or design. The case of $h = 0$ corresponds to a homogeneous product market.

Consumer search is sequential, with a constant marginal cost of search $c > 0$, common across consumers. Each consumer chooses whether to initiate search, and a consumer who decides to search can always choose not to buy the product of the firm sampled and continue to search for another product. Consumer search is random, without replacement and with perfect and costless recall. That is, at each step of the sequential search process, a consumer samples firms' products randomly, but she does not sample the same product more than once, and she can always recall previously sampled products. Consumers do not discount the product value due to a delay in consumption. We assume that when indifferent between searching or not, consumers will search (so that a zero search cost would imply full information).

When a firm refers a consumer who visits it to other firms, the consumer can costlessly follow the referral. This is consistent with the idea that search costs come mainly in the form of learning about product characteristics, and therefore referrals substantially lower consumers' search costs. The assumption is not crucial though. Although we allow a perfect recall of past referrals, no recalls of past referrals would occur in equilibrium because a consumer would either purchase at the first firm she visits or at the firm referred by the first firm. We also assume that *whenever a firm knows that it cannot sell its own product to a consumer given its own price and other firms' prices, and thus is indifferent whether to provide a referral or not and where to refer the consumer, it refers the consumer to the best-matching firm, and that whenever a firm knows that it can sell its own product given its own price and other firms' prices, it does not make any referral.* This assumption of "*honest referrals*" resolves firms' indifference and is in the spirit of the honor codes in the professional service industries. Note that honest referrals are in fact incentive compatible in the sense that firms do not gain by deviating unilaterally from the equilibrium with honest referrals. Suppose a firm were to deviate and refer a departing consumer to a poorly matching firm. The consumer would follow the referral because she expects it to be valuable in the equilibrium. If the referred product is better for her than the product sold by the referring firm, she would purchase the referred product, even though it may not be the best-matching product for her. Otherwise, if she learns that the product is of poor match, she would engage in additional random search. Thus, the consumer would not purchase the dishonest firm's product anyway.

We start in section 3.1 with the analysis of price competition in a random search market without referrals. The strategy of a firm in such a market is a nondiscriminatory price. A symmetric (pure-strategy) equilibrium is an equilibrium in which all firms charge the same price p^*. We derive the equilibrium price that characterizes the symmetric equilibrium with consumer participation, which we call *the random search equilibrium*. Then, in section 3.2 we study the search model with referrals and no referral fees. In the model, firms simultaneously set prices and referral policies. Our focus is again on the symmetric pure-strategy equilibria in which all firms use the same price and referral policy. We will show that since a firm refers a consumer if only if the consumer would otherwise leave the firm to engage in random search, firm i's referral rule is characterized by the same critical distance as the consumer stopping rule. Hence, a symmetric referral equilibrium is described by a common price p_R^* and a referral rule x_R^*, such that if the distance between a consumer's ideal product and a firm is more than x_R^*, the firm refers the consumer to the best-matching firm. We derive the symmetric equilibrium with consumer participation, (p_R^*, x_R^*), which we call *the referral equilibrium.*

3.1 Random Search

In this section, we assume that firms do not make referrals, and so the only way for consumers to receive information about a product's location and price is to engage in sequential random search. We will support an equilibrium with a symmetric price p by following Wolinsky's (1983, 1986) techniques. In order to find the equilibrium in the random search model, we assume that all firms except firm i charge price p and that consumers rationally expect the common price level, and then derive the best response price for firm i in a market with an arbitrary large number of firms.[4] By setting the best response price equal to p, we obtain a symmetric equilibrium in the monopolistically competitive market.

First note that there is always a symmetric equilibrium without consumer participation in the market. In such an equilibrium, all firms charge a prohibitively high price, and consumers do not search correctly expecting the market price to be very high. This is an equilibrium because consumers only receive price information by engaging in search, and if a firm were to deviate by reducing its price it would not be able to convey the price information to consumers. Hence, no deviation is profitable.

To sidestep such an uninteresting equilibrium, we assume $c \leq \frac{h}{4}$, and later we will show that this condition is necessary for the existence of a symmetric equilibrium with consumer participation (Lemma 2). When consumers participate in the market, firm i's profit depends on the optimal stopping rule for a consumer engaged in random sequential search. Lemma 1 derives the optimal stopping rule.

Lemma 1. *Assume that $c \leq \frac{h}{4}$ and the number of firms n is very large. If firms charge symmetric prices, then the optimal stopping rule for a consumer engaged in random search is to stop searching if and only if she draws a product that is less than distance $x^* = \sqrt{\frac{c}{h}}$ from her ideal product. When all firms except firm i set a common price p and firm i sets a price p_i, then the optimal stopping rule for a consumer engaged in random search is to stop searching at firm i if and only if firm i's product is less than distance*

$$x(p_i, p) = x^* + \left(\frac{p - p_i}{h}\right) \tag{1}$$

from her ideal product.

In a symmetric equilibrium, the probability that a consumer stops her search at firm i is $2x^* = 2\sqrt{\frac{c}{h}} \leq 1$ because the optimal stopping rule tells her to stop searching once she samples a product closer than distance x^* "on both sides" of her ideal product on the

[4] The assumption that n is very large assures that even if a consumer visits a number of firms, the remaining unvisited firms are still approximately evenly distributed over the unit circle. We could have taken the finiteness of n seriously by analyzing an n-firm search market, but this would complicate the analysis and make the model intractable. For a justification of the approach see Wolinsky (1983) and, in particular, his footnote 2 in p. 276.

circumference. If firm i were to set a different price, $p_i \neq p$, then it could affect consumers' search behavior somewhat. If firm i were to surprise consumers by charging a higher price, the firm would prompt more consumers to continue random searches.

When $c > \frac{h}{4}$, the optimal stopping rule in a symmetric equilibrium is for a consumer to always stop searching regardless of the product found since for such parameter values $x^* = \sqrt{\frac{c}{h}} > \frac{1}{2}$. However, without sequential search, firms have an incentive to raise prices above the expected price level, and therefore a symmetric equilibrium with consumer participation does not exist when $c > \frac{h}{4}$. Lemma 2's proof makes this argument formally.

Lemma 2. *Assume that $c > \frac{h}{4}$ and the number of firms n is very large. Then, in the random search model there is no (pure-strategy) symmetric equilibrium with consumer participation.*

Lemma 2 is a version of the so-called Wernerfelt paradox. Wernerfelt (1994) considered a *monopolist*'s price setting problem with $h = 0$ (homogeneous goods), $c > 0$, and consumer valuations v uniformly distributed over an interval $[0, 1]$. He showed that due to a hold-up problem the market collapses. If price p is expected by consumers, then the consumers whose product valuations are more than or equal to $p + c$ engage in searches. However, then the firm has an incentive to raise the price from p to $p + c$. The incentive to raise the price exists whenever it is below $1 - c$, and for higher prices, consumers do not enter the market. Thus, when $c > 0$, there cannot exist a symmetric equilibrium with consumer participation.

This type of "*non-participation*" equilibrium is not specific to the monopolist case. If $c > \frac{h}{4}$, then consumers would never conduct sequential search in a symmetric equilibrium of the random search model because $2x^* > 1$. Thus, they visit at most one firm, and a hold-up problem occurs. This is reminiscent of the Diamond paradox (Diamond, 1971). If a buyer buys at most one unit, as in this model, firms try to extract all surplus, which results in non-participation of consumers. However, such an equilibrium is rather uninteresting. Thus, we will restrict parameter values to focus on interesting cases.

We will therefore assume that $c \leq \frac{h}{4}$ and show that there is a unique symmetric pure-strategy equilibrium with consumer participation, which we call the *random search equilibrium.* Note that in a symmetric random search equilibrium, it is strictly dominated for firm i to set a price p_i such that $x(p_i, p) > \frac{1}{2}$ or $x(p_i, p) < 0$. In case $x(p_i, p) > \frac{1}{2}$, which occurs when

p_i is below $p - h\left(\frac{1}{2} - x^*\right)$, all consumers who visit firm i buy there since i's offer is better than engaging in further random search, and a firm could increase its price slightly without losing any customers. In case $x(p_i, p) < 0$, which occurs when p_i is above $p + hx^*$, no consumer would purchase firm i's product, but firm i could slightly reduce its price to generate positive sales and profits. Thus, we can restrict our attention to $p_i \in \left[p - h\left(\frac{1}{2} - x^*\right), p + hx^*\right]$.

To derive firm i' demand when it sets a price p_i and all other firms charge p, we need to first find the probability that a consumer samples firm i on a kth random draw, stops search to buy firm i's product, and then sum over all k, $1 \leq k \leq n$. The probability of drawing firm i as the kth firm is $\frac{1}{n}$ for a consumer since all firms are symmetric and at each step in the sequential search, the next store is chosen randomly with equal probability and without correlation.[5] The probability that a consumer visits firm i on a kth random draw is $(1 - 2x^*)^{k-1}$ which, in accordance with the optimal stopping rule, is the probability that all previous $k - 1$ draws were further than distance $x^* = \sqrt{\frac{c}{h}}$ on both sides from her ideal position. If firm i charges p_i, the probability that a consumer purchases at firm i upon the visit is $2x(p_i, p)$. As a result, with a very large n, firm i's *demand function per searching consumer* is

$$D_i^n(p_i, p) = \frac{1}{n} \sum_{k=1}^{n} (1 - 2x^*)^{k-1} \, 2x(p_i, p). \tag{2}$$

By rewriting, we obtain

$$D_i^n(p_i, p) = \frac{1}{n} \left(\frac{1 - (1 - 2x^*)^n}{1 - (1 - 2x^*)} \right) 2x(p_i, p). \tag{3}$$

If consumer population is kept constant, firm i's demand falls to zero as the number of firms n grows. To avoid this problem, *we replicate the consumer population n times as the number of firms n grows*, thus keeping consumer population per firm constant.

For zero marginal costs, the profit function of firm i per searcher is then $\tilde{\pi}_i^n(p_i, p) =$

[5] With n firms, firm i is selected first with probability $1/n$. It is selected on a second draw if it is not selected first (probability $1 - 1/n$), and if it is selected as the second firm among the remaining $n - 1$ firms (probability $1/(n-1)$). Thus, the probability of firm i being selected on the second draw is also $1/n$. Similarly, we can show that firm i is selected on the kth draw with probability $1/n$ (see Wolinsky, 1986). Strictly speaking, once a consumer samples a firm, the equidistant location of firms on the circle is not preserved. Thus, the optimal stopping rule needs to be modified at each step. Following Wolinsky (1983), we assume away this possibility by assuming that n is very large. For the same reason we ignore the possibility of a consumer not being able to find a firm located within the critical distance x^*.

$np_i D_i^n(p_i, p)$ in the n-firm market. In the limit as $n \to \infty$, the profit function becomes

$$\tilde{\pi}_i(p_i, p) = \lim_{n\to\infty} \tilde{\pi}_i^n(p_i, p) = p_i \left(1 + \frac{p - p_i}{\sqrt{hc}}\right). \tag{4}$$

Firm i's profit is the measure of searchers times the profit function per searcher. Let $\bar{v}$ denote the valuation of the consumer indifferent between initiating search and staying outside the market (the critical valuation for market participation). Only consumers with valuations $v \in [\bar{v}, 1]$ enter the market. Then, firm i's profit is

$$\pi_i(p_i, p) = (1 - \bar{v})\tilde{\pi}_i(p_i, p). \tag{5}$$

A firm's price does not affect consumers' market participation decisions and thus does not affect the number of consumers who visit the firm. This is because consumers' decision to engage in search is based on price expectations, and not the actual prices set by firms. Therefore, the measure of searchers, $(1 - \bar{v})$, does not depend on p_i, and firm i chooses p_i to maximize its profit per searcher, $\tilde{\pi}_i(p_i, p)$. The properties of the unique random search equilibrium and comparative statics results are stated in Proposition 1.

Proposition 1. *Consider the limit case of $n \to \infty$. When $c \leq \min\left\{\frac{1}{4h}, \frac{h}{4}\right\}$, there exists a unique symmetric random search equilibrium with price $p^* = \sqrt{hc}$, the critical valuation for market participation $\bar{v} = 2\sqrt{hc}$, and profits $\pi^* = (1 - \bar{v})\sqrt{hc}$. When $c \leq \min\left\{\frac{1}{4h}, \frac{h}{4}\right\}$ is violated, there is no symmetric equilibrium with consumer participation. The equilibrium price increases and consumer market participation decreases in search cost, c, and product heterogeneity, h. Profits can increase or decrease in c and h.*

As we have seen in Lemma 2, condition $c \leq \frac{h}{4}$ is necessary for sustaining sequential consumer search in a symmetric equilibrium. The other condition $c \leq \frac{1}{4h}$ is needed to guarantee that the equilibrium price is not prohibitively high. If this condition is violated, then $\bar{v} = 2\sqrt{hc} > 1$, and all consumers expect a negative payoff and stay away from the market.

The comparative statics analysis of Proposition 1 shows that, as expected, the equilibrium price in the random search equilibrium increases with search cost and product heterogeneity. Interestingly, the equilibrium profits may increase or decrease with c and h depending on their levels. The equilibrium profits increase in c and h for sufficiently low levels of the parameters, $hc < \frac{1}{16}$. For other values of c and h, the decline in market participation is not compensated for by the higher price associated with larger c and h. In other words, for $\frac{1}{16} \leq hc \leq \frac{1}{4}$, sellers prefer to operate in search markets with lower search costs and lower product heterogeneity.

[Figure 1 HERE]

The equilibrium consumer decisions to engage in search, buy at a firm located at a distance x, or engage in sequential search are illustrated in Figure 1. Only consumers whose product valuations are at least $\bar{v} = 2\sqrt{hc}$ engage in search. Consumers visiting a firm closer than $x^* = \sqrt{\frac{c}{h}}$ from their ideal positions buy the product, while others continue to search.

3.2 Search with Referrals

In this section, we allow firms to provide referrals to consumers visiting them. We assume that there are no referral fees and that a firm (salesperson) can observe a customer's ideal position x from having a conversation with her, but cannot observe her willingness-to-pay v. Therefore, the referral that a firm offers its customer can only be conditioned on the observed consumer's ideal product.

In the model, firms simultaneously set nondiscriminatory prices and referral policies, while consumers hold rational expectations and select the optimal stopping rule for their sequential search with costless and perfect recall of products and referrals. In a search market with referrals, a consumer chooses whether to start a random search given her product valuation v; after each draw the consumer learns the firm's product and decides whether to follow the firm's referral if it is given, continue random search, purchase the best examined item, or leave the market without purchasing any product.

As is discussed in Section 3, we assume that firms' referrals are honest in the sense that whenever a firm knows that it cannot sell its own product to a consumer at the given prices, it refers her to the best-matching firm, and whenever a firm knows that it can sell its own product, it does not make any referral. Given this, firm i's referral policy is formally written as $r_i : [0, \frac{1}{2}] \to \{0, 1\}$, where the domain is the distance between firm i's location and a consumer's ideal location, and the range is the binary choice between refer, 1, or not, 0. For example, $r_i(x) = 1$ means that firm i refers a consumer to her best-matching firm if the distance between firm i and the consumer's ideal location is x.

Our focus is again on the symmetric pure-strategy equilibria (i.e., equilibria in which firms choose the same price and referral policy). A firm would refer a consumer (to her best-matching firm) if and only if the consumer's ideal position is sufficiently far, so that the consumer prefers engaging in further random search to buying at the firm. The optimal stopping rule in sequential search with referrals is for a consumer visiting firm i to stop searching at firm i if and only if firm i's product is closer to her ideal product than distance $x_R(p_i, p)$. Since a firm refers a consumer if only if the consumer would otherwise leave the firm to engage in additional random search, firm i's referral rule is characterized by the same critical distance as the consumer stopping rule. The symmetric equilibrium referral rule $r^* : [0, \frac{1}{2}] \to \{0, 1\}$ states that if the distance between a customer's position and a firm is more than $x_R^* = x_R(p, p)$, the firm refers the consumer to her best-matching firm. Formally, $r^* : [0, \frac{1}{2}] \to \{0, 1\}$ is $r^*(x) = 0$ for all $x < x_R^*$, and $r^*(x) = 1$ for all $x \geq x_R^*$. Hence, a symmetric referral equilibrium can be characterized by a pair $(p_R^*,\ x_R^*)$.

Figure 2 illustrates search with referrals. A consumer randomly chooses the first firm to visit, say, firm i, located at distance x from her ideal position (point C). When the distance between the consumer and firm i is at least $x_R(p_i, p)$, the consumer is referred by firm i to the best-matching firm (firm j), and she has to decide whether to buy at firm i, follow the referral, continue random search, or leave the market. When firm i is located closer than $x_R(p_i, p)$ from the consumer's ideal position, the consumer does not receive a referral and has to either buy at firm i, continue random search, or leave the market. In the symmetric equilibrium, the consumer follows the referral and buys at firm j when firm i is located further away than $x_R^* = x_R(p, p)$, and buys at firm i immediately otherwise.

To find the equilibrium $(p_R^*,\ x_R^*)$, consider firm i choosing a price, p_i, to maximize its profits, and suppose that all other firms set a common price p and referral rule x_R^*. Lemma 3 derives the equilibrium referral rule x_R^* and the optimal stopping rule $x_R(p_i, p)$ for a consumer engaged in search with referrals.

Lemma 3. *Assume that* $c \leq \frac{h}{4}$ *and the number of firms* n *is very large. The symmetric equilibrium referral rule is* $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(c/h)}$*. Suppose firm* i *sets a price* p_i *and other firms set a common price* p *and use referral rule* x_R^**. The optimal stopping rule for a consumer engaged in search with referrals is to stop searching at firm* i *if and only if firm* i*'s product is closer to her ideal product than distance* $x_R(p_i, p) = x_R^* + \left(\frac{p - p_i}{h}\right)$.

As in the case of search without referrals, $c > \frac{h}{4}$, the optimal stopping rule in a symmetric equilibrium is for a consumer to always stop searching regardless of the product found (Lemma 2). Thus, consumers search at most one firm, and since referrals are given only to consumers who would otherwise engage in additional random search, consumers do not receive referrals. With no effective referral taking place in a symmetric equilibrium, a version of the Wernerfelt paradox comes back, resulting in a non-participation equilibrium. Hence, a symmetric equilibrium with consumer participation does not exist in search markets with referrals when $c > h/4$.

Next, we calculate the demand function of firm i assuming that other firms choose a symmetric price p and make referrals to customers located further than x_R^*. We can restrict our attention to $p_i \in \left[p + hx_R^* - \frac{1}{2}h, p + hx_R^*\right]$ because it is strictly dominated for firm i to price outside this interval. In case $p_i < p + hx_R^* - \frac{1}{2}h$, we have $x_R(p_i, p) > \frac{1}{2}$, and all consumers who visit firm i buy there since i's offer is better than engaging in further random search or following the referral. But then firm i would rather set a slightly higher price. In case $p_i > p + hx_R^*$, we have $x_R(p_i, p) < 0$, and no consumer visiting firm i purchases its product. Moreover, $v - p_i < v - p - hx_R^*$ holds for any v, and even consumers who are referred to firm i do not purchase from it. Thus, demand for firm i in this case is zero, and firm i would benefit by reducing its price to generate positive demand and profits.

For $p_i \in \left[p + hx_R^* - \frac{1}{2}h, p + hx_R^*\right]$, consumers located at $x < x_R(p_i, p) \in [0, \frac{1}{2}]$ buy from firm i and firm i's *demand function per searcher* when there are n firms is

$$\begin{aligned} D_{Ri}^n(p_i, p) &= \left(\frac{1}{n}\right) 2x_R(p_i, p) + \left(\frac{n-1}{n}\right) \times \frac{1}{n-1}(1 - 2x_R^*) \\ &= \frac{1}{n}\left(1 + \frac{2(p - p_i)}{h}\right), \end{aligned} \tag{6}$$

where $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(c/h)}$ and n is very large. The first term represents demand from consumers who visit firm i first, while the second term represents demand from consumers who visit other firms first and are referred to firm i.

Given this, firm i's profit function per searcher with n-replicated consumer population is

$$\tilde{\pi}_{Ri}(p_i, p) = \lim_{n \to \infty} p_i n D^n_{Ri}(p_i, p) = p_i \left(1 + \frac{2(p - p_i)}{h}\right). \tag{7}$$

Let $\bar{v}_R$ denote the valuation of the consumer who is indifferent between initiating sequential search and staying outside the market in the referral equilibrium (the critical valuation for market participation). When firm i sets a price p_i and all firms are expected to set price p and use referral rule x^*_R, firm i's profit function is

$$\pi_{Ri}(p_i, p) = (1 - \bar{v}_R)\tilde{\pi}_{Ri}(p_i, p). \tag{8}$$

In Proposition 2 we describe the unique pure-strategy symmetric equilibrium with consumer participation, which we call the *referral equilibrium*.

Proposition 2. *Consider the limit case of $n \to \infty$. When $c \leq \frac{h}{4}$ and $h-1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$, there exists a unique symmetric referral equilibrium with price $p^*_R = \frac{h}{2}$, critical valuation for market participation $\bar{v}_R = p^*_R + hx^*_R$, and profits $\pi^*_R = p^*_R(1 - \bar{v}_R)$, where $x^*_R = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\,(c/h)}$. When either $c \leq \frac{h}{4}$ or $h - 1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$ is violated, there is no symmetric referral equilibrium with consumer participation. Referral intensity decreases with consumer search costs and increases with product heterogeneity. The equilibrium price is perfectly insensitive to search costs, whereas it increases as product heterogeneity increases. Consumers' market participation and profits decrease with search cost, c, and they can increase or decrease with product heterogeneity, h.*

According to Proposition 2, the symmetric referral equilibrium exists if and only if $c \leq \frac{h}{4}$ and $h - 1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$. The first inequality is necessary for sustaining sequential consumer search in a symmetric equilibrium. The second inequality is equivalent to $\bar{v}_R \leq 1$, where $\bar{v}_R = p^*_R + hx^*_R = h - \frac{h}{2}\sqrt{1 - 4\,(c/h)}$ is the critical valuation for market participation.

Figure 3 illustrates the equilibrium consumer decisions for different realizations of v and x. Only those consumers whose willingness-to-pay is higher than $\bar{v}_R$ engage in search. Consumers visiting a firm located closer than x^*_R buy the product, while others follow the firm's referral and buy from the referred seller.

[Figure 3 HERE]

Why does search cost c not matter in the determination of price p_R^* in this case? By lowering its price, firm i can increase its sales only through an increase in the retention rate $2x_R(p_i, p) = 2x_R^* - 2(p_i - p)/h$ of consumers who visit firm i first. However, a change in the retention rate, which equals $-2/h$, is not affected by search cost c since sequential search never takes place in the equilibrium. Thus, in search markets with referrals the equilibrium price is determined only by heterogeneity parameter h. We can see an analogy with the Diamond paradox: in both cases, sequential search does not occur, and the equilibrium price is independent of the level of search cost (as long as it is positive). In the current model, however, there is still competition among firms trying to retain initial customers, and the monopoly price does not prevail as the equilibrium price.

3.3 Comparison of the Random Search and Referral Equilibria

So far, we have described the random search and referral equilibria. Proposition 3 below compares the regions of their existence, while Proposition 4 compares the properties of the equilibria, and shows that unless search cost c is very low relative to heterogeneity parameter h, the referral equilibrium Pareto dominates the random search equilibrium.

Proposition 3. *Consider the limit case of $n \to \infty$. If $h \leq 1$, then both the random search and referral equilibria exist if and only if $c \leq \frac{h}{4}$. However, if $1 \leq h \leq \frac{5}{3}$ then the referral equilibrium exists for a larger range of search cost c, whereas if $h \geq \frac{5}{3}$, then the random search equilibrium exists for a larger range of search cost c than the referral equilibrium (in particular, the referral equilibrium cannot exist for $h > 2$).*

Figure 4 illustrates Proposition 3 by showing the regions of parameter values of h and c for which the random search and referral equilibria exist. The random search equilibrium exists for any level of product heterogeneity provided search cost is sufficiently low. In contrast, a referral equilibrium may exist only for a sufficiently low heterogeneity parameter, $h \leq 2$. As product heterogeneity approaches the critical level of 2, the market collapses because the equilibrium price $p_R^* = \frac{h}{2}$ approaches unity and thus becomes prohibitively high for consumers, regardless the magnitude of search cost.

[Figure 4 HERE]

Intuitively, when the product is highly heterogeneous, firms have greater market power to set high prices in the random search and referral equilibria, which discourages consumers' market participation. Even for a very high h, a random search market would not close down for a sufficiently low search cost since the low search cost imposes competitive pressure on firms in the market. In the referral equilibrium, prices are not affected by search costs, and the search market with referrals does not open for $h > 2$.

Next we compare the random search and referral equilibria in the region of parameter values for which both of them exist. We show that for a relatively high search cost, the referral equilibrium *Pareto-dominates* the random search equilibrium. That is, the ex ante expected payoffs of each consumer and each firm in the referral equilibrium are higher for some agents and at least as high for other agents than in the random search equilibrium.

Proposition 4. *Consider the limit case of $n \to \infty$. Whenever both the random search and referral equilibria exist, we have $p_R^* \geq p^*$ and $x_R^* \leq x^*$. Consumers are better off and the market participation is larger ($\bar{v} \geq \bar{v}_R$) in the referral equilibrium if and only if $c \geq 0.09h$, i.e. search cost is not very low relative to product heterogeneity. Thus, $c \geq 0.09h$ guarantees that firms earn more profits in the referral equilibrium, and the referral equilibrium Pareto-dominates the random search equilibrium.*

These results are not surprising. If search cost c is very low, consumers surely prefer the random search equilibrium, since the random search equilibrium price p^* is low while the referral equilibrium price p_R^* is insensitive to c. If search cost is not very low, the referral equilibrium is favored by consumers, and since the equilibrium price is higher in the referral equilibrium than in the random search equilibrium, firms favor this equilibrium as well. The referral equilibrium Pareto-dominates the random search equilibrium when the gain to consumers due to higher matching quality and lower search costs is higher than the loss due to increased price.

$>$From the welfare perspective, referrals handle the information problem of matching buyers and sellers but worsen the hold-up problem of high prices in search markets. The reason the equilibrium prices are higher under referrals is that in the referral equilibrium, a fraction $1-2x_R = \sqrt{1-4(c/h)}$ of consumers are referred to their best-matching products. Not surprisingly, the firm faces a more inelastic demand. Recall that firm i's demand per searcher with n-replicated consumer population is $\left(1+2\frac{p-p_i}{h}\right)$ and $\left(1+\frac{p-p_i}{\sqrt{hc}}\right)$ in search markets with and without referrals, respectively. It is easy to confirm that the elasticity of demand a firm faces is $\left(\frac{p_i}{\frac{h}{2}+(p-p_i)}\right)$ and $\left(\frac{p_i}{\sqrt{hc}+(p-p_i)}\right)$ in search markets with and without referrals. Since $\sqrt{hc} \leq \frac{h}{2}$ holds for $c \leq \frac{h}{4}$, firm i's demand elasticity in a search market with referrals is lower in absolute value.

The fact that prices are higher in the referral equilibrium than in the random search equilibrium indicates that firms benefit more from the presence of referrals than consumers. That is, if consumer welfare is higher under referrals, then firms find them beneficial as well, but the reverse is not true. It could be the case that firms find the referral equilibrium more profitable, while the referrals make consumers worse off.[6]

3.3.1 Social Welfare

We next study how social welfare depends on the parameters of the model in the presence and absence of referrals. We define social welfare as a sum of consumer welfare and profits. Suppose a critical valuation for market participation is $\widehat{v}$ in a search market. The expected utility of a consumer who values the product at $v \geq \bar{v}$ is $EU(v) = v - \widehat{v}$, and consumer welfare is $\int_{\widehat{v}}^{1} EU(v)dv = \frac{1}{2}(1-\widehat{v})^2$. Therefore, $\widehat{v}$ fully determines consumer welfare, and higher consumer market participation is equivalent to higher consumer welfare.

In search markets with or without referrals social welfare is a function of price p and a critical valuation for market participation $\widehat{v}$ because it can be written as

$$W(p,\widehat{v}) = (1-\widehat{v})\,p + \frac{1}{2}(1-\widehat{v})^2 . \tag{9}$$

Proposition 5 summarizes the comparative statics results for social welfare in the random search and the referral equilibria.

Proposition 5. *Social welfare in the random search and referral equilibria is higher when*

[6] Although firms usually would benefit from establishing a norm of referring the consumers who leave them to the best-matching firms, there are some parameter values for which profits are lower in the referral equilibrium than in the random search equilibrium. For these parameter values, it must be that consumers' market participation is lower (and consumers are worse off) because the referral equilibrium price is higher. Therefore, the referral equilibrium can be Pareto dominated by the random search equilibrium.

search cost is lower. The effect of product heterogeneity on social welfare is negative in the random search equilibrium, and it is ambiguous in the referral equilibrium.

The comparative statics results for the random search equilibrium indicate that social welfare is higher in markets with low search costs and product heterogeneity. Similarly, for the referral equilibrium, social welfare is higher in markets with low search costs, but the effect of product heterogeneity on social welfare is ambiguous. The results are quite intuitive. An increase in search cost does not affect the equilibrium price, but it reduces referral intensity and market participation. Therefore, higher search costs are detrimental from the point of view of both the consumer and social welfare. In contrast, higher product heterogeneity stimulates referral activity and can improve social welfare despite a price increase.

4 Referral Fees

Consider a referral fee in the form of a commission (an ad valorem referral fee). Let $\gamma \in (0, 1)$ be a common referral commission level, which can arise as an outcome of competition for referrals or be a norm in the industry. Since all firms offer the same commission γ, a referring firm cannot do better than to make honest referrals. As before, we assume that firms cannot price-discriminate between consumers and look for a symmetric equilibrium.[7] Commissions give firms strictly positive incentives to make referrals, and therefore they could help sustain the referral equilibrium. At the same time, the equilibrium referral rule x_R^* remains unchanged because a consumer's search strategy is based only on prices and distances between the consumer's ideal position and firms. In the symmetric equilibrium, commissions do not affect consumers' search strategies. First, we analyze the case without firms that specialize in making referrals (without selling products), then we consider the case with such firms.

4.1 Without Third Party Referrals

We will now analyze how the equilibrium price and market participation are affected by commissions. With a commission γ, firm i's profit function is a sum of profits from consumers

[7] In fact, non-discrimination can be one of the conditions for referral fees between lawyers to be allowed under state codes of professional responsibility.

buying from firm i on their first visit, on their visit by referral, and the payments from other firms for the referrals the firm makes. A measure $(1-p-hx_R^*)$ of consumers participates in the market since only consumers with values exceeding $p+hx_R^*$ initiate search. Firm i sells to $2x_R(p_i,p)$ of the searchers who visit firm i first and collects referral payments γp for the rest of consumers who visit firm i first. A measure $(1-2x_R^*)$ of consumers who first visit another firm are referred to firm i, and firm i receives $(1-\gamma)p_i$ for each of the referral customers. Therefore, the profit of firm i can be written as

$$\begin{aligned}\pi_R(p_i,p;\gamma) &= (1-p-hx_R^*)\left(2p_ix_R(p_i,p)+\gamma p\left(1-2x_R(p_i,p)\right)+(1-2x_R^*)(1-\gamma)p_i\right)\\ &= (1-p-hx_R^*)\left(p_i+\frac{2\left(p-p_i\right)}{h}\left(p_i-p\gamma\right)+\gamma(p-p_i)\left(1-2x_R^*\right)\right). \qquad (10)\end{aligned}$$

Proposition 6 describes the referral equilibrium in the presence of commissions.

Proposition 6. *Consider the limit case of $n\to\infty$. Suppose that there is a commission $\gamma\in(0,1)$ for referrals. If $h\leq 2$, $c\leq\frac{h}{4}$ and γ is small, then there exists a unique symmetric referral equilibrium, in which $p_R^*(\gamma)=\frac{h}{2}\left(\frac{1-\gamma\sqrt{1-4(c/h)}}{1-\gamma}\right)$, $\bar{v}_R(\gamma)=p_R^*(\gamma)+hx_R^*$, and $\pi_R^*(\gamma)=(1-\bar{v}_R(\gamma))p_R^*(\gamma)$, where $x_R^*=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\left(c/h\right)}$. The equilibrium price increases with γ and c, and it increases with h whenever c is sufficiently small relative to h. Consumers' market participation decreases with γ. An increase in γ can increase or decrease the firms' equilibrium profits.*

The proof of Proposition 6 shows that for the referral equilibrium to exist, the commission γ has to be sufficiently small to satisfy $\gamma\leq 1-\frac{1}{2/t-1}\left(1-\sqrt{1-4\left(s/t\right)}\right)$. This condition is needed to guarantee consumer participation, $\bar{v}_R(\gamma)=p_R^*(\gamma)+tx_R^*\leq 1$. Proposition 6 states that a part of the commission is borne by consumers in the form of higher prices. That the equilibrium price increases with γ is an intuitive result, since in the presence of referral fees it is less costly for a firm to raise its price; even if a firm cannot sell a commodity to a customer it can still obtain the commission. Consumers are clearly worse off under high commissions because the optimal consumer search is not affected by γ, while the equilibrium prices are higher.

The effect of commission γ on the equilibrium price is largest when $c=\frac{h}{4}$ because $\frac{\partial^2 p_R^*(\gamma)}{\partial c\partial\gamma}=\left(\sqrt{1-4(c/h)}\right)^{-1}(\gamma-1)^{-2}>0$ for $c\in(0,\frac{h}{4})$. If $c=\frac{h}{4}$, then $p_R^*(\gamma)=\frac{h}{2}\left(\frac{1}{1-\gamma}\right)$.

Therefore, the equilibrium price increases at most by factor $\left(\frac{1}{1-\gamma}\right)$ in the presence of a commission γ. On the other hand, when $c \to 0$ (and $x_R^* \to 0$) the price is not affected by γ. When firm i increases its price, the benefit of a higher commission revenue from referrals is just offset by higher commission fees for referred consumers in the limiting case $c \to 0$.

So far, we assumed that a referral free is ad valorem: a fraction γ of price of a product. In order to check the robustness of the analysis, we now introduce a flat referral fee irrespective of the product price. Let $f > 0$ be a common flat referral fee, and assume that firms making referrals act honestly (as long as they are offered the same referral fees). Again, we assume that firms cannot price-discriminate between referred consumers and those engaged in random search, and we look for a symmetric equilibrium. We find a qualitatively similar equilibrium to the one obtained in case of commissions.

Proposition 6'. *Consider the limit case of $n \to \infty$. Suppose there is a flat referral fee $f > 0$. If $c \leq \frac{h}{4}$ and f is small, then there exists a unique symmetric referral equilibrium, in which $p_R^*(f) = \frac{h}{2} + f$, $\bar{v}_R(f) = p_R^*(f) + hx_R^*$, $\pi_R^*(f) = (1 - \bar{v}_R(f))p_R^*(f)$, and $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(c/h)}$. The equilibrium price increases in f and market participation decreases in f. An increase in f can increase or decrease firms' equilibrium profits.*

This proposition is of some interest, since it says that referral fees are borne 100% by consumers. That is, consumers are clearly worse off by having a high referral fee f. The equilibrium price is higher by f, implying $\bar{v}_R(f) > \bar{v}_R$, and the consumers' market participation declines. This result is intuitive since the cost of raising the price is reduced by the presence of referral fees. That is, even if a firm cannot sell a commodity to a customer it can still obtain a referral fee f.

4.2 With Third Party Referrals

Let us now introduce third-party referral agents (brokers), who do not sell products but specialize in referral services, receiving a commission $\gamma \in (0, 1)$ for referrals. Suppose that there are third-party referral agents with measure $\alpha > 0$. Then the total measure of firms and brokers is $1 + \alpha$, and consumers visit them randomly. To maintain symmetry of search costs, we assume that the first visit to a firm or a broker costs $c > 0$. If a consumer visits a broker, she receives an honest referral with probability one, whereas if she visits a firm she does not receive a referral with probability $2x^*_{RT}(\alpha)$ (and in that case she purchases the firm's product). Although the presence of brokers affects the equilibrium referral rule, brokers are passive players in this model. A broker gives honest referrals to consumers who visit her, and charges referral commission γ to firms. In contrast, firm i maximizes the following profit function

$$\pi_{RTi}(p_i, p; \gamma, \alpha) = \frac{1 - p - hx^*_{RT}(\alpha)}{1+\alpha}\left[p_i + (1 - 2x^*_{RT}(\alpha))\gamma(p - p_i) + \tfrac{2}{h}(p_i - \gamma p)(p - p_i) + \alpha(1-\gamma)p_i\right]. \tag{11}$$

The following proposition characterizes the referral equilibrium in the presence of referral payments and third-party brokers.

Proposition 7. *Consider the limit case of $n \to \infty$. Suppose that there are referral brokers with measure α and the commission rate $\gamma \in (0,1)$ is small. If $c \leq \frac{(1+\alpha)h}{4}$, then $p^*_{RT}(\gamma,\alpha) = \frac{h}{2}\left(\frac{1-\gamma\sqrt{1-4c/h}}{1-\gamma} + \alpha\right)$, $x^*_{RT}(\alpha) = \frac{1+\alpha}{2} - \frac{1+\alpha}{2}\sqrt{1 - \frac{4c}{(1+\alpha)h}}$, and $\bar{v}_{RT}(\gamma,\alpha) = p^*_{RT}(\gamma,\alpha) + hx^*_{RT}(\alpha)$. The critical referral distance $x^*_{RT}(\alpha)$ decreases in the measure of brokers, α, while the equilibrium price $p^*_{RT}(\gamma,\alpha)$ increases in α. Consumers' market participation decreases in α in most cases. An increase in α can increase or decrease the firms' equilibrium profits.*

Proposition 6 shows that the equilibrium price increases even further with third-party referrals. The reason is that the demand curve a firm faces becomes more inelastic, since the firm has more perfect-match consumers for whom it does not compete. Anderson and Renault (2000) first observed this effect in an economy with horizontally differentiated search goods à la Wolinsky (1986). Note that this result follows even in the absence of commissions ($\gamma = 0$). Although there are cost-saving gains for consumers as a result of having third-party referrals, an increase in α usually hurts consumers through the equilibrium price increases unless α is small and search cost is relatively high. The presence of brokers affects the referral intensity in an intuitive way: all else equal, brokers render consumer search more attractive, prompting firms to refer more consumers to other firms.

>From a welfare point of view, there is a trade-off between the incentive problem in

referral activity and the hold-up problem in pricing. Third-party referrals ameliorate the incentive problem in referral activity since the third-party brokers always immediately match their customers with firms that sell their ideal products, while the ordinary firms may try to sell their products to consumers even if their products are not perfect matches. At the same time, third-party referrals worsen the hold-up problem present in search markets. Since firms cannot precommit to prices before consumers incur search costs, they are tempted to charge a high price. Consumers anticipate that event in the equilibrium and thus do not initiate search in the first place. For very high search costs, consumers favor markets with brokers because in such markets the gains in referral efficiency outweigh the losses of a price increase.

In a number of industries, members of professional organizations resist paying referral fees to nonmembers, arguing that industry profits will be siphoned by outsiders. According to Proposition 7, however, the existence of brokers can increase the market price; the effect on profit of each (nonbroker) firm is ambiguous, since the market size is likely to be reduced by an increase in α, but the equilibrium price (profit margin) will increase. Thus, as long as γ is not too high, each firm's profit can in fact be higher in the presence of brokers.

Finally, we conduct a similar analysis for search markets flat referral fees and third-party referral brokers. Firm i's profit in such markets is

$$\begin{aligned}\pi_i(p_i, p; f, \alpha) &= \frac{1 - p - hx^*_{RT}(\alpha)}{1+\alpha}\left[2x_{RT}(p_i, p)p_i + \alpha(p_i - f) + (1 - 2x^*_{RT}(\alpha))(p_i - f)\right] \\ &= \frac{1 - p - hx^*_{RT}(\alpha)}{1+\alpha}\left[p_i\left(1 + \frac{2(p - p_i)}{h}\right) + f\frac{2(p_i - p)}{h} + \alpha(p_i - f)\right] \qquad (12)\end{aligned}$$

The properties of the referral equilibrium in search markets with flat referral fees and brokers are outlined in Proposition 7'.

Proposition 7'. *Suppose that there are referral brokers with measure α and a flat referral fee $f > 0$ is small. If $c \leq \frac{(1+\alpha)h}{4}$, then $p^*_{RT}(f, \alpha) = \frac{h(1+\alpha)}{2} + f$, $x^*_{RT}(\alpha) = \frac{1+\alpha}{2} - \frac{1+\alpha}{2}\sqrt{1 - \frac{4c}{(1+\alpha)h}}$, and $\bar{v}_{RT}(f, \alpha) = p^*_{RT}(f, \alpha) + hx^*_{RT}(\alpha)$. The critical referral distance $x^*_{RT}(\alpha)$ decreases in the measure of brokers, α, while the equilibrium price $p^*_{RT}(f, \alpha)$ increases in α. Consumers' market participation decreases in α in most cases. An increase in α can increase or decrease the firms' equilibrium profits.*

The proof is very similar to that of Proposition 7, and it is therefore omitted.

5 Conclusion

We consider an environment where consumers need to conduct a costly search to gain information on products or services offered in a horizontally differentiated market. Examples range from law firms and other professional services to an antique market or high-tech consumer products. In the framework of the search market introduced by Wolinsky (1983), we study the practice of referring consumers to competing sellers. In the basic model, there are no referral fees, and thus it seems unlikely that a firm could benefit from referring a consumer to its competitor. We find that the custom of giving honest referrals whenever a firm cannot sell its product to a consumer raises the equilibrium prices and thus can be favored by sellers and trade organizations. Although, given set prices, referrals benefit consumers because they improve the product-match quality and decrease their search costs, in the equilibrium such referrals can actually hurt consumers if search costs are very low relative to product heterogeneity. The reason is that in the search market without referrals, the equilibrium price becomes trivial for very low search costs, whereas in the referral equilibrium prices are insensitive to search costs. The possibility that referrals hurt consumers is somewhat paradoxical, and it is only realized for very low search costs. Referrals increase profits and improve social welfare even under broader conditions – as long as the market does not contract too much in response to higher prices.

Referral fees give sellers a strong incentive to refer a consumer to other sellers when they cannot sell to the consumer, but the equilibrium price increases with the fees. Common referral fees (commissions or flat referral fees) do not affect the referral intensity. Thus consumers and consumer protection agencies would like to see low commission rates, provided entry into the market is not discouraged. We also look at the role of third-party brokers who do not sell the product but simply match buyers and sellers in exchange for a referral payment. Brokers further improve the match quality because they always provide consumers with the best match, and their presence further increases the equilibrium price. Hence, there is a welfare trade-off between an efficiency gain due to the lessened incentive problem in referrals and an efficiency loss due to a strengthening of the problem of high prices in search markets.

Finally, we will provide an argument for referral fees by modifying our model slightly. Suppose that a firm observes a customer's ideal product with some inaccuracy: the observed ideal may be a bit off from the true ideal product. Suppose that such noises are independently and identically distributed with a zero mean. In this case, without a referral fee, firms are not eager to give referrals unless there is absolutely no chance for them to sell their products to the customers. With a referral fee, they give referrals to their customers if the chance of selling their own products is low. Moreover, firms have an incentive to make truthful referrals, i.e. referrals that are as accurate as possible, given the noise. In contrast, without the noise, we need to assume "honest referrals" (as is done in this paper), since firms are indifferent between referring a consumer to any competitor that would sell a buyable product to the customer. When referrals are sufficiently untrustworthy so that consumers rationally discount them, referral equilibrium prices are lower, resulting in the random search equilibrium. Thus, introducing (a bit of) referral fees may improve welfare in the Pareto sense as long as consumer search costs are not too low relative to product heterogeneity.

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Appendix A: Proofs

Proof of Lemma 1. In a symmetric equilibrium consumers expect all firms to charge the same price p. To derive the optimal stopping rule for consumers, consider the decision of a consumer visiting a firm to purchase at the firm or continue search. If she continues, she will draw a product sold at price p and located at distance x from her ideal position, where x is (approximately) uniformly distributed on the unit circle when the number of firms n is very large. It is a well-known result in search theory that the fixed reservation value search is the optimal consumer strategy for the consumer in this case (see Wolinsky 1983 and 1986). That is, there is a reservation (critical) distance, x^*, and the associated reservation utility, $u(x^*, v)$, such that a consumer stops search whenever she samples a product located within distance x^* from her ideal position. The reservation utility can be determined by equalizing the expected gain from exactly one more search and the cost of search for the consumer,

$$2\int_0^{x^*} (u(x,v) - u(x^*, v))\, dx = c, \tag{A1}$$

where $u(x, v) = v - hx$ is the consumer utility gross of price and search costs. We multiply the integral in (A1) by 2 because we integrate over all locations that are closer than distance x^* "on both sides" of a consumer's ideal location on the circumference, and products located at distance x to the left and to the right are equally desirable for the consumer.

Simplifying this, we obtain

$$2\int_0^{x^*} h\,(x^* - x)\, dx = c, \tag{A2}$$

and therefore the critical distance and reservation utility are $x^* = \sqrt{\frac{c}{h}}$ and $w^*(v) \equiv u(x^*, v) = v - \sqrt{hc}$.

Next we derive the optimal stopping rule assuming that consumers expect all firms to charge a price p, but one firm deviates by charging a different price. Consider the decision of a consumer visiting firm i located at distance x from her ideal position and selling at price p_i to purchase at firm i or continue search. A consumer stops searching at firm i if x satisfies: $v - p_i - hx > u(x^*, v) - p$, or $x < x(p_i, p)$, where $x(p_i, p) = \sqrt{\frac{c}{h}} + \frac{p - p_i}{h} = x^* + \frac{p - p_i}{h}$.■

Proof of Lemma 2. Assume that $c > \frac{h}{4}$ and suppose that, to the contrary, there exists an equilibrium with a symmetric price p such that a positive measure of consumers engage in sequential search. Given consumers expect price p at every firm, a consumer visiting firm i located at a distance x would purchase from the firm as long as $x < x(p_i, p) = x^* + \frac{p-p_i}{h}$, where $x^* = \sqrt{\frac{c}{h}} > \frac{1}{2}$ (Lemma 1). All consumers who visit firm i continue to purchase from it as long as $x^* + \frac{p-p_i}{h} > \frac{1}{2}$. But then each firm has an incentive to raise its price slightly above the expected price p without losing any consumers because only consumers with values $v \geq p + c$ enter the market and a sufficiently small price increase would not encourage consumers to engage in additional search. This argument holds for any $p < 1$. Therefore, the symmetric equilibrium with consumer participation does not exist if $c > \frac{h}{4}$.■

Proof of Proposition 1. In the random search model, firm i maximizes its profit function $\tilde{\pi}_i(p_i, p) = p_i \left(1 + \frac{p-p_i}{\sqrt{hc}}\right)$ by choosing $p_i = \frac{p+\sqrt{ch}}{2}$. Thus, the symmetric equilibrium price is $p^* = \sqrt{hc}$.

For which parameter values does the random search equilibrium (with consumer participation) exist? From the proof of Lemma 1, the reservation utility (gross of price) is $w^*(v) = v - \sqrt{hc}$. In order for some consumers to have a nonnegative equilibrium expected utility and engage in search, price p^* must be no higher than $w^*(v)$ for some $v \in [0, 1]$, namely $v = 1$. Thus, $p^* = \sqrt{hc} \leq 1 - \sqrt{hc}$, which can be written as $\sqrt{hc} \leq \frac{1}{2}$ or $c \leq \frac{1}{4h}$. For some consumers to search beyond the first firm in the equilibrium, we also need $x^* = \sqrt{\frac{c}{h}} \leq \frac{1}{2}$, or $c \leq \frac{h}{4}$. Lemma 2 shows that there is no symmetric pure-strategy equilibrium with consumer participation if $c > \frac{h}{4}$. To summarize, condition $c \leq \min\{\frac{1}{4h}, \frac{h}{4}\}$ on search costs ensures that the random search equilibrium exists.

Consumers whose willingness-to-pay v is greater than or equal to a critical value $\bar{v} = 2\sqrt{hc}$ engage in search. Indeed, if $v \geq \bar{v}$ then $v - p^* - \sqrt{hc} \geq 0$, and such a consumer would follow the optimal stopping rule, stopping whenever the distance from the ideal position is less than x^*. This implies that a fraction $1 - \bar{v} = 1 - 2\sqrt{hc}$ of consumers engage in search, and each firm's profit can be written as $\pi^* = (1 - \bar{v})\, \tilde{\pi}_i(p^*, p^*) = \left(1 - 2\sqrt{hc}\right) p^* = (1 - 2\sqrt{hc})\sqrt{hc}$.

The comparative statics results are as follows: $\partial p^*/\partial c > 0$, $\partial p^*/\partial h > 0$, $\partial (1 - \bar{v}) / \partial c < 0$, $\partial (1 - \bar{v}) / \partial h < 0$. For profits, $\partial \pi^*/\partial h = \frac{1}{2}\sqrt{c/h} - 2c = \frac{1}{2}\sqrt{c/h}\left(1 - 4\sqrt{hc}\right)$, and $\partial \pi^*/\partial c =$

$\frac{1}{2}\sqrt{h/c}\left(1-4\sqrt{hc}\right)$, and therefore $\partial\pi^*/\partial h > 0$ and $\partial\pi^*/\partial c > 0$ if and only if $hc < \frac{1}{16}$.■

Proof of Lemma 3. Consider a symmetric equilibrium with price p and referral rule x_R^*. Let us start the analysis with a second-round search. Suppose that a consumer has randomly sampled firm i located at distance x from her ideal position. Her utility (gross of search cost) from purchasing commodity there is $v - p - hx$, where p is the symmetric price charged by firms. In the symmetric referral equilibrium, a consumer can get a referral with a probability $1-2x_R^*$. If she gets a referral and follows it, her utility will be $v-p$. A consumer who visited a firm located at a distance $x < x_R^*$ from her ideal position will not get a referral. Then, her choice is one of the following three: (i) make no purchase, receiving $(-c)$; (ii) purchase the product, receiving $(v - p - hx - c)$; or (iii) engage in additional sequential search.

As we will show below, only consumers with willingness-to-pay v such that $v-p-hx_R^* \geq 0$ enter the market in equilibrium, and since $v - p - hx \geq 0$ for all $x < x_R^*$, it is better for the searcher to purchase rather than not. If she engages in further sequential search, she visits a random firm (firm $j \neq i$), which is located at distance $\widetilde{x}$ from the consumer and charges price p. At firm j, the consumer receives and follows a referral with probability $(1 - 2x_R^*)$, recalls firm i's offer, or buys firm j's product.

The expected payoff from engaging in one additional search is then

$$
\begin{aligned}
&\Delta EU(x;x_R^*)\\
= \;& (1-2x_R^*)(v-p) + 2(x_R^*-x)(v-p-hx) + 2\int_0^x (v-p-h\widetilde{x})d\widetilde{x} - c - (v-p-hx)\\
= \;& (1-2x_R^*)hx + hx^2 - c. \qquad\qquad \text{(A3)}
\end{aligned}
$$

The terms in equation (A3) are explained as follows. When searching once again (i.e., by visiting firm j), the consumer will be referred by firm j to her ideal product with probability $1 - 2x_R^*$. She will find firm j's product to be a poorer match than firm i's product and will recall firm i's offer with probability $2(x_R^* - x)$, and otherwise, she will buy firm j's product. The subtracted terms in (A3) are an additional search cost and the utility level she obtains by purchasing firm i's product. It is easy to see that for any $x_R^* \in [0, \frac{1}{2}]$ and any $x \in [0, x_R^*]$, we have $\partial\Delta EU(x;x_R^*)/\partial x = h\,(1 - 2x_R^* + 2x) \geq 0$. This means that as long as $\Delta EU(x_R^*;x_R^*) \leq 0$, consumers who have $x < x_R^*$ would not engage in further search.

A consumer who visited a firm located at distance x_R^* from her ideal position is indifferent between searching and not searching (given that all other firms are using referral rule x_R^*): $\Delta EU(x_R^*; x_R^*) = 0$. It follows from

$$\begin{aligned} \Delta EU(x_R^*; x_R^*) &= (1 - 2x_R^*)hx_R^* + hx_R^{*\,2} - c \\ &= hx_R^* - hx_R^{*2} - c = 0 \end{aligned} \tag{A4}$$

that $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(c/h)}$ for $c \leq \frac{h}{4}$. The value x_R^* describes the symmetric equilibrium referral rule. If a consumer gets a referral and $v \geq p$, then she follows the referral. If she does not get a referral, it means that sequential search is not beneficial, and as a result, she either purchases (when $v \geq p + hx$) or goes home without purchase.

If $c > \frac{h}{4}$ holds, then no referrals will take place in a symmetric equilibrium, since $\Delta EU(x_R^*, x_R^*) > 0$ for all $x_R^* \in [0, \frac{1}{2}]$, and consumers stop searching regardless of the product found. Hence, if $c > \frac{h}{4}$ then there will be no referral equilibrium (with consumer participation) for the same reason that there does not exist a symmetric random search equilibrium (Lemma 2).

Next, we find the consumer's optimal stopping rule when she observes a price p_i at firm i that is different from p. Let x be the distance between a consumer's ideal point and the location of firm i. We look for the threshold distance for firm i, $x = x_R(p_i, p)$, such that a consumer at distance $x_R(p_i, p)$ from the ideal point is indifferent between purchasing i's product and searching further.

Suppose that an additional search after visiting firm i matches the consumer with a product at firm $j \neq i$ at distance $\tilde{x}$ from her ideal point. There are two cases: (i) $\tilde{x} \geq x_R^*$ and she gets a referral, or (ii) $\tilde{x} < x_R^*$ and she does not get firm j's referral (in this case, she will not search further). A consumer prefers to buy firm i's rather than firm j's product if and only if $v - p_i - hx < v - p - h\tilde{x}$ (or $\tilde{x} > x + \frac{p_i - p}{h}$). One additional round of search results in consumer utility no less than $v - p - hx_R^*$.

First, assume that the consumer utility from firm i's product is at least as high as that under the worst realization of an additional search: $v - p_i - hx \geq v - p - hx_R^*$ (or $x + \frac{p_i - p}{h} \leq x_R^*$, or $p_i \leq p + hx_R^* - hx$). In this case, the consumer may recall firm i's product under some realizations of $\tilde{x}$. If $\tilde{x} > x_R^*$, she gets firm j's referral and obtains $(v - p)$. If

$x+\frac{p_i-p}{h} \leq \tilde{x} \leq x_R^*$, she recalls firm i's product. Finally, if $\tilde{x} < x+\frac{p_i-p}{h}$, she buys at firm j. Therefore, a customer's gain from engaging in an additional search when firm i charges p_i and other firms charge p is as follows:

$$\begin{aligned} & (1-2x_R^*)(v-p)+2\left(x_R^*-x+\frac{p-p_i}{h}\right)(v-p_i-hx)+2\int_0^{x+\frac{p_i-p}{h}}(v-p-h\tilde{x})d\tilde{x} \\ & -c-(v-p_i-hx) \qquad \text{(A5)} \\ = \; & h\left[x+\left(\frac{1}{2}\sqrt{1-4\,(c/h)}+\frac{p_i-p}{h}\right)\right]^2-\frac{h}{4}. \end{aligned}$$

By setting the above gain to zero, we obtain the threshold distance $x_R(p_i,p)$ for firm i: $x_R(p_i,p)=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\,(c/h)}+\frac{p-p_i}{h}$. Note that $x_R(p_i,p)=x_R^*+\frac{p-p_i}{h}$, and that the condition we assumed is satisfied for $x=x_R(p_i,p)$.

Second, suppose that $v-p_i-hx < v-p-hx_R^*$ (or $x > x_R(p_i,p)$). In that case, the consumer leaves firm i to search further.■

Proof of Proposition 2. Since we are analyzing a symmetric equilibrium, we assume that consumers expect that every firm charges price p, and that the common referral rule is described by $x_R^*=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\,(c/h)}$. Consider a consumer whose willingness-to-pay v satisfies $v-hx_R^*-p \geq 0$. She has the following expected utility from the initial search (given the optimal stopping rule characterized by x_R^*):

$$\begin{aligned} EU_R(v,p) &= 2\int_0^{x_R^*}(v-hx-p)dx+(1-2x_R^*)(v-p)-c \qquad \text{(A6)} \\ &= (v-p)-hx_R^{*\,2}-c \\ &= v-hx_R^*-p, \end{aligned}$$

where $hx_R^*-hx_R^{*\,2}-c=0$ follows from the definition of x_R^* in Lemma 3. That is, a consumer whose willingness-to-pay v satisfies $v-hx_R^*-p \geq 0$ obtains a nonnegative expected utility, $EU_R(v,p) \geq 0$. On the other hand, if a consumer's willingness-to-pay v satisfies $v-hx_R^*-p < 0$, it is easy to see that $EU_R(v,p) < 0$ for any stopping rule. Thus, given that p is a prevailing symmetric price, a consumer engages in the initial search if and only if her willingness-to-pay v is not less than $\bar{v}_R(p)=hx_R^*+p$.

We will next show that there is a symmetric equilibrium when $\bar{v}_R \leq 1$. Since π_i is concave in p_i, the first-order condition evaluated at $p_i = p$ characterizes the symmetric equilibrium. The first-order condition is $\partial \pi_{Ri}(p_i, p)/\partial p_i = (1 - p - hx_R^*)(1 - (4p_i - 2p)/h) = 0$. Thus, there is a unique symmetric equilibrium price $p_R^* = \frac{h}{2}$. The value of $\bar{v}_R$ can be found by substituting p_R^* into $v_R(p) = p + hx_R^*$. We obtain $\bar{v}_R = h - \frac{h}{2}\sqrt{1 - 4(c/h)}$. Finally, $\bar{v}_R \leq 1$ if and only if $h - 1 \leq \frac{h}{2}\sqrt{1 - 4(c/h)}$.

The equilibrium price $p_R^* = \frac{h}{2}$ increases in h and does not depend on c. Referral intensity $1 - 2x_R = \sqrt{1 - 4(c/h)}$ decreases with consumer search costs and increases with product heterogeneity. Consumers' market participation, measured by $(1 - \bar{v}_R)$, decreases in search cost, c, because $\partial \bar{v}_R/\partial c = 1/\sqrt{1 - 4(c/h)} > 0$. For product heterogeneity, h, $\partial \bar{v}_R/\partial h = \frac{1}{2}\left(2\sqrt{1 - 4(c/h)} - (1 - 2(c/h))\right)\frac{1}{\sqrt{1-4(c/h)}} > 0$, which is equivalent to $2\sqrt{1 - 4(c/h)} > (1 - 2(c/h))$. Since $1 - 2(c/h) > 0$, this is equivalent to $4(1 - 4(c/h)) > (1 - 2(c/h))^2$, or $4(c/h)^2 + 12(c/h) - 3 < 0$. The last inequality holds if and only if $c/h < \left(\sqrt{3} - \frac{3}{2}\right)$. Hence, $\partial \bar{v}_R/\partial h > 0$ and consumers' market participation decreases in h for $c < \left(\sqrt{3} - \frac{3}{2}\right) h \approx 0.23h$. However, there is a region of parameter values for which the opposite is true. When $\left(\sqrt{3} - \frac{3}{2}\right) h \leq c \leq h/4$, then $\partial \bar{v}_R/\partial h < 0$ and consumers' market participation increases in h in the equilibrium. Equilibrium profits $\pi_R^* = \frac{h}{2}(1 - h + \frac{h}{2}\sqrt{1 - 4(c/h)})$ decrease in search cost, $\partial \pi_R^*/\partial c < 0$. The effect of h on the profits is ambiguous. Profits increase in h if and only if $h - 3c + (1 - 2h)\sqrt{1 - 4(c/h)} > 0$. For example, it suffices to require $h \leq \frac{1}{2}$. In contrast, for relatively large h, profits decrease with a further increase in product heterogeneity.■

Proof of Proposition 3. According to Proposition 1, conditions $c \leq \frac{1}{4h}$ and $c \leq \frac{h}{4}$ are necessary and sufficient for the existence of the random search equilibrium. From Proposition 2, the referral equilibrium exists if and only if $c \leq \frac{h}{4}$ and $h - 1 \leq \frac{h}{2}\sqrt{1 - 4(c/h)}$. The latter inequality holds for $h \leq 1$, and for $h > 1$ it is equivalent to $c \leq \frac{1}{4h}(2 - h)(3h - 2)$ or $c \leq 2 - \frac{3}{4}h - \frac{1}{h}$. When $h \leq 1$, both equilibria exist for $c \leq \frac{1}{4h}$. For $h > 1$, $c \leq \frac{1}{4h}$ always implies $c \leq \frac{h}{4}$, and $\frac{1}{4h} < 2 - \frac{3}{4}h - \frac{1}{h}$ holds if and only if $1 < h < \frac{5}{3}$.■

Proof of Proposition 4. The referral equilibrium price is higher than the random search equilibrium price if and only if $p^* = \sqrt{hc} \leq \frac{h}{2} = p_R^*$. This is equivalent to $c \leq \frac{h}{4}$. Thus, if both types of equilibria exist, then $p^* \leq p_R^*$. From $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1-4(c/h)}$ and $x^* = \sqrt{c/h}$ it follows that $x_R^* \leq x^*$ if and only if $2\sqrt{\frac{c}{h}} - 1 \geq \sqrt{1 - \frac{4c}{h}}$, or $4\frac{c}{h} - 4\sqrt{\frac{c}{h}} + 1 \geq 1 - \frac{4c}{h}$. This is again equivalent to $c \leq \frac{h}{4}$.

Next, we compare the willingness-to-pay of consumers who are indifferent between participating or not under two equilibria. In the random search equilibrium, $\bar{v} = 2\sqrt{hc}$ is a threshold value, while in the referral equilibrium, $\bar{v}_R = h - \frac{h}{2}\sqrt{1-4(c/h)}$. Consumers are better off in the referral equilibrium if and only if $\bar{v}_R \leq \bar{v}$, or $\sqrt{h} - 2\sqrt{c} \leq \frac{1}{2}\sqrt{h-4c}$. This is equivalent to $\left(\sqrt{h} - 2\sqrt{c}\right)^2 \leq \frac{1}{4}\left(\sqrt{h} - 2\sqrt{c}\right)\left(\sqrt{h} + 2\sqrt{c}\right)$, or $3\sqrt{h} - 10\sqrt{c} \leq 0$, or $c \geq 0.09h$. Hence, consumer welfare is higher in the referral equilibrium whenever $c \geq 0.09h$.■

Proof of Proposition 5. The social welfare in the random search equilibrium is $W^*(c,h) = \frac{1}{2} - \sqrt{hc} \geq 0$ for $c \leq \frac{1}{4h}$. Thus, it follows that $\partial W^*/\partial c < 0$ and $\partial W^*/\partial h < 0$.

The social welfare in the referral equilibrium can be written as

$$\begin{aligned} W_R^*(c,h) &= (1-\bar{v}_R)\,p_R^* + \frac{1}{2}(1-\bar{v}_R)^2 \\ &= \frac{1}{2}\left(1 + \frac{h}{2}\sqrt{1-4\,(c/h)} - h\right)\left(1 + \frac{h}{2}\sqrt{1-4\,(c/h)}\right). \end{aligned} \tag{A7}$$

It follows that $\partial W_R^*/\partial c < 0$. In contrast, the sign of $\partial W_R^*/\partial h$ is ambiguous. In the proof of Proposition 2, it is established that for sufficiently large c relative to h, $\left(\sqrt{3} - \frac{3}{2}\right) \leq \frac{c}{h} \leq \frac{1}{4}$, market participation increases in h in the equilibrium. Hence, for these parameter values, social welfare definitely increases with increasing product heterogeneity. On the other hand, for sufficiently small search costs, the effect is negative since W_R^* is continuously differentiable in c and h for $h \in (0, 4c)$, $\partial W_R^*/\partial h = -\frac{1}{4}\theta(h,c)/\sqrt{h\,(h-4c)}$, where $\theta(h,c) \equiv 2\,(c\,(2-3h) + h\,(h-1)) + \sqrt{h\,(h-4c)}\,(2-h+2c)$, and for $c \to 0$, $\partial W_R^*/\partial h \to -\frac{1}{4}h < 0$.■

Proof of Proposition 6. Using firm i's profit $\pi_{Ri}(p_i, p; \gamma)$, the first-order condition is

$$\frac{\partial \pi_{Ri}}{\partial p_i} = (1-p-hx_R^*)\left(1-(1-2x_R^*)\gamma + 2\left(\frac{p-p_i}{h}\right) - 2\left(\frac{p_i-\gamma p}{h}\right)\right) = 0. \tag{A8}$$

Thus, the equilibrium price, given commission γ, is $p_R^*(\gamma) = \frac{h}{2(1-\gamma)}(1-(1-2x_R^*)\gamma) = \frac{h}{2}\left(\frac{1-\gamma\sqrt{1-4(c/h)}}{1-\gamma}\right)$. It is easy to see that $dp_R^*(\gamma)/d\gamma > 0$ and $\partial p_R^*(\gamma)/\partial c < 0$. The sign of $\partial p_R^*(\gamma)/\partial h$ is the same as the sign of $\sqrt{1-4(c/h)} + (2(c/h)-1)\gamma$. If $2(c/h)-1>0$, then $\partial p_R^*(\gamma)/\partial h > 0$. If $2(c/h)-1 \leq 0$, then $\sqrt{1-4(c/h)} - (1-2(c/h))\gamma > \sqrt{1-4(c/h)} - 1 + 2(c/h) \geq 0$. Solving the inequality, we find that $\partial p_R^*(\gamma)/\partial h > 0$ if and only if $(c/h) < \frac{1}{2\gamma^2}\sqrt{1-\gamma^2}\left(1-\sqrt{1-\gamma^2}\right)$. The critical level for search cost relative to product heterogeneity decreases in γ, varying from $\frac{1}{4}$ for $\gamma \to 0$ to 0 for $\gamma \to 1$.■

Proof of Proposition 6'. With a fixed referral fee f, firm i's profit function is written as the sum of profits from consumers buying from firm i on their first visit, on their visit by referral, and the payments from other firms for the referrals firm i makes. A measure $(1-p-hx_R^*)$ of consumers participates in the market, since only consumers with values exceeding $p+hx_R^*$ initiate search. Firm i sells to $2x_R(p_i,p)$ of the searchers who visit firm i first and collects referral fee f for the rest of consumers it refers. A measure $(1-2x_R^*)$ of consumers who first visit another firm is referred to firm i, and firm i receives (p_i-f) for each of the referral customers. Therefore, the profits can be written as

$$\begin{aligned}\pi_{Ri}(p_i,p;f) &= (1-p-hx_R^*)\left(2p_i x_R(p_i,p) + (1-2x_R^*)(p_i-f) + f\,(1-2x_R(p_i,p))\right)\\ &= (1-p-hx_R^*)\left(p_i\left(1-\frac{2(p_i-p)}{h}\right) + f\times\frac{2(p_i-p)}{h}\right).\end{aligned} \tag{A9}$$

By taking the first-order condition, we have

$$\frac{\partial \pi_{Ri}}{\partial p_i} = (1-p-hx_R^*)\left(1+\frac{2p-4p_i}{h}+\frac{2f}{h}\right). \tag{A10}$$

Thus, the equilibrium price, given a referral fee f, is $p_R^*(f) = \frac{h}{2}+f$. Since a consumer who is indifferent about participating in the market has a willingness-to-pay $\bar{v}_R(f) = p_R^*(f)+hx_R^*$, an increase in f reduces consumers' market participation and consumer welfare. For the equilibrium to exist, some consumers have to engage in search: $\bar{v}_R(f) = f+h-\frac{1}{2}h\sqrt{1-4(c/h)} \leq 1$, or $f \leq 1-h+\frac{1}{2}h\sqrt{1-4(c/h)}$.

The equilibrium profit is described by $\pi^*_R(f) = (1 - p^*_R(f) - hx^*_R)p^*_R(f)$, where $x^*_R = \frac{1}{2} - \frac{1}{2}\sqrt{1-4(c/h)}$. This implies that a marginal increase in f improves firms' profits if and only if $p^*_R(f) < \frac{1-hx^*_R}{2}$ (the monopoly price given hx^*_R), or $f < \frac{1}{2} - \frac{3h}{4} + \frac{h}{4}\sqrt{1-4(c/h)}$. Therefore, for $f \in \left(0, \frac{1}{2} - \frac{3h}{4} + \frac{h}{4}\sqrt{1-4(c/h)}\right)$, profits increase with referral fees, while profits decrease with f for $\frac{1}{2} - \frac{3h}{4} + \frac{h}{4}\sqrt{1-4(c/h)} \le f \le 1 - h + \frac{h}{2}\sqrt{1-4(c/h)}$.■

Proof of Proposition 7. From the expression for firm i's profits $\pi_{RTi}(p_i, p; \gamma, \alpha)$, the first-order condition in the case of a commission γ includes an additional marginal benefit $\alpha(1-\gamma)$, and the price is $\alpha\frac{h}{2}$ higher than $p^*_R(\gamma)$ of Proposition 6. The equilibrium price given a commission γ is $p^*_{RT}(\gamma, \alpha) = \frac{h}{2}\left(\frac{1-\gamma\sqrt{1-4c/h}}{1-\gamma} + \alpha\right)$. The equilibrium referral rule, $x^*_{RT}(\alpha)$, is affected by the existence of third-party brokers.

The expected payoff from engaging in one additional search is

$$\begin{aligned}
&\Delta EU(x; x^*_{RT}; \alpha) \\
= \ & \frac{\alpha}{1+\alpha}(v-p) + \frac{1}{1+\alpha}\left[(1-2x^*_{RT})(v-p) + 2(x^*_{RT}-x)(v-p-hx) + 2\int_0^x (v-p-hx')dx'\right] \\
& -c - (v-p-hx) \qquad \text{(A11)} \\
= \ & (1 - \frac{1}{1+\alpha}2x^*_{RT})hx + \frac{1}{1+\alpha}hx^2 - c.
\end{aligned}$$

It is easy to see that for any $x^*_{RT} \in [0, \frac{1+\alpha}{2}]$ and any $x \in [0, x^*_{RT}]$, we have $\partial\Delta EU(x; x^*_{RT})/\partial x = h\left(1 - 2x^*_{RT}/(1+\alpha) + 2x/(1+\alpha)\right) \ge 0$. This means that as long as $\Delta EU(x^*_{RT}; x^*_{RT}) \le 0$, every consumer who has $x < x^*_{RT}$ would not engage in an additional search.

To find the symmetric equilibrium referral rule, let $\Delta EU(x^*_{RT}; x^*_{RT}; \alpha) = 0$. That is, a consumer who visited a firm x^*_{RT} apart from her ideal position is indifferent about searching (given that all other firms choose referral rule x^*_{RT}):

$$\begin{aligned}
\Delta EU(x^*_{RT}; x^*_{RT}; \alpha) &= (1 - \frac{1}{1+\alpha}2x^*_{RT})hx^*_{RT} + \frac{1}{1+\alpha}h{x^*_{RT}}^2 - c \qquad \text{(A12)} \\
&= hx^*_{RT} - \frac{1}{1+\alpha}h{x^*_{RT}}^2 - c = 0,
\end{aligned}$$

or

$$x^*_{RT}(\alpha) = \frac{1+\alpha}{2} - \frac{1+\alpha}{2}\sqrt{1 - \frac{4c}{(1+\alpha)h}} \qquad \text{(A13)}$$

for $c \leq (1+\alpha)h/4$. This valuation $x^*_{RT}(\alpha)$ describes the symmetric equilibrium referral rule. Now, we calculate the expected utility from the initial search:

$$\begin{aligned} EU_R(v,p;\alpha) &= \frac{\alpha}{1+\alpha}(v-p) + \frac{1}{1+\alpha}\left[2\int_0^{x^*_{RT}(\alpha)}(v-hx-p)dx + (1-2x^*_{RT}(\alpha))(v-p)\right] - c \\ &= \frac{\alpha}{1+\alpha}(v-p) + \frac{1}{1+\alpha}\left[(v-p) - h(x^*_{RT}(\alpha))^2\right] - c \\ &= v - p - \frac{h(x^*_{RT}(\alpha))^2}{1+\alpha} - c \\ &= v - p - hx^*_{RT}(\alpha) \end{aligned} \tag{A14}$$

Thus, the critical value for market participation is $\bar{v}_{RT}(\gamma,\alpha) = p^*_{RT}(\gamma,\alpha) + hx^*_{RT}(\alpha)$. To determine how $\bar{v}_R(\gamma,\alpha)$ is affected by α, we first examine how $x^*_{RT}(\alpha)$ responds to an increase in α. Calculations reveal that

$$\frac{dx^*_{RT}(\alpha)}{d\alpha} = \frac{1}{2}\left[1 - \frac{\sqrt{(1+\alpha)^2h^2 - 4c(1+\alpha)h + 4c^2}}{\sqrt{(1+\alpha)^2h^2 - 4c(1+\alpha)h}}\right] < 0. \tag{A15}$$

Finally, we determine the effects on $\bar{v}_R(\gamma;\alpha)$. Since $\frac{dp^*_{RT}(\gamma;\alpha)}{d\alpha} = \frac{h}{2}$, we have

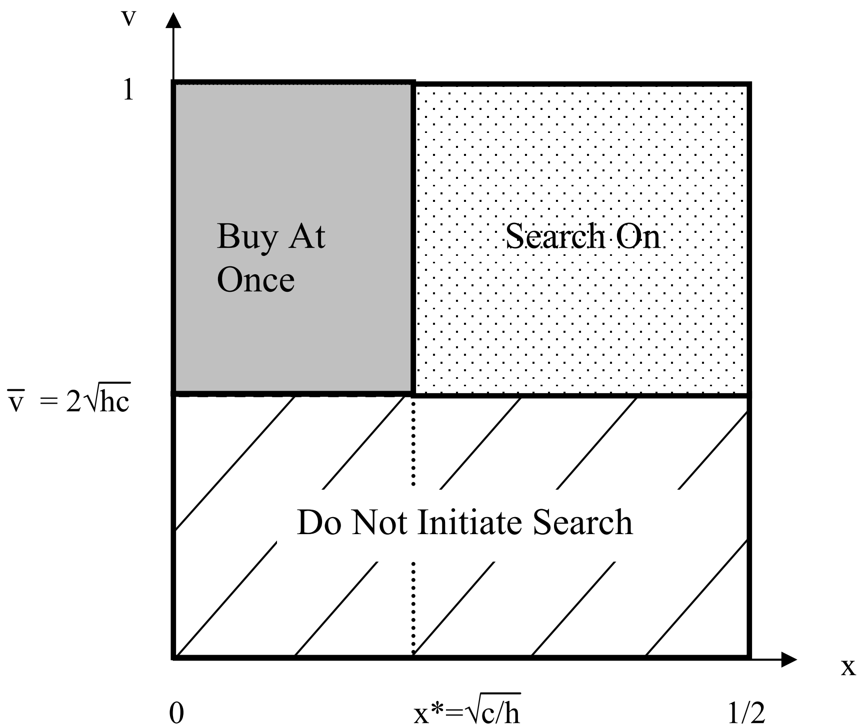
$$\frac{d\bar{v}_{RT}(\gamma,\alpha)}{d\alpha} = \frac{h}{2}\left[2 - \frac{\sqrt{(1+\alpha)^2h^2 - 4c(1+\alpha)h + 4c^2}}{\sqrt{(1+\alpha)^2h^2 - 4c(1+\alpha)h}}\right]. \tag{A16}$$

Thus, $\frac{d\bar{v}_{RT}(\gamma,\alpha)}{d\alpha} \gtreqless 0$ if and only if $4((1+\alpha)^2h^2 - 4c(1+\alpha)h) - ((1+\alpha)^2h^2 - 4c(1+\alpha)h + 4c^2) \gtreqless 0$. Straightforward calculations yield:

$$\begin{aligned} & 4((1+\alpha)^2h^2 - 4c(1+\alpha)h) - ((1+\alpha)^2h^2 - 4c(1+\alpha)h + 4c^2) \qquad \text{(A17)} \\ &= 4((1+\alpha)h - 2c)^2 - 16c^2 - ((1+\alpha)h - 2c)^2 \\ &= 3\left[((1+\alpha)h - 2c)^2 - \frac{16}{3}c^2\right] \\ &= 3((1+\alpha)h - 2c - \frac{4}{\sqrt{3}}c)((1+\alpha)h - 2c + \frac{4}{\sqrt{3}}c). \end{aligned}$$

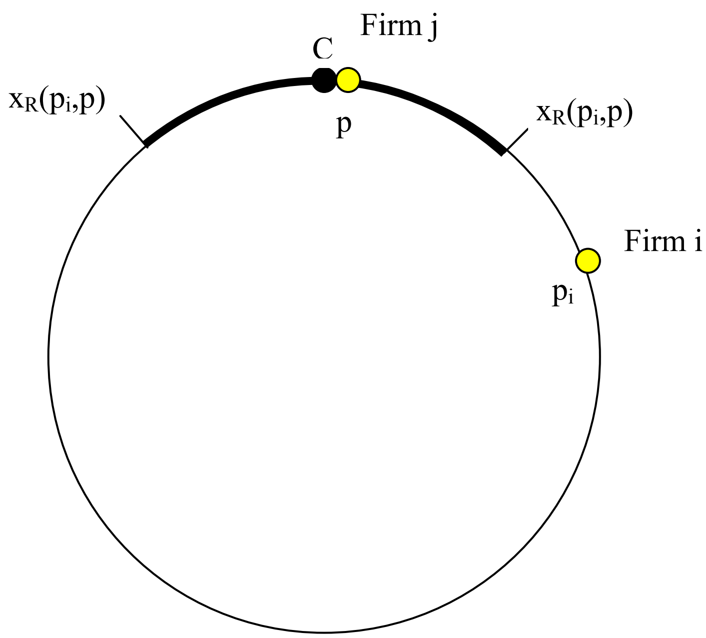
Since $(1+\alpha)h - 4c \geq 0$ by the equilibrium condition, $\frac{d\bar{v}_{RT}(\gamma,\alpha)}{d\alpha} \gtreqless 0$ if and only if $(1+\alpha)h - (2+\frac{4\sqrt{3}}{3})c \gtreqless 0$. Thus, if $c < \frac{(1+\alpha)h}{2+\frac{4\sqrt{3}}{3}} = \left(\sqrt{3} - \frac{3}{2}\right)(1+\alpha)h$, then the critical valuation goes up and market participation goes down with α. However, if $c \in ((\sqrt{3} - \frac{3}{2})(1+\alpha)h, \frac{(1+\alpha)}{4}h] \approx (0.23h(1+\alpha), 0.25h(1+\alpha)]$, market participation increases with α.■

Figure 1. Consumer Decisions in the Random-Search Equilibrium



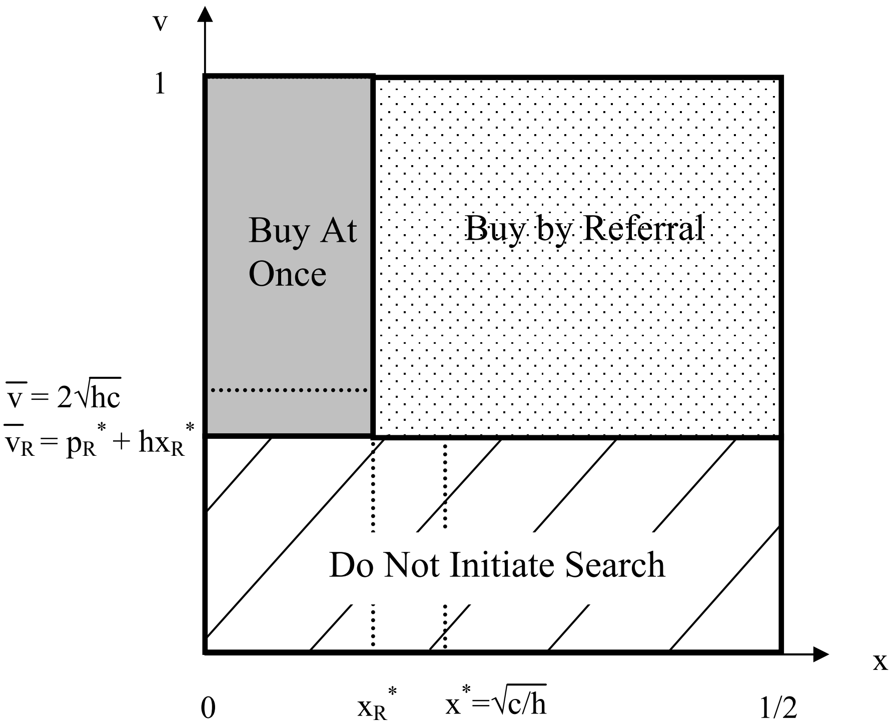
Notes: In the region labeled "Buy At Once" consumers receive utility (gross of search costs) $v - hx - p^*$. In the region labeled "Search On" consumers receive utility $v - \sqrt{hc} - p^*$. In the region labeled "Do Not Initiate Search" consumers receive zero utility.

Figure 2. Random Search with Referrals



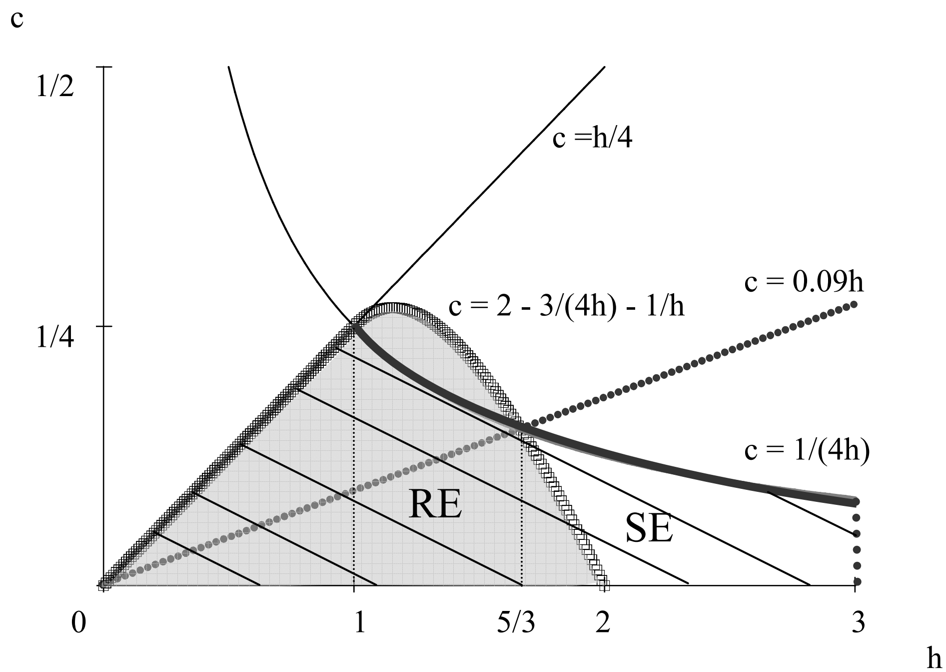
Notes: A consumer located at point C randomly samples firm *i* and has to decide whether to follow firm *i*'s referral to the best-matching product sold by firm *j*, buy at firm *i*, continue random search, or leave the market. Firm *i* refers the consumer to firm *j* if and only if the consumer is located at least $x_R(p_i,p)$ away.

Figure 3. Consumer Decisions in the Referral Equilibrium.



Notes: In the region labeled "Buy At Once" consumers receive utility (gross of search costs) $v - hx - p_R^*$. In the region labeled "Buy by Referral" consumers receive utility $v - p_R^*$. In the region labeled "Do Not Initiate Search" consumers receive zero utility.

Figure 4. Existence of the Search and Referral Equilibria



Notes: The random search equilibrium (SE) exists for c≤ min{1/(4h), h/4}, while the referral equilibrium (RE) exists whenever i) h≤1 and c≤ h/4 or ii) h>1 and c≤ 2 – 3/(4h) - 1/h. When the two equilibria exist, referrals lead to a Pareto improvement for c≥0.09h.