Referrals in Search Markets*

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Abstract

This paper compares the equilibrium outcomes in search markets a la Wolinsky (1984) with and without referrals. Although it seems clear that consumers would benefit from referrals, it is not at all clear whether firms would unilaterally provide information about competing offers since such information could encourage consumers to purchase the product elsewhere. In a model of a horizontally differentiated product and sequential consumer search, we show that valuable referrals can arise in the equilibrium: a firm will give referrals to consumers whose ideal product is sufficiently far from the firm's offering. We allow firms to price-discriminate among consumers, and consumers to misrepresent their tastes. It is found that the equilibrium profits tend to be higher in markets with referrals than in the ones without, even though there is a continuum of equilibrium prices with referrals. Under a certain parameter range, referrals lead to a Pareto improvement.

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1 Introduction

We often observe referral practices in industries where consumers purchase goods and services infrequently, or else the product characteristics are difficult to assess. Examples include professional services in areas of law, accounting, real estate, and health care, as well as specialized services and high-tech electronic products. As a result, in these industries a typical consumer needs to conduct a costly search in order to find the products that match her tastes. In contrast, firms know the industry well, and they gather information on goods and services provided by their competitors simply by engaging in their everyday business and participating in business groups. Through conversations with their customers, firms can also ascertain the customers' preferences and inform them about where they can find the products they are looking for. In such industries, referrals can reduce consumer search costs and improve matching outcomes.[1] However, in the absence of referral fees, a firm does not want to refer a consumer to another seller if there is a chance for the firm to sell its product to the consumer. Then, a natural question arises of whether or not there is any economic motive to making referrals without referral fees.[2]

Under what circumstances would firms inform consumers about products offered by other firms? Giving referrals can be a way to build goodwill with potential customers.[3] It is also a way to build referral relationships with business contacts. The reciprocity in referral relationships can be further supported by professional groups and associations that promote networking between members of a profession for the purpose of exchanging customer leads. Referral clubs aim specifically to "provide a structured and positive environment for businesses to pass referrals to one another" (e.g. *Gold Star Referral Clubs* and *Referral Key*).

Referrals affect consumers' search behavior, and as a result, firms' pricing behavior as well. In this paper, we first investigate the effects of referrals on equilibrium prices, profits,

[1] Sales clerks in retail markets may also offer referrals. This happens when a consumer searching for a suitable product (a digital camera, a piece of furniture, and a collection item) does not find it in a store, asks for referral, and is referred to a store that sells it.

[2] To avoid potential conflicts of interest in referral activity, professional associations in law, accounting, and real estate regulate referrals by establishing codes of honor, which may prohibit the payment of referral fees.

[3] One may argue that a store's sales clerk chooses to provide a valuable referral to a consumer purely out of goodwill. However, if being helpful to customers in that way is a norm for sellers, then we would like to know the impact of such a norm on the market outcome.

and consumer welfare. Then, we move on to the fundamental questions about the economic reasons for referrals to arise in search markets: Why are referrals made? How can the social norm for a firm to refer poorly matched potential consumers to the best-matching competitors arise in an equilibrium? We show that the social norm of making referrals can benefit firms, and this provides an explanation for existing referral practices in various industries.

To analyze the role of referrals in search markets we introduce referrals into a model of competition between firms producing a horizontally differentiated product. The product space is a unit circle, with firms evenly and consumers uniformly distributed over it. Consumer utility from a product decreases as a mismatch increases between a firm's product and the consumer's ideal product. A consumer can engage in sequential search by randomly sampling products at a constant marginal search cost. Upon a visit to a firm, the consumer learns the features of the product offered by the firm. The basic set up is as in Wolinsky's symmetric oligopolistic price competition model (Wolinsky, 1984), except that we introduce heterogenous consumer valuation for the product in order to endogenize consumer market participation. We then evaluate the effect of referrals in such markets. We allow firms to refer consumers who visit them to other sellers. A firm does not have to provide consumers with referrals if the firm would rather sell them its own product. However, when a firm knows that a consumer would not purchase its product, it may as well refer her to another firm. In this paper, we assume that when indifferent, a firm always refers a consumer to the best-matching product. Since the model is rather complicated, we first derive equilibria with and without referrals in the benchmark model of no price discrimination by firms and no misrepresentation of locations by consumers. We relax these assumptions in the main model and compare equilibria with and without referrals in this perhaps more realistic case.

The results are interesting. In the benchmark model, the symmetric equilibrium with or without referral is unique (Propositions 1 and 2 respectively). The norm of referring consumers to the best-matching competitors increases prices, and firms tend to earn more profits in the referral equilibrium than in equilibrium without referrals (Proposition 3). This result can be understood as follows. When a firm refers consumers to the rival firm offering them the highest utility, the referring firm is actually weakening the incentives of the rival firm to cut prices. Referrals make the rival's demand more inelastic and thus they function as a pro-collusive device. This suggests that although referrals provide consumers with valuable information that decreases search costs and improves product match, consumers may be worse off in the presence of referrals.[4] We show that indeed for sufficiently low search costs relative to the degree of product heterogeneity, consumers prefer markets without referrals. At the same time, referrals can lead to a Pareto improvement in search markets (Proposition 3).

In contrast, in the main model with consumer preference misrepresentation and firm price discrimination, a consumer may choose to report her location truthfully or to misrepresent it in order to get a referral, depending on the distance between the product sold by the firm she visits and her ideal one (Proposition 4).[5] Although the symmetric random search equilibrium (without referrals) is unique, there is a continuum of symmetric referral equilibria with different prices (Proposition 4). This indeterminacy of equilibrium comes from a kinked demand function generated by consumers' preference misrepresentation. Somewhat surprisingly, under most parameter values and referral equilibrium prices, firms are still better off in the referral equilibria than in the random search equilibrium (Proposition 5). In particular, if the optimal equilibrium prices are chosen, firms always prefer the referral equilibrium for any parameter values that assure the existence of the equilibria (Proposition 5). That is, the tendency for firms to prefer the establishment of referral practices in the industry is robust, and this explains why referrals arise in many industries, even in the absence of referral fees. We find that when search costs are not too low, consumers tend to be better off in the presence of referrals. We also find that depending on the parameter values and the equilibrium price, the random search equilibrium can Pareto-dominate or be Pareto-dominated by the referral equilibrium (Proposition 5).

Given differences in opinion about referral policies and their prominence in many industries, it is perhaps surprising that there is not much economic theory on the topic. One

[4] Such a negative externality on consumers from improved information was first demonstrated by Anderson and Renault (2000) in an economy with horizontally differentiated search goods.

[5] Propositions 4 and 5 deal with the referral equilibria under preference misrepresentation, yet *without* price discrimination. Proposition 6 shows that as long as consumers can misrepresent their preferences, the equilibrium is intact even if firms can price-discriminate among consumers based on their reported preferences.

exception is a study by Garicano and Santos (2004), which examines referrals between vertically differentiated firms (*vertical referrals*). Due to complementarity between the values of opportunities and firms' skills, efficient matching involves assigning more valuable opportunities to high-quality firms. The authors show that flat referral fees can support efficient referrals from high-quality to low-quality firms but not vice versa. The moral hazard problem arises because a low-quality firm has an incentive to keep the best opportunities to itself rather than refer them to a high-quality firm.[6] However, the model does not allow the firms' clients to conduct searches on their own. In contrast, we do not deal with the moral hazard problem in this paper. Instead, we focus on the role of *horizontal referrals* in search markets where consumers can always randomly visit a firm and at a cost learn about the firm's product and price.

The rest of the paper is organized as follows. Section 2 outlines the fundamental features of our basic environment. Section 3 considers price competition between firms selling a horizontally differentiated product in search markets without referrals and describes the random search equilibrium that arises in the model. In Section 4, we analyze search markets with referrals and derive the referral equilibria in such markets, first assuming that consumers cannot misrepresent their locations and then relaxing this assumption. Section 5 compares the referral and random search equilibria. In Section 6, we show that the results of the main model carry over to the case of price discrimination based on reported consumer locations and in the presence of referral fees. Section 6 offers concluding comments. The Appendix contains all the proofs.

2 The Basic Environment

We model competition between firms that produce a horizontally differentiated product closely following Wolinsky (1983, 1984, 1986).[7] We assume that there is a large number of

[6] For an early study on the effects of fee splitting on referrals in health care see Pauly (1979). An empirical study by Spurr (1990) on referral practices among lawyers examines the proportion of cases referred between lawyers, as dependent on the value and nature of a claim, advertising activity, and other factors.

[7] The closest model (especially our random search case) is Wolinsky (1984), but he assumes fixed price in the paper for the sake of simplicity of the analysis. The market price is endogeneously determined in Wolinsky (1983) in a similar spatial model, though the analysis is only briefly described there. Wolinsky (1986) offers the most thorough analysis on the determination of oligopolistic price, though he assumes away spatial structure in that paper, relying on i.i.d. random willingness-to-pay by consumers. We analyze a

firms, n, of which locations are symmetrically located (equidistant) on a circle of unit circumference and producing the product of the location at a zero marginal cost. As the number of firms goes to infinity, firms will be distributed uniformly over the circle. For analytical simplicity, ignore integer problems assuming that consumers regard the distribution of firms is uniform over the unit circle.[8] There is unit mass of consumers per firm, and each consumer has a unit demand and is characterized by a valuation of the product (willingness-to-pay) and preferences over the horizontally differentiated products. Consumers' ideal positions are uniformly distributed over the unit circle. Independently of spatial preferences, each consumer has a value v for her ideal product (the product that is a perfect match with her tastes). Although Wolinsky (1984) assumes that v is common to all consumers, we assume that v is uniformly distributed over an interval $[0, 1]$ in order to determine the equilibrium consumer participation endogenously in the model. Each consumer's ideal position and valuation are independently distributed (double uniform distribution over the surface of a cylinder with unit circumference with unit height), and she knows her v and her ideal position on a circle. As a result, consumers decide if they participate in the market based on her value of v (endogenous participation decision).

Firms know the positions on the circle of all other firms in the industry, whereas consumers do not know which firms offer which products. When a consumer visits a firm, she learns of the firm's position and its price, while the firm learns of the consumer's ideal product (tastes). The firm remains uncertain about the consumer's willingness-to-pay even after her visit. This assumption captures the idea that firms may find it easier to extract information about a consumer's ideal product than about the consumer's willingness-to-pay for the ideal product.

Consider a consumer whose product valuation is v and who learns that the firm's product is located at distance x from her ideal position. Let the consumer's utility for the firm's product (gross of price and search costs) be $u(x, v) = v - hx$, where $h > 0$ is the taste parameter that denotes the degree of product heterogeneity and hx represents the disutility a consumer receives from consuming a product located at a distance x in the product space

variant of the model in Wolinsky (1984) using the analysis in Wolinsky (1986).

[8] We follow the approach taken by Wolinsky (1983, 1984).

from the consumer's ideal product. For example, a high h corresponds to markets where consumers feel strongly about product characteristics, e.g. color, weight, or design. The case of $h = 0$ corresponds to a homogeneous product market.

Consumer search is sequential, with a constant marginal cost of search $c > 0$, common across consumers. Each consumer chooses whether to initiate search, and a consumer who decides to search can always choose not to buy the product of the firm sampled and continue to search for another product. We assume that consumer search is random with replacement.[9] That is, at each step of the sequential search process, a consumer samples firms' products randomly, but she does not sample the same product more than once, and she can always recall previously sampled products. We assume that when indifferent between searching or not, consumers do not search.

A firm's referral policy can be conditional on consumer location. At the same time, firms are assumed to set nondiscriminatory prices in the benchmark model with or without referrals. When a firm refers a customer to another firm, the customer can costlessly follow the referral. This is consistent with the idea that search costs come mainly in the form of learning about product characteristics, and therefore referrals substantially lower consumers' search costs. The assumption is not crucial, though. We also assume that whenever a firm knows that it cannot sell its own product to a consumer given its own price and other firms' prices, and thus is indifferent whether to provide a referral or not and where to refer the consumer, it refers the consumer to the best-matching firm. This assumption of *honest referrals* resolves firms' indifference and is in the spirit of the honor codes adopted in markets for professional services. Note that honest referrals are in fact incentive compatible in the sense that firms do not gain by deviating unilaterally from the equilibrium in which honest referrals are supported.[10]

We start in Section 3 with the analysis of price competition in a random search market without referrals. We assume that firms set nondiscriminatory prices, and we then derive

[9] We assume search with replacement in order to have an invariant distribution of firms as search proceeds for the case of a finite number of firms. As Wolinsky (1984) correctly states, as the number of firms goes to infinity, it becomes irrelevant whether sampling process is with or without replacement.

[10] Suppose a firm were to deviate and refer a departing consumer to a poorly matching firm. The consumer would follow the referral because she expects it to be valuable in the equilibrium. Then, she would either purchase the referred product, or if she learns that the product is of poor match, she would engage in additional random search. Thus, the consumer would not purchase the dishonest firm's product anyway.

the equilibrium price p^* that characterizes the unique symmetric (pure-strategy) equilibrium with consumer participation, which we call *the random search equilibrium*.

3 Random Search

In this section, we assume that firms do not make referrals, and so the only way for consumers to receive information about a product's location and price is to engage in sequential random search. We will support the equilibrium with a symmetric price p by following Wolinsky's (1984, 1986) techniques. To find the equilibrium in the random search model, we assume that all firms except firm i charge price p and that consumers rationally expect the common price level, and we then derive the best-response price for firm i in a market with an arbitrary large number of firms.[11] By setting the best-response price equal to p, we obtain a symmetric equilibrium in the monopolistically competitive market.

To allow for the possibility of an equilibrium with consumer participation, we assume $c \leq \frac{h}{4}$, and later we will show that this condition is necessary for the existence of a symmetric equilibrium with consumer participation (Proposition 1).[12] For analytical simplicity, we ignore the integer problem due to finiteness of firms, so consumers regard firms' locations are uniformly distributed. When consumers participate in the market, firm i's profit depends on the optimal stopping rule for a consumer engaged in random sequential search. Lemma 1 derives the optimal stopping rule (similar to Wolinsky 1986: the proof is omitted).

Lemma 1. *Assume that $c \leq \frac{h}{4}$. When all firms except firm i set a common price p and firm i sets a price p_i, then the optimal stopping rule for a consumer engaged in random search is to stop searching at firm i if and only if firm i's product is within distance*

$$x(p_i, p) = x^* + \left(\frac{p - p_i}{h}\right) \tag{1}$$

[11] We could have taken the finiteness of n seriously by analyzing an n-firm search market, but this would greatly complicate the analysis because the distribution of unsampled products changes as search proceeds. Following Wolinsky (1984), we assume away this possibility by letting n be very large (see footnote 2 on page 276 in Wolinsky (1983) for a justification of the approach). For very large n we can also ignore the possibility of a consumer not being able to find a firm located within the critical distance x^*.

[12] Note that there is always a symmetric equilibrium without consumer participation in the market. In such an equilibrium, all firms charge a prohibitively high price, and consumers do not search correctly expecting the market price to be very high. This is an equilibrium because consumers only receive price information by engaging in search, and if a firm were to deviate by reducing its price it would not be able to convey the price information to consumers. Hence, no deviation is profitable.

from her ideal product, where $x^* = \sqrt{\frac{c}{h}}$ *is the critical distance in the symmetric equilibrium. If in case all firms are searched in the above manner while could not find any product that is close enough to satisfy the condition, then choose the firm that offers the highest net surplus as long as it is nonnegative,* $\max_j(v - x_j - p_j) \geq 0$.

In a symmetric equilibrium, the probability that a consumer stops her search at firm i is $2x^* = 2\sqrt{\frac{c}{h}} \leq 1$ because the optimal stopping rule tells her to stop searching once she samples a product within distance x^* "on both sides" of her ideal product on the circumference. For example, if firm i were to set a different price, $p_i \neq p$, it could then affect consumers' search behavior somewhat. If firm i were to surprise consumers by charging a higher price, firm i would prompt more consumers to continue random searches.

Note that in a symmetric random search equilibrium, it is strictly dominated for firm i to set a price p_i such that $x(p_i, p) > \frac{1}{2}$ or $x(p_i, p) < 0$. In case $x(p_i, p) > \frac{1}{2}$, which occurs when p_i is below $p - h\left(\frac{1}{2} - x^*\right)$, all consumers who visit firm i buy there since i's offer is better than engaging in further random search, and a firm could increase its price slightly without losing any customers. In case $x(p_i, p) < 0$, which occurs when p_i is above $p + hx^*$, no consumer would purchase firm i's product, but firm i could slightly reduce its price to generate positive sales and profits. Thus, we can restrict our attention to $p_i \in \left[p - h(\frac{1}{2} - x^*), p + hx^*\right]$.

To derive firm i' demand when it sets a price p_i and all other firms charge p, we need to first find the probability that a consumer samples firm i on a kth random draw, stops search to buy firm i's product, and then sum over all k, $1 \leq k \leq n$. The probability of drawing firm i as the kth firm is $\frac{1}{n}$ for a consumer, since all firms are symmetric and at each step in the sequential search, the next store is chosen randomly with equal probability and without correlation.[13] The probability that a consumer searches at least k times is $(1 - 2x^*)^{k-1}$ (which, in accordance with the optimal stopping rule, is the probability that all previous $k - 1$ draws were farther than distance $x^* = \sqrt{\frac{c}{h}}$ on both sides from her ideal position). If firm i charges p_i, the probability that a consumer purchases at firm i upon the visit is $2x(p_i, p)$. Note that firms are located in equi-distanced manner. As a result, with

[13] With n firms, firm i is selected first with probability $1/n$. It is selected on a second draw if it is not selected first (probability $1 - 1/n$), and if it is selected as the second firm among the remaining $n - 1$ firms (probability $1/(n-1)$). Thus, the probability of firm i being selected on the second draw is also $1/n$. Similarly, we can show that firm i is selected on the kth draw with probability $1/n$ (see Wolinsky, 1986).

a very large n, each consumer finds a firm that is close enough to her position before all n firms have been exhausted.[14] Thus, firm i's *demand per searcher* is

$$\begin{aligned}\widetilde{D}_i^n(p_i, p) &= \frac{1}{n}\sum_{k=1}^{n}(1-2x^*)^{k-1}\,2x(p_i,p) \qquad (2)\\ &= \frac{1}{n}\left(\frac{1-(1-2x^*)^n}{1-(1-2x^*)}\right)2x(p_i,p).\end{aligned}$$

The last term of the demand function is the case where consumers searched all firms while none of the firms offer To finish the derivation of firm i's demand function, we need to find the measure of searchers. Let v^* denote the critical valuation for market participation, which is the valuation of the consumer indifferent between initiating search and staying outside the market. Only consumers with valuations $v \in [v^*, 1]$ enter the market. Since there is unit mass of consumers per firm, the total measure of searchers is $n(1-v^*)$, and firm i's *demand function* is

$$D_i^n(p_i, p) = (1-v^*)\left(\frac{1-(1-2x^*)^n}{1-(1-2x^*)}\right)2x(p_i,p). \qquad (3)$$

For zero marginal costs, the profit function of firm i is then $\pi_i^n(p_i, p) = p_i D_i^n(p_i, p)$ in the n-firm market. In the limit as $n \to \infty$, firm i's profit function becomes

$$\pi_i(p_i, p) = \lim_{n\to\infty}\pi_i^n(p_i, p) = (1-v^*)\left[p_i\left(1+\frac{p-p_i}{\sqrt{hc}}\right)\right]. \qquad (4)$$

A firm's price does not affect consumers' market participation decisions and thus does not affect the number of consumers who visit the firm. This is because consumers' decisions to engage in search are based on price expectations, not on the actual prices set by firms. Therefore, the measure of searchers does not depend on p_i, and firm i chooses p_i to maximize the expression in the square brackets in (4). The properties of the unique random search equilibrium and comparative statics results are stated in Proposition 1.

Proposition 1. *Consider the limit case of $n \to \infty$. There exists a unique symmetric random search equilibrium if and only if $c \le \min\left\{\frac{1}{4h}, \frac{h}{4}\right\}$. The equilibrium price is $p^* = \sqrt{hc}$, critical valuation for market participation is $v^* = 2\sqrt{hc}$, and profits are $\pi^* = (1-v^*)p^*$.*

[14] In Wolinsky (1986), demand function has a second term that is composed by the demand by consumers who search all firms without satisfying the criterion described by the optimal stopping rule. This is because in Wolinsky's model there is only n times to try searching. In our model (with replacement: in order to preserve stationary distribution), we may visit the same firm again and again randomly. So, even if the number of firms is n, the number of search can be much larger until all firms have been exhausted.

The equilibrium price increases and consumer market participation decreases in search cost, c, and product heterogeneity, h. Profits can increase or decrease in c and h.

Condition $c \leq \frac{h}{4}$ is necessary for sustaining sequential consumer search in a symmetric equilibrium.[15] The other condition $c \leq \frac{1}{4h}$ is needed to guarantee that the equilibrium price is not prohibitively high. If this condition is violated, then $v^* = 2\sqrt{hc} > 1$, and all consumers expect a negative payoff and stay away from the market.

The comparative statics analysis of Proposition 1 shows that, as expected, the equilibrium price in the random search equilibrium increases with search cost and product heterogeneity. Interestingly, the equilibrium profits may increase or decrease with c and h depending on their levels. The equilibrium profits increase in c and h for sufficiently low levels of the parameters, $hc < \frac{1}{16}$. For other values of c and h, the decline in market participation is not compensated for by the higher price associated with larger c and h. In other words, for $\frac{1}{16} \leq hc \leq \frac{1}{4}$, sellers prefer to operate in search markets with lower search costs and lower product heterogeneity. Intuitively, if search cost c goes up, then the profits may go down despite a higher equilibrium price p^* because the price increase reduces equilibrium consumer participation. Janssen et al. (2005) and Konishi (2005) similarly find that the equilibrium profits are influenced by these two effects.

[**Figure 1 here**]

The equilibrium consumer decisions to engage in search, buy at a firm located at a distance x, or engage in sequential search are illustrated in Figure 1. Only consumers whose product valuations are at least $v^* = 2\sqrt{hc}$ engage in search. Consumers visiting a firm located within distance $x^* = \sqrt{\frac{c}{h}}$ from their ideal positions buy the product, while others

[15] When $c > \frac{h}{4}$, there is no symmetric pure-strategy equilibrium with consumer participation in the random search model. When $c > \frac{h}{4}$, the optimal stopping rule in a symmetric equilibrium is for a consumer to always stop searching regardless of the product found since for such parameter values $x^* = \sqrt{\frac{c}{h}} > \frac{1}{2}$. However, without sequential search, firms have an incentive to raise prices above the expected price level, and therefore a symmetric equilibrium with consumer participation does not exist when $c > \frac{h}{4}$. This type of *non-participation* equilibrium is described in Stiglitz (1979) and is also reminiscent of the Diamond paradox (Diamond, 1971). However, such an equilibrium is rather uninteresting, and we will restrict parameter values to focus on equilibria with consumer participation.

continue to search. Figure 1 also depicts the expected utility of a consumer who first draws a product located at distance x from the consumer.

4 Search with Referrals

The goal of this section is to analyze search models with referrals and no referral fees. In the model, firms simultaneously set uniform prices and choose referral policies. Our focus is again on the symmetric pure-strategy equilibria in which all firms use the same price and referral policy. We will show that since a firm refers a consumer if and only if the consumer would otherwise leave the firm to engage in random search, firm i's referral rule is characterized by the same critical distance as the consumers' stopping rule. Hence, a symmetric referral equilibrium is described by a common price p_R^* and a referral rule x_R^*, such that if the distance between a consumer's ideal product and a firm is greater than x_R^*, the firm refers the consumer to the best-matching firm. We derive the unique symmetric equilibrium with consumer participation, (p_R^*, x_R^*), which we call *the referral equilibrium.*

Firms do not observe consumer locations but they know the locations of all products. Given that firms do not have an incentive to refer consumer located close to them, consumers may want to misrepresent their locations in order to receive a referral. This suggests that the model that allows consumers to strategically manipulate their preferences may be preferable. However, for the sake of clarity of presentation, we start in Section 4.1 with the benchmark case of referrals with no consumer manipulation. Then, in Section 4.2 we analyze the main case of referrals with consumer misrepresentation. The latter is significantly more complicated with qualitatively different characteristics such as indeterminacy of the market equilibrium price.

4.1 No misrepresentation of preferences

In this section, we allow firms to provide referrals to consumers. We assume that there are no referral fees and that a firm (salesperson) can observe a customer's ideal position x from having a conversation with her, but cannot observe her willingness-to-pay v. Therefore, the referral that a firm offers its customer can only be conditioned on the observed consumer's ideal product. We continue to assume in the benchmark model that firms charge nondiscriminatory prices, but we will relax this assumption in Section 6.

In the model, firms simultaneously set nondiscriminatory prices and referral policies, while consumers hold rational expectations and select the optimal stopping rule for their sequential search with costless and perfect recall of products and referrals. In a search market with referrals, a consumer chooses whether to start a random search given her product valuation v; after each draw the consumer learns the firm's product and decides whether to follow the firm's referral if it is given, continue random search, purchase the best examined item, or leave the market without purchasing any product. Our focus is again on symmetric pure-strategy equilibria (i.e., equilibria in which firms choose the same price and referral policy).

As discussed in Section 2, we assume that firms' referrals are honest in the sense that whenever a firm knows that it cannot sell its own product to a consumer at the given prices, it refers her to the best-matching firm. A firm would then refer a consumer if and only if the consumer's ideal position is sufficiently far away, so that the consumer prefers engaging in further random search to buying at the firm. The optimal stopping rule in sequential search with referrals is for a consumer visiting firm i to stop searching at firm i if and only if firm i's product is within distance $x_R(p_i, p)$ from her ideal product. Since a firm refers a consumer if and only if the consumer would otherwise leave the firm to engage in additional random search, firm i's referral rule is characterized by the same critical distance as the consumer stopping rule. The symmetric equilibrium referral rule states that if the distance between a customer's position and a firm is more than $x_R^* = x_R(p, p)$, the firm refers the consumer to her best-matching firm. Hence, a symmetric referral equilibrium can be characterized by a pair $(p_R^*,\ x_R^*)$.

To find the equilibrium $(p_R^*,\ x_R^*)$, consider firm i choosing an arbitrary price p_i, and suppose that all other firms set a common price p and referral rule x_R^*. The optimal referral threshold $x_R(p_i, p)$ for firm i is based on the location of a consumer who is indifferent between buying from firm i at price p_i and continuing to search at random among firms charging p. Lemma 2 derives the equilibrium referral rule x_R^* and optimal stopping rule $x_R(p_i, p)$ for a consumer engaged in search with referrals.

Lemma 2. *Assume that $c \leq \frac{h}{4}$. Suppose firm i sets a price p_i and other firms set a common* *price p and use referral rule x_R^*. The optimal stopping rule for a consumer engaged in search with referrals is to stop searching at firm i if and only if firm i's product is within distance $x_R(p_i, p) = x_R^* + \left(\frac{p - p_i}{h}\right)$ from her ideal product, where $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(c/h)}$ is the equilibrium referral rule.*

[**Figure 2 here**]

Figure 2 illustrates the optimal consumer search and firm i's referral policy, assuming that firm i sets a price p_i and other firms set a common price p and use referral rule x_R^*. A consumer randomly chooses the first firm to visit, say, firm i, located at distance x from her ideal position. When the distance between the consumer and firm i is greater than $x_R(p_i, p)$, as in the case of consumer B, the consumer is referred by firm i to the best-matching firm (firm j), and she has to decide whether to buy at firm i, follow the referral, continue random search, or leave the market. When firm i is located within distance $x_R(p_i, p)$ from the consumer's ideal position, as in the case of consumer A, the consumer does not receive a referral and has to either buy at firm i, continue random search, or leave the market. In the symmetric equilibrium, a consumer follows the referral and buys at firm j when firm i is located farther away than $x_R^* = x_R(p, p)$, and she buys at firm i immediately otherwise.

In case $c > \frac{h}{4}$, the optimal stopping rule in a symmetric equilibrium is for a consumer to always stop searching regardless of the product found. This is the same as in the case of no referrals. Thus, consumers search at most one firm, and since referrals are given only to consumers who would otherwise engage in additional random search, consumers do not receive referrals. Hence, just as in the model without referrals, a symmetric equilibrium with consumer participation does not exist in search markets with referrals when $c > h/4$.

Next, we calculate the demand function of firm i assuming that other firms choose a symmetric price p and give referrals to customers located farther than x_R^*. We can restrict our attention to $p_i \in \left[p + hx_R^* - \frac{1}{2}h, p + hx_R^*\right]$ because it is strictly dominated for firm i to price outside this interval. In case $p_i < p + hx_R^* - \frac{1}{2}h$, we have $x_R(p_i, p) > \frac{1}{2}$, and all consumers who visit firm i buy there since i's offer is better than engaging in further random search or following the referral. But then firm i would rather set a slightly higher price. In case $p_i > p + hx_R^*$, we have $x_R(p_i, p) < 0$, and no consumer visiting firm i purchases its product. Moreover, $v - p_i < v - p - hx_R^*$ holds for any v, and even consumers who are referred to firm i do not purchase from it. Thus, demand for firm i in this case is zero, and firm i would benefit by reducing its price to generate positive demand and profits.

For $p_i \in \left[p + hx_R^* - \frac{1}{2}h, p + hx_R^*\right]$, consumers located at $x \leq x_R(p_i, p) \in [0, \frac{1}{2}]$ buy from firm i. Assuming firm i follows the optimal referral policy $x_R(p_i, p)$, its *demand per searcher* is

$$\begin{aligned} \widetilde{D}_{Ri}^n(p_i, p) &= \left(\frac{1}{n}\right) 2x_R(p_i, p) + \left(\frac{n-1}{n}\right) \times \frac{1}{n-1}(1 - 2x_R^*) \\ &= \frac{1}{n}\left(1 + \frac{2(p - p_i)}{h}\right), \end{aligned} \tag{5}$$

where $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\,(c/h)}$ and the number of firms n is very large. The first term represents demand from consumers who visit firm i first, while the second term represents demand from consumers who visit other firms first and are then referred to firm i.

Let v_R^* denote the critical valuation for market participation, which is the valuation of the consumer who is indifferent between initiating sequential search and staying outside the market in the referral equilibrium. Since there is unit mass of consumers per firm, the total measure of participating consumers is $n(1 - v_R^*)$, and firm i's *demand function* is

$$D_{Ri}^n(p_i, p) = (1 - v_R^*)\left(1 + \frac{2(p - p_i)}{h}\right). \tag{6}$$

Hence, when firm i sets a price p_i and all firms are expected to set price p and use referral rule x_R^*, firm i's profit function in the limit as $n \to \infty$ is

$$\pi_{Ri}(p_i, p) = \lim_{n \to \infty} p_i D_{Ri}^n(p_i, p) = (1 - v_R^*)\left[p_i\left(1 + \frac{2(p - p_i)}{h}\right)\right]. \tag{7}$$

As before, a firm's price does not affect consumers' market participation decisions and, therefore, $(1 - v_R^*)$ does not depend on p_i. Given this, firm i chooses p_i to maximize the expression in the square brackets in (7). In Proposition 2 we describe the unique pure-strategy symmetric equilibrium with consumer participation, which we call the *referral equilibrium.*

Proposition 2. *Consider the limit case of $n \to \infty$. There exists a unique symmetric referral equilibrium if and only if $c \leq \frac{h}{4}$ and either $h \leq 1$ or $c \leq 2 - 3h/4 - 1/h$. The equilibrium price is $p_R^* = \frac{h}{2}$, referral rule is $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\,(c/h)}$, critical valuation for market participation is $v_R^* = p_R^* + hx_R^*$, and profits are $\pi_R^* = p_R^*(1 - v_R^*)$. Referral intensity decreases with consumer search cost and increases with product heterogeneity. The equilibrium price is perfectly insensitive to search cost, whereas it increases as product heterogeneity increases. Consumers' market participation and profits decrease with search cost, c, and they can increase or decrease with product heterogeneity, h.*

According to Proposition 2, the symmetric referral equilibrium exists if and only if $c \leq \frac{h}{4}$ and either $h \leq 1$ or $c \leq 2 - 3h/4 - 1/h$. The inequality $c \leq \frac{h}{4}$ is necessary for sustaining sequential consumer search in a symmetric equilibrium. The second condition obtains from $h - 1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$, which is equivalent to $v_R^* \leq 1$, where $v_R^* = p_R^* + hx_R^* = h - \frac{h}{2}\sqrt{1 - 4\,(c/h)}$ is the critical valuation for market participation.

Figure 3 illustrates the equilibrium consumer decisions and the expected consumer utility for different realizations of v and x. Only those consumers whose product valuations are at least v_R^* engage in search. Consumers visiting a firm located within distance x_R^* buy the product, while others follow the firm's referral and buy from the referred seller.

[**Figure 3 here**]

Why does search cost c not matter in the determination of price p_R^* in this case? By lowering its price, firm i can increase its sales only through an increase in the retention rate $2x_R(p_i, p) = 2x_R^* - 2(p_i - p)/h$ of consumers who visit firm i first. However, a change in the retention rate, which equals $-2/h$, is not affected by search cost c since sequential search never takes place in the equilibrium. Thus, in search markets with referrals the equilibrium price is determined only by heterogeneity parameter h. We can see an analogy with the Diamond paradox (Diamond 1971): in both cases, sequential search does not occur, and the equilibrium price is independent of the level of search cost (as long as it is positive). In the current model, however, there is still competition among firms trying to retain initial customers, and the monopoly price does not prevail as the equilibrium price.

Proposition 3. *Whenever both random search and referral equilibria exist, we have* $p_R^* \geq p^*$ *and* $x_R^* \leq x^*$. *All consumers are better off in the referral equilibrium if and only if* $c \geq 0.09h$, *i.e. search cost is not very low relative to product heterogeneity. Thus,* $c \geq 0.09h$ *guarantees that firms earn more profits in the referral equilibrium, and the referral equilibrium Pareto-dominates the random search equilibrium.*

4.2 Misrepresentation of preferences

In the benchmark case, we assumed that firms can correctly identify consumers' ideal locations. However, a consumer whose true location is outside a firm's referral region but close to the referral border x_R^* (say, $x_R^* - \Delta$, where $\Delta > 0$ is small) has an incentive to report her position to be $x_R^* + \varepsilon$ so that she can receive a referral, where $\varepsilon > 0$ is arbitrarily small. Although such manipulation would result in an imperfect referral, it is beneficial as long as $x_R^* - \Delta > \Delta$. To capture the possibility of consumer manipulation, we will assume that (1) as before, firms know other firms' positions, but they cannot observe the location of a consumer who visits them, and that (2) although consumers cannot tell firms' positions prior to their visits, they can immediately identify a firm's position upon a visit to the firm.

Maintaining the assumption of nondiscriminatory pricing, we obtain results similar to those of the benchmark model. Figure 5 illustrates the equilibrium referrals obtained by consumers depending on their distance from a sampled firm (firm i). In the symmetric equilibrium firms provide referrals for anybody outside of $[-x_R^{**}, x_R^{**}]$. Consumers located farther than x_R^{**} from firm i truthfully report their location and obtain perfect referrals. Consumers located in the region $[-x_R^{**}, -\frac{x_R^{**}}{2}) \cup (\frac{x_R^{**}}{2}, x_R^{**}]$ have an incentive to misrepresent their position in order to receive an imperfect referral. These consumers report their location to be $x_R^{**} + \varepsilon$ (where $\varepsilon > 0$ is arbitrary small) in order to receive a referral to a firm located at $x_R^{**} + \varepsilon$, which is less than $\frac{x_R^{**}}{2}$ from their ideal position. Firm i can sell its product only to new (not referred by other firms) consumers in region $[-\frac{x_R^{**}}{2}, \frac{x_R^{**}}{2}]$. However, by symmetry, firm i receives referrals from other firms on average of the measure of customers equivalent to $[-x_R^{**}, -\frac{x_R^{**}}{2}) \cup (\frac{x_R^{**}}{2}, x_R^{**}]$.

[**Figures 5 and 6 here**]

Figure 6 illustrates consumer decisions in the referral equilibrium with consumer misrepresentation and the equilibrium utility enjoyed by consumers, as dependent on the distance from the first firm they visit.

Lemma 2 can be modified by replacing the equilibrium referral rule $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4c/h}$ with $x_R^{**} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8c/h}$. We refer to this statement as Lemma 3 and use it to prove Proposition 6.

Lemma 3. *Assume that $c \leq \frac{h}{8}$. Suppose firm i sets a price p_i and other firms set a price p and use referral rule x_R^{**}. The optimal stopping rule for a consumer engaged in search with referrals is to stop searching at firm i if and only if firm i's product is within distance $x_B(p_i, p) = \frac{1}{2}x_R^{**} + \left(\frac{p - p_i}{h}\right)$ from her ideal product, where $x_R^{**} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8\,(c/h)}$ is the symmetric equilibrium referral rule.*

Note that unlike in Lemma 2, consumers purchase a firm's product in the equilibrium only if they are located within distance $x_B(p, p) = \frac{1}{2}x_R^{**}$ from the firm. Although this appears to be a minor modification, the demand curve can be kinked in this case depending on the assumption of . This is coming from the imperfect referral demand. On the one hand, if $p_i > p$, then some imperfect referral consumers may leave firm i to engage in further search.[16] On the other hand, if $p_i < p$, then all imperfect referral consumers clearly purchase the firm's product. However, since the measure of consumers arriving at a firm depends only on expected rather than actual price. That is, consumers with imperfect referral are simply pleasantly surprised, and the price cutting does not generate additional demand from any referred consumers because price cutting cannot increase the number of consumers who visit the firm by referrals. This asymmetry causes a kinked demand curve, which results in a continuum of symmetric equilibrium prices, as described in Proposition 6.

Proposition 4. *Suppose that consumers can misrepresent their locations. Consider the limit case of $n \to \infty$. When $c \leq \frac{h}{8}$, and either $h \leq \frac{4}{3}$ or $c \leq \frac{h}{8}\left(1 - \left(3 - \frac{4}{h}\right)^2\right)$, there exists a continuum of symmetric referral equilibria with prices $p_R^{**} \in [\frac{h}{4}, \frac{h}{2}]$, referral rule $x_R^{**} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8\,(c/h)}$, critical valuation for market participation $v_R^{**} = p_R^{**} + \frac{1}{2}hx_R^{**}$, and profits*

[16] Perfectly matched referred consumers purchase product anyway unless p_i is much higher than p, since the match is perfect.

$\pi_R^{**} = p_R^{**}(1 - v_R^{**})$. *Referral intensity is lower in the case of consumer misrepresentation, and it decreases with consumer search costs and increases with product heterogeneity.*

We could compare the referral equilibria without and with preference misrepresentation outlined in Propositions 2 and 6 respectively. The main differences are that under misrepresentation of consumer preferences the total referral activity is lower, some referrals are imperfect, and there is a range of prices supported in the referral equilibrium. Figure 7 illustrates how the region of parameter values for which the referral equilibrium exists would contract in the presence of manipulative consumers.

[**Figure 7 here**]

5 Comparison of the Random Search and Referral Equilibria

In this section we will compare the regions of their existence of the equilibria,and welfare properties of the random search and referral equilibria under preference misrepresention. Since there is a continuum of equilibria in the referral case, the comparison of equilibria is involved. In the following proposition and figure, for each possible referral equilibrium price $p \in [\frac{h}{4}, \frac{h}{2}]$, we identify the range of parameter values under which equilibrium exists.

Figure 7 illustrates Proposition 3 by showing the regions of parameter values of h and c for which the random search and referral equilibria exist. When $p_R^{**} = \frac{h}{2}$ (the highest equilibrium price), the range for the referral equilibrium is a subset of the one for random search equilibrium, but this is not the case for other equilibrium prices. The random search equilibrium exists for any level of product heterogeneity, provided search cost is sufficiently low. In contrast, a referral equilibrium may exist only for a sufficiently low heterogeneity parameter, $h \leq 2$. As product heterogeneity approaches the critical level of 2, the market collapses because the equilibrium price $p_R^{**} = \frac{h}{2}$ approaches unity and thus becomes prohibitively high for consumers, regardless of the magnitude of search cost.

Intuitively, when the product is highly heterogeneous, firms have greater market power to set high prices in the random search and referral equilibria, which discourages consumers' market participation. Even for a very high h, a random search market would not close down for a sufficiently low search cost, since the low search cost imposes competitive pressure on firms in the market. In the referral equilibrium, prices are not affected by search costs, and the search market with referrals does not open for $h > 2$.

We next compare the random search and referral equilibria in the region of parameter values for which both of them exist. Consumer surplus is directly related to the value of the marginal consumer – the consumer who is indifferent between entering and staying out of the market.[17] We use the critical values for market participation v^* and v_R^* to show that $v^* \geq v_R^*$, and therefore all consumer types are better off and market participation is larger in the referral equilibrium, as long as search cost is not very low relative to product heterogeneity. We further show that for relatively high search cost, the referral equilibrium *Pareto-dominates* the random search equilibrium. That is, the ex ante expected payoffs of each consumer and each firm in the referral equilibrium are higher for some agents and at least as high for other agents than in the random search equilibrium.

Proposition 5. *Denote the prices in the random search equilibrium (SE) and referral equilibrium (RE**) by p^* and p_R^{**}, respectively.*

*1. For any price $p_R^{**} \in [\frac{h}{4}, \frac{h}{2}]$, there exists a referral equilibrium if and only if (i) $c \leq \frac{h}{8}$, and (ii) either $h \leq 4(1 - p_R^{**})$ or $c \leq \frac{h}{8}\left(1 - (1 - (4/h)(1 - p_R^{**}))^2\right)$. As p_R^{**} decreases, the parameter range that assures the existence of RE** expands monotonically. The parameter range that assures the existence of SE includes the one that assures the existence of RE** with the highest price $p_R^{**} = \frac{h}{2}$, but this is not the case for a lower price.*

2. Consider the parameter range in which both random search and referral equilibria exist.

a) If $c \leq \frac{h}{16}$, then $p^* \leq \frac{h}{4}$ and for any $p_R^{**} \in [\frac{h}{4}, \frac{h}{2}]$, $p_R^{**} \geq p^*$. If $\frac{h}{16} < c \leq \frac{h}{8}$, then $p^* \in (\frac{h}{4}, \frac{h}{2})$ and the referral equilibrium prices can higher or lower than the random search equilibrium price.

*b) Consumers prefers RE** to SE if search cost c is high relative to product heterogeneity*

[17] To see this, suppose a critical valuation for market participation in a search market is $\widehat{v}$. The expected utility of a consumer who values the product at $v \geq \widehat{v}$ is $EU(v) = v - \widehat{v}$, and consumer welfare is $\int_{\widehat{v}}^{1} EU(v)dv = \frac{1}{2}(1 - \widehat{v})^2$. Therefore, $\widehat{v}$ fully determines the expected utility of any consumer and the total consumer welfare. Higher consumer market participation is equivalent to higher consumer benefits.

h. As the referral equilibrium price p_R^{**} goes up, the parameter range that consumers prefer RE^{**} to SE shrinks, and consumers always (weakly) prefer SE to RE^{**} with $p_R^{**} = \frac{h}{2}$.

c) Firms always prefer RE^{**} to SE for an optimally chosen price $p_R^{**} \in [\frac{h}{4}, \frac{h}{2}]$.

These results are not surprising. If search cost c is very low, consumers surely prefer the random search equilibrium, since the random search equilibrium price p^* is low while the referral equilibrium price p_R^* is insensitive to c. If search cost is not very low, the referral equilibrium is favored by consumers; and since the equilibrium price is higher in the referral equilibrium than in the random search equilibrium, firms favor this equilibrium as well. The referral equilibrium Pareto-dominates the random search equilibrium when the gain to consumers due to higher matching quality and lower search costs is higher than the loss due to increased price. From the welfare perspective, referrals handle the information problem of matching buyers and sellers but lead to higher prices in search markets. The reason the equilibrium prices are higher under referrals is that in the referral equilibrium, a fraction $1 - 2x_R = \sqrt{1 - 4(c/h)}$ of consumers are referred to their best-matching products. Not surprisingly, the firm faces a more inelastic demand.[18]

The fact that prices are higher in the referral equilibrium than in the random search equilibrium indicates that firms benefit more than consumers from the presence of referrals. That is, if consumer welfare is higher under referrals, then firms find them beneficial as well, but the reverse is not true. It could be the case that firms find the referral equilibrium more profitable, while the referrals make consumers worse off.[19] Since firms usually enjoy higher profits in the referral equilibrium, it should be selected over the random search equilibrium by firms and professional associations whenever both equilibria exist.

We next study how social welfare depends on the parameters of the search model. We

[18] Recall that firm i's demand is proportional to $\left(1 + 2\frac{p-p_i}{h}\right)$ and $\left(1 + \frac{p-p_i}{\sqrt{hc}}\right)$ in search markets with and without referrals, respectively. Hence, for $c \leq \frac{h}{4}$, firm i's demand is more inelastic in a search market with referrals.

[19] Although firms usually would benefit from establishing honest referrals, there are some parameter values for which profits are lower in the referral equilibrium than in the random search equilibrium. For these parameter values, it must be that consumer market participation is lower (and consumers are worse off) because the referral equilibrium price is higher. Therefore, the referral equilibrium can be Pareto dominated by the random search equilibrium.

define social welfare as a sum of consumer welfare and profits. According to footnote 10, consumer welfare is $\int_{\widehat{v}}^{1} EU(v)dv = \int_{\widehat{v}}^{1} (v - \widehat{v})\, dv = \frac{1}{2}(1-\widehat{v})^2$, which is only a function of the critical valuation for market participation $\widehat{v}$. The social welfare in the search model with and without referrals is then a function of price p and $\widehat{v}$:

$$W(p, \widehat{v}) = (1-\widehat{v})\, p + \frac{1}{2}(1-\widehat{v})^2 . \tag{8}$$

Proposition 5 summarizes the comparative statics results for social welfare in the random search and the referral equilibria.

Proposition 5. *Social welfare in the random search and referral equilibria is higher when search cost is lower. The effect of product heterogeneity on social welfare is negative in the random search equilibrium, and it is ambiguous in the referral equilibrium.*

The comparative statics results for the random search equilibrium indicate that social welfare is higher in markets with low search costs and low product heterogeneity. Similarly, for the referral equilibrium, social welfare is higher in markets with low search costs, but the effect of product heterogeneity on social welfare is ambiguous. The results are quite intuitive. An increase in search cost does not affect the equilibrium price, but it reduces referral intensity and market participation. Therefore, higher search costs are detrimental from the point of view of both the consumer and social welfare. In contrast, higher product heterogeneity stimulates referral activity and can improve social welfare despite a price increase.

6 Extensions

6.1 Price Discrimination by Firms

Next, we consider the possibility of price discrimination by firms. In the main analysis, we assumed that firms must charge the same (uniform) price to all consumers, even though firms can observe a consumer's preferred location and refer the consumer to the most suitable product. The reader may wonder whether firms would try to use their information about consumer location to price-discriminate between consumers based on the distance, x, between the consumer's ideal position and the firm's product. We will address this possibility by assuming that firms can engage in such price discrimination. That is, we let each firm i

choose a price function $p_i(x)$ instead of the uniform price p_i and then examine how our results are affected by this modification.

We find that as long as consumers can misrepresent their location, it does not matter whether or not firms are allowed to price-discriminate between consumers based on location, since consumers can always obtain the lowest price from firms even under price discrimination. However, the result is drastically different if consumers cannot misrepresent their location.

Proposition 7. *Suppose firms can price-discriminate on consumer location. Then (a) if consumers can misrepresent themselves, the equilibria in Proposition 6 remain to be the referral equilibria, and (b) if consumers cannot misrepresent their location, then there is no symmetric pure-strategy equilibrium with consumer participation in search markets with or without referrals.*

The second statement of Proposition 7 is a bit more involved although it is an elaborate version of the hold up problem pointed out by Stiglitz (1979). The argument is as follows. When a firm sees a consumer who has chosen to search, it knows that her willingness to pay is above some threshold. Consider the firm that proposes the weakly lowest full price in equilibrium as measured by the sum of price and transport cost. Then it is easy to show that a consumer with the threshold valuation must have zero surplus with that firm and at the corresponding distance. Indeed, if this consumer's surplus is strictly positive, then the firm increases its price slightly since it will not lose any consumer to the outside option. Furthermore, if the price increase is less than the search cost, then the firm will not lose the consumer to any other firm, since the consumer expects the full price to be at least as large anywhere else. Hence deviating to a strictly higher price improves profit. But then if a consumer with valuation expects to earn at most zero surplus with all firms, she will not be willing to search. This leads to market closure, since however high is a consumer's willingness-to-pay is, the firm has an incentive to raise its price (Diamond, 1971, and Stiglitz, 1979).

6.2 Referral Fees and Commissions

The activities of real estate agents can illustrate the role of referrals in search markets and motivate the introduction of commissions into the model. A home buyer contacts a real state agent in a realtor's office. The real state agent obtains a full commission if she can sell a property handled by the realtor's office as a seller broker (usually about 6% of the home sale price in the U.S. real estate industry). However, if the agent has no properties suitable for the buyer, the agent will refer the buyer to a property handled by an agent at another realtor's office to obtain a half-commission as a buyer broker. The agent obviously prefers to sell a property as a seller broker, but if she knows that a buyer will not purchase any of the properties she handles, she works hard to find a property suitable for her buyer-client for a half-commission. Without this system of "referrals," a buyer would need to visit many realtor's offices before finding a suitable property. Thus this referral practice would seem to reduce buyers' search costs substantially. However, it also affects consumer demand and market prices.

Suppose that, as in the real estate market, there is a common commission level. As in the benchmark model, we assume that firms cannot price-discriminate between consumers, and we look for a symmetric equilibrium. A firm's profit function in this case is a sum of profits from consumers buying at the firm on their first visit, on their visit by referral, and the payments by other firms for the referrals the firm makes. Since all firms offer the same commission, a referring firm cannot do better than to refer a consumer to the best-matching firm.

We can show that there exists a symmetric referral equilibrium when the commission is small enough to guarantee consumer participation. We obtain the same type of referral equilibrium with modified symmetric equilibrium price and consumer participation. The equilibrium price is higher under commissions. This is not surprising. Referral payments encourage a firm to raise its price because it can obtain those payments when it does not sell its product. Consumers are clearly worse off under referral payments because the optimal consumer search is not affected by their presence, while the equilibrium prices are higher. The equilibrium referral rule remains unchanged because a consumer's search strategy is based only on prices and distances between the consumer's ideal position and firms. Finally, commissions give firms strictly positive incentives to make referrals, and therefore they could help sustain the referral equilibrium. In fact, when it is costly for firms to make referrals, referral payments that compensate firms for the effort are needed to support the referral equilibrium. The same results are true for common referral fees. The results of the formal analysis are available from the authors upon request.

7 Conclusion

We consider an environment in which consumers need to conduct a costly search to gain information on products or services offered in a horizontally differentiated market. In the framework of the search market introduced by Wolinsky (1983), we study the practice of referring consumers to competing sellers. Businesses that are highly dependent on others for referral of their clients include many service providers and professionals, such as real estate agents, lawyers, health care providers, home inspectors, and construction workers, but the model could be applied to some retail markets as well.

In the basic model, there are no referral fees, and thus it seems unlikely that a firm could benefit from referring a consumer to its competitor. We find that the custom of giving honest referrals when a firm cannot sell its product to a consumer raises the equilibrium prices and thus can be favored by sellers and trade organizations. Although, given set prices, referrals benefit consumers because they improve the product-match quality and decrease their search costs, in the equilibrium such referrals can actually hurt consumers if search costs are very low relative to product heterogeneity. The reason is that in the search market without referrals, the equilibrium price becomes trivial for very low search costs, whereas in the referral equilibrium prices are insensitive to search costs. The possibility that referrals hurt consumers is somewhat paradoxical, and it is only realized for very low search costs. Referrals increase profits and improve social welfare even under broader conditions – as long as the market does not contract too much in response to higher prices.

Similar results obtain when we allow consumers to misrepresent their location. We show that in this case it does not matter whether or not firms are allowed to price-discriminate between consumers based on their location. However, if consumers cannot misrepresent their location while firms can price-discriminate, the referral equilibrium collapses for the same reason as in Diamond (1971) and Stiglitz (1979). Common referral fees and commissions give sellers a strong incentive to refer a consumer to other sellers when they cannot sell to the consumer. That is, introducing small referral fees may improve welfare in the Pareto sense if they help sustain the referral equilibrium. At the same time, referral payments increase the equilibrium price and do not affect the referral intensity. Therefore, consumers and consumer protection agencies would like to see low commissions and referral rates, provided entry into the market is not discouraged.

Our analysis should be viewed as the first step in modeling referrals between competing sellers. One interesting extension is to examine an endogenous formation of professional referral networks. Since it is usually costly to establish and maintain referral partnerships, sellers would be in contact with a few associates. Our model can be viewed as describing what happens when the costs of establishing links between members of a profession circle are negligible. In reality, although the costs may have decreased with the use of online referral clubs, it may still be the case that a seller forms strategic partnerships with a small number of sellers. Then, an "honest" referral would need to be reinterpreted as a referral to the partner that matches the client's preferences to the fullest extent possible.

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Appendix: Proofs

Proof of Proposition 1.

First assume that $c > \frac{h}{4}$ and suppose that, to the contrary, there exists an equilibrium with a symmetric price p such that a positive measure of consumers engage in sequential search. Given that consumers expect price p at every firm, a consumer visiting firm i located at a distance x would purchase from the firm as long as $x \leq x(p_i, p) = x^* + \frac{p-p_i}{h}$, where $x^* = \sqrt{\frac{c}{h}} > \frac{1}{2}$ (Lemma 1). All consumers who visit firm i continue to purchase from it as long as $x^* + \frac{p-p_i}{h} \geq \frac{1}{2}$. But then each firm has an incentive to raise its price slightly above the expected price p without losing any consumers because only consumers with values $v \geq p + c$ enter the market and a sufficiently small price increase would not encourage consumers to engage in additional search. This argument holds for any $p < 1$. Therefore, the symmetric equilibrium with consumer participation does not exist if $c > \frac{h}{4}$.

Next, assume that $c \leq \frac{h}{4}$. In the random search model, firm i maximizes its profit function $\pi_i(p_i, p) = (1 - v^*)\, p_i \left(1 + \frac{p-p_i}{\sqrt{hc}}\right)$ by choosing $p_i = \frac{p+\sqrt{ch}}{2}$. Thus, the symmetric equilibrium price is $p^* = \sqrt{hc}$.

For which parameter values does the random search equilibrium (with consumer participation) exist? From the proof of Lemma 1, the reservation utility (gross of price) is $w^*(v) = u(x^*, v) = v - \sqrt{hc}$. For some consumers to have a nonnegative equilibrium expected utility and engage in search, price p^* must be no higher than $w^*(v)$ for some $v \in [0, 1]$, namely $v = 1$. Thus, $p^* = \sqrt{hc} \leq 1 - \sqrt{hc}$, which can be written as $\sqrt{hc} \leq \frac{1}{2}$ or $c \leq \frac{1}{4h}$. For some consumers to search beyond the first firm in the equilibrium, we also need $x^* = \sqrt{\frac{c}{h}} \leq \frac{1}{2}$, or $c \leq \frac{h}{4}$. To summarize, condition $c \leq \min\{\frac{1}{4h}, \frac{h}{4}\}$ on search costs ensures that the random search equilibrium exists.

Consumers whose willingness-to-pay v is greater than or equal to a critical value $v^* = 2\sqrt{hc}$ engage in search. Indeed, if $v \geq v^*$ then $v - p^* - \sqrt{hc} \geq 0$, and such a consumer would follow the optimal stopping rule, stopping whenever the distance from the ideal position is no more than x^*. This implies that a fraction $1 - v^* = 1 - 2\sqrt{hc}$ of consumers engage in search, and each firm's profit can be written as $\pi^* = \pi_i(p^*, p^*) = (1 - v^*)\, p^* = (1 - 2\sqrt{hc})\sqrt{hc}$.

The comparative statics results are as follows: $\partial p^* / \partial c > 0$, $\partial p^* / \partial h > 0$, $\partial\left(1 - v^*\right) / \partial c <$

0, $\partial(1-v^*)/\partial h < 0$. For profits, $\partial\pi^*/\partial h = \frac{1}{2}\sqrt{c/h} - 2c = \frac{1}{2}\sqrt{c/h}\left(1-4\sqrt{hc}\right)$, and $\partial\pi^*/\partial c = \frac{1}{2}\sqrt{h/c}\left(1-4\sqrt{hc}\right)$, and therefore $\partial\pi^*/\partial h > 0$ and $\partial\pi^*/\partial c > 0$ if and only if $hc < \frac{1}{16}$.■

Proof of Lemma 2. Consider a symmetric equilibrium with price p and referral rule x_R^*. Let us start the analysis with a second-round search. Suppose that a consumer has randomly sampled firm i located at distance x from her ideal position. Her utility (gross of search cost) from purchasing the product there is $v-p-hx$, where p is the symmetric price charged by firms. In the symmetric referral equilibrium, a consumer can obtain a referral with a probability $1-2x_R^*$. If she obtains a referral and follows it, her utility will be $v-p$. A consumer who visited a firm located at a distance $x \leq x_R^*$ from her ideal position will not receive a referral. Then, her choice is one of the following three: (i) make no purchase, receiving $(-c)$; (ii) purchase the product, receiving $(v-p-hx-c)$; or (iii) engage in additional sequential search.

As we will show below, only consumers with willingness-to-pay v such that $v-p-hx_R^* \geq 0$ enter the market in equilibrium, and since $v-p-hx \geq 0$ for all $x \leq x_R^*$, it is better for the searcher to purchase rather than not. If she engages in further sequential search, she visits a random firm (firm $j \neq i$), which is located at distance $\widetilde{x}$ from the consumer and charges price p. At firm j, the consumer receives and follows a referral with probability $(1-2x_R^*)$, recalls firm i's offer, or buys firm j's product.

The expected payoff from engaging in one additional search is then

$$\begin{aligned}
&\Delta EU(x;x_R^*)\\
=\ &(1-2x_R^*)(v-p)+2(x_R^*-x)(v-p-hx)+2\int_0^x(v-p-h\widetilde{x})d\widetilde{x}-c-(v-p-hx)\\
=\ &(1-2x_R^*)hx+hx^2-c. \qquad\qquad \text{(A3)}
\end{aligned}$$

The terms in equation (A3) are explained as follows. When searching once again (by visiting firm j), the consumer will be referred by firm j to her ideal product with probability $1-2x_R^*$. She will find firm j's product to be a poorer match than firm i's product and will recall firm i's offer with probability $2(x_R^*-x)$, and otherwise, she will buy firm j's product. The subtracted terms in (A3) are an additional search cost and the utility level she obtains by purchasing firm i's product. It is easy to see that for any $x_R^* \in [0, \frac{1}{2}]$ and any $x \in [0, x_R^*]$, we have $\partial \Delta EU(x; x_R^*)/\partial x = h\,(1 - 2x_R^* + 2x) \geq 0$. This means that as long as $\Delta EU(x_R^*; x_R^*) \leq 0$, consumers who have $x \leq x_R^*$ would not engage in further search.

A consumer who visited a firm located at distance x_R^* from her ideal position is indifferent between searching and not searching (given that all other firms are using referral rule x_R^*): $\Delta EU(x_R^*; x_R^*) = 0$. It follows from

$$\begin{aligned} \Delta EU(x_R^*; x_R^*) &= (1 - 2x_R^*)hx_R^* + hx_R^{*2} - c \\ &= hx_R^* - hx_R^{*2} - c = 0 \end{aligned} \tag{A4}$$

that $x_R^* = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4(c/h)}$ for $c \leq \frac{h}{4}$. The value x_R^* describes the symmetric equilibrium referral rule. If a consumer receives a referral and $v \geq p$, then she follows the referral. If she does not receive a referral, it means that sequential search is not beneficial, and as a result, she either purchases (when $v \geq p + hx$) or goes home without purchase.

If $c > \frac{h}{4}$ holds, then no referrals will take place in a symmetric equilibrium, since $\Delta EU(x_R^*, x_R^*) < 0$ for all $x_R^* \in [0, \frac{1}{2}]$, and consumers stop searching regardless of the product found. Hence, if $c > \frac{h}{4}$ then there will be no referral equilibrium (with consumer participation) for the same reason that there is no symmetric random search equilibrium.

Next, we find the consumer's optimal stopping rule when she observes a price p_i at firm i that is different from p. Let x be the distance between a consumer's ideal point and the location of firm i. We look for the threshold distance for firm i, $x = x_R(p_i, p)$, such that a consumer at distance $x_R(p_i, p)$ from the ideal point is indifferent between purchasing i's product and searching further.

Suppose that an additional search after visiting firm i matches the consumer with a product at firm $j \neq i$ at distance $\tilde{x}$ from her ideal point. There are two cases: (i) if $\tilde{x} > x_R^*$ then she receives a referral, and (ii) if $\tilde{x} \leq x_R^*$ then she does not receive firm j's referral and does not search further. A consumer prefers to buy firm i's if and only if $v - p_i - hx \geq v - p - h\tilde{x}$ (or $\tilde{x} \geq x + \frac{p_i - p}{h}$). One additional round of search results in consumer utility no less than $v - p - hx_R^*$.

First, assume that the consumer utility from firm i's product is at least as high as that under the worst realization of an additional search: $v - p_i - hx \geq v - p - hx_R^*$ (or

$x+\frac{p_i-p}{h}\le x_R^*$, or $p_i\le p+hx_R^*-hx$). In this case, the consumer may recall firm i's product under some realizations of $\tilde{x}$. If $\tilde{x}>x_R^*$, she receives firm j's referral and obtains $(v-p)$. If $x+\frac{p_i-p}{h}\le\tilde{x}\le x_R^*$, she recalls firm i's product, and if $\tilde{x}<x+\frac{p_i-p}{h}$, she buys at firm j. Therefore, a customer's gain from engaging in an additional search when firm i charges p_i and other firms charge p is as follows:

$$\begin{aligned}&(1-2x_R^*)(v-p-hx_R^*)+2\left(x_R^*-x+\frac{p-p_i}{h}\right)(v-p_i-hx)+2\int_0^{x+\frac{p_i-p}{h}}(v-p-h\tilde{x})d\tilde{x}\\&-c-(v-p_i-hx) \qquad\qquad (\text{A5})\\=\;&h\left[x+\left(\frac{1}{2}\sqrt{1-4\,(c/h)}+\frac{p_i-p}{h}\right)\right]^2-\frac{h}{4}.\end{aligned}$$

By setting the above gain to zero, we obtain the critical referral distance $x_R(p_i,p)$ for firm i: $x_R(p_i,p)=x_R^*+\frac{p-p_i}{h}$, where $x_R^*=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\,(c/h)}$. Note that the condition $v-p_i-hx\ge v-p-hx_R^*$ we assumed is satisfied for $x\le x_R(p_i,p)$.

Second, suppose that $v-p_i-hx<v-p-hx_R^*$ (or $x>x_R(p_i,p)$). In this case, the consumer leaves firm i to search further.■

Proof of Proposition 2. Since we are analyzing a symmetric equilibrium, we assume that consumers expect that every firm charges price p and follows a common referral rule, which is, according to Lemma 2, described by $x_R^*=\frac{1}{2}-\frac{1}{2}\sqrt{1-4\,(c/h)}$. Consider a consumer whose willingness-to-pay v satisfies $v-hx_R^*-p\ge 0$. She has the following expected utility from the initial search (given the optimal stopping rule characterized by x_R^*):

$$\begin{aligned}EU_R(v,p)&=2\int_0^{x_R^*}(v-hx-p)dx+(1-2x_R^*)(v-p)-c \qquad (\text{A6})\\&=(v-p)-hx_R^{*2}-c\\&=v-hx_R^*-p,\end{aligned}$$

where $hx_R^*-hx_R^{*2}-c=0$ follows from the definition of x_R^* in Lemma 2. That is, a consumer whose willingness-to-pay v satisfies $v-hx_R^*-p\ge 0$ obtains a nonnegative expected utility, $EU_R(v,p)\ge 0$. On the other hand, if a consumer's willingness-to-pay v satisfies $v-hx_R^*-p<0$, it is easy to see that $EU_R(v,p)<0$ for any stopping rule. Thus, given that p is a prevailing symmetric price, a consumer engages in the initial search if and only if her willingness-to-pay v is not less than $v_R(p) = hx_R^* + p$.

We will next show that there is a symmetric equilibrium when $v_R^* \leq 1$. Since firm i's profit $\pi_{Ri}(p_i, p) = (1 - p - hx_R^*)p_i\,(1 + 2(p - p_i)/h)$ is concave in p_i, the first-order condition evaluated at $p_i = p$ characterizes the symmetric equilibrium. The first-order condition is $\partial\pi_{Ri}(p_i, p)/\partial p_i = (1 - p - hx_R^*)\,(1 - (4p_i - 2p)\,/h) = 0$. Thus, there is a unique symmetric equilibrium price $p_R^* = \frac{h}{2}$. The value of v_R^* can be found by substituting p_R^* into $v_R(p) = p + hx_R^*$. We obtain $v_R^* = h - \frac{h}{2}\sqrt{1 - 4\,(c/h)}$. Finally, $v_R^* \leq 1$ if and only if $h - 1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$. Hence, referral equilibrium with consumer participation exists whenever $c \leq \frac{h}{4}$ and $h - 1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$, which hold if and only if $c \leq \frac{h}{4}$ and either $h \leq 1$ or $c \leq 2 - 3h/4 - 1/h$.

The equilibrium price $p_R^* = \frac{h}{2}$ increases in h and does not depend on c. Referral intensity $1 - 2x_R = \sqrt{1 - 4\,(c/h)}$ decreases with consumer search costs and increases with product heterogeneity. Consumers' market participation, measured by $(1 - v_R^*)$, decreases in search cost, c, because $\partial v_R^*/\partial c = 1/\sqrt{1 - 4\,(c/h)} > 0$. For product heterogeneity, h, $\partial v_R^*/\partial h = \frac{1}{2}\left(2\sqrt{1 - 4\,(c/h)} - (1 - 2\,(c/h))\right)\frac{1}{\sqrt{1-4(c/h)}} > 0$, which is equivalent to $2\sqrt{1 - 4\,(c/h)} > (1 - 2\,(c/h))$. Since $1 - 2\,(c/h) > 0$, this is equivalent to $4\,(1 - 4\,(c/h)) > (1 - 2\,(c/h))^2$, or $4\,(c/h)^2 + 12\,(c/h) - 3 < 0$. The last inequality holds if and only if $c/h < \left(\sqrt{3} - \frac{3}{2}\right)$. Hence, $\partial v_R^*/\partial h > 0$ and consumers' market participation decreases in h for $c < \left(\sqrt{3} - \frac{3}{2}\right)h \approx 0.23h$. However, there is a region of parameter values for which the opposite is true. When $\left(\sqrt{3} - \frac{3}{2}\right)h \leq c \leq h/4$, then $\partial v_R^*/\partial h < 0$ and consumers' market participation increases in h in the equilibrium. Equilibrium profits $\pi_R^* = \frac{h}{2}(1 - h + \frac{h}{2}\sqrt{1 - 4\,(c/h)})$ decrease in search cost, $\partial\pi_R^*/\partial c < 0$. The effect of h on the profits is ambiguous. Profits increase in h if and only if $h - 3c + (1 - 2h)\sqrt{1 - 4\,(c/h)} > 0$. For example, it suffices to require $h \leq \frac{1}{2}$. In contrast, for relatively large h, profits decrease with an increase in product heterogeneity.■

Proof of Proposition 4. According to Proposition 6, conditions $c \leq \frac{1}{4h}$ and $c \leq \frac{h}{4}$ are necessary and sufficient for the existence of the random search equilibrium. From Proposition 2, the referral equilibrium exists if and only if $c \leq \frac{h}{4}$ and $h - 1 \leq \frac{h}{2}\sqrt{1 - 4\,(c/h)}$. The latter inequality holds for $h \leq 1$, and for $h > 1$ it is equivalent to $c \leq \frac{1}{4h}(2-h)(3h-2)$ or $c \leq 2 - \frac{3}{4}h - \frac{1}{h}$. When $h \leq 1$, both equilibria exist for $c \leq \frac{1}{4h}$. For $h > 1$, $c \leq \frac{1}{4h}$ always implies $c \leq \frac{h}{4}$, and $\frac{1}{4h} < 2 - \frac{3}{4}h - \frac{1}{h}$ holds if and only if $1 < h < \frac{5}{3}$.

Next, we compare the willingness-to-pay of consumers who are indifferent between participating or not under two equilibria. In the random search equilibrium, $v^* = 2\sqrt{hc}$ is a threshold value, while in the referral equilibrium, $v_R^* = h - \frac{h}{2}\sqrt{1-4(c/h)}$. Consumers are better off in the referral equilibrium if and only if $v_R^* \leq v^*$, or $\sqrt{h} - 2\sqrt{c} \leq \frac{1}{2}\sqrt{h-4c}$. This is equivalent to $\left(\sqrt{h}-2\sqrt{c}\right)^2 \leq \frac{1}{4}\left(\sqrt{h}-2\sqrt{c}\right)\left(\sqrt{h}+2\sqrt{c}\right)$, or $3\sqrt{h} - 10\sqrt{c} \leq 0$, or $c \geq 0.09h$. Hence, consumer welfare is higher in the referral equilibrium whenever $c \geq 0.09h$.■

Proof of Proposition 5. The social welfare in the random search equilibrium is $W^*(c,h) = \frac{1}{2} - \sqrt{hc} \geq 0$ for $c \leq \frac{1}{4h}$. Thus, it follows that $\partial W^*/\partial c < 0$ and $\partial W^*/\partial h < 0$.

The social welfare in the referral equilibrium can be written as

$$\begin{aligned} W_R^*(c,h) &= (1-v_R^*)\,p_R^* + \frac{1}{2}(1-v_R^*)^2 \\ &= \frac{1}{2}\left(1+\frac{h}{2}\sqrt{1-4(c/h)}-h\right)\left(1+\frac{h}{2}\sqrt{1-4(c/h)}\right). \end{aligned} \tag{A7}$$

It follows that $\partial W_R^*/\partial c < 0$. In contrast, the sign of $\partial W_R^*/\partial h$ is ambiguous. In the proof of Proposition 2, it is established that for sufficiently large c relative to h, $\left(\sqrt{3}-\frac{3}{2}\right) \leq \frac{c}{h} \leq \frac{1}{4}$, market participation increases in h in the equilibrium. Hence, for these parameter values, social welfare definitely increases with increasing product heterogeneity. On the other hand, for sufficiently small search costs, the effect is negative since W_R^* is continuously differentiable in c and h for $h \in (0, 4c)$, $\partial W_R^*/\partial h = -\frac{1}{4}\theta(h,c)/\sqrt{h(h-4c)}$, where $\theta(h,c) \equiv 2(c(2-3h)+h(h-1)) + \sqrt{h(h-4c)}\,(2-h+2c)$, and for $c \to 0$, $\partial W_R^*/\partial h \to -\frac{1}{4}h < 0$.■

Proof of Lemma 3. Consider a symmetric equilibrium with price p and referral rule x_R^{**} (i.e., a firm makes a referral if and only if a consumer's reported location is farther than x_R^{**}). The main difference from Lemma 2 is that consumers outside the a firm's referral region may not purchase the firm's product. Consumers located within the region $[-x_R^{**}, x_R^{**}]$ but closer to the region's border would rather report their location to be $x_R^{**} + \varepsilon$ to receive a referral to the firm located at $x_R^{**} + \varepsilon$, where ε is arbitrary small. To see this, note that if a consumer located at $x_R^{**}-\Delta$ purchases at the firm, her payoff is $v-h(x_R^{**}-\Delta)-p$, while if she reports $x_R^{**}+\varepsilon$ and receives a referral, her payoff is $v-h(\Delta+\varepsilon)-p$. A consumer who is indifferent between purchasing and receiving referral is located as $\frac{x_R^{**}}{2}$ (ignoring an arbitrary small ε).

Consider a consumer located at distance $x\in[0,\frac{x_R^{**}}{2}]$ from firm i. She can purchase firm i's product to receive utility $v-hx-p$ (gross of a sunk search cost c). Suppose that she conducts an additional random search and samples the next firm j located at distance $\tilde{x}$ from her, $\tilde{x}\in\left[0,\frac{1}{2}\right]$. There are four cases: (i) if $\tilde{x}\in[-x,x]$ then she purchases firm j's product, (ii) if $\tilde{x}\in(x,x_R^{**}-x]\cup[-(x_R^{**}-x),-x)$ then she purchases firm i's product, (iii) if $\tilde{x}\in(x_R^{**}-x,x_R^{**}]\cup[-x_R^{**},-(x_R^{**}-x))$ then she reports her location to be $x_R^{**}+\varepsilon$ (or $-x_R^{**}-\varepsilon$) to receive a referral to the firm at that location and purchase there, and (iv) if $\tilde{x}\in(x_R^{**},\frac{1}{2}]\cup[-\frac{1}{2},-x_R^{**})$ then she reports the true location to receive a referral to her best-matching firm and purchase there. The expected payoff from engaging in one additional search is then

$$\begin{aligned}\Delta EU(x;x_R^{**}) &= (1-2x_R^{**})(v-p)+4\left(\frac{x_R^{**}}{2}-x\right)(v-p-hx)+4\int_0^x(v-p-h\widetilde{x})d\widetilde{x}\\ &\quad -c-(v-p-hx)= \qquad\qquad (A8)\\ &= (1-2x_R^{**}+4x)hx-2hx^2-c\end{aligned}$$

It is easy to see that for any $x_R^{**}\in[0,\frac{1}{2}]$ and any $x\in[0,x_R^{**}]$, we have $\partial\Delta EU(x;x_R^{**})/\partial x = h\left(1-2x_R^{**}+4x\right)\geq 0$ as before. This means that as long as $\Delta EU(\frac{1}{2}x_R^{**};x_R^{**})\leq 0$, consumers who have $x\leq\frac{1}{2}x_R^{**}$ would not engage in further search but would instead purchase firm i's product.

A consumer who visited a firm located at distance $\frac{1}{2}x_R^{**}$ from her ideal position is indifferent between searching and not searching (given that all other firms are using referral rule x_R^{**}): $\Delta EU(\frac{1}{2}x_R^{**};x_R^{**})=0$. It follows from

$$\begin{aligned}\Delta EU(\frac{1}{2}x_R^{**};x_R^{**}) &= \frac{1}{2}(1-2x_R^{**}+2x_R^{**})hx_R^{**}-\frac{1}{2}hx_R^{**2}-c \qquad (A9)\\ &= \frac{1}{2}hx_R^{**}-\frac{1}{2}hx_R^{**2}-c=0\end{aligned}$$

that $x_R^{**}=\frac{1}{2}-\frac{1}{2}\sqrt{1-8(c/h)}$ describes the symmetric equilibrium referral rule for $c\leq\frac{h}{8}$. If a consumer receives a referral and $v\geq p$, then she follows the referral. If she does not receive a referral, it means that sequential search is not beneficial, and as a result, she either purchases or leaves the market without a purchase. If $c > \frac{h}{8}$ holds, then no referrals will take place in a symmetric equilibrium.

Next, we find the consumer's optimal stopping rule. Suppose firm i charges p_i while other firms charge p and follow the referral rule $x_R^{**} = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 8(c/h)}$. Let x be the distance between a consumer's ideal point and the location of firm i. We look for the threshold distance for firm i, $x = x_B(p_i, p)$, such that a consumer at distance $x_B(p_i, p)$ from the ideal point is indifferent between purchasing i's product and searching further. Note that while x_R^{**} describes firm's equilibrium referral decision, $x_B(p, p)$ describes a consumer's equilibrium purchase decision at the firm she visits. To find $x = x_B(p_i, p)$, we look at consumer who is indifferent between buying at firm i and conducting an additional search. We will show that $x_B(p, p) = \frac{1}{2}x_R^{**}$.

Suppose that an additional search after visiting firm i matches the consumer with a product at firm $j \neq i$ located at distance $\tilde{x}$ from her ideal point. There are three cases: (i) if $\tilde{x} > x_R^{**}$ then she receives a perfect referral, (ii) if $\tilde{x} \leq \frac{1}{2}x_R^{**}$ then she does not receive firm j's referral and does not search further, and (iii) if $\frac{1}{2}x_R^{**} < \tilde{x} \leq x_R^{**}$ then she pretends to be located at $x_R^{**} + \varepsilon$ to obtain an imperfect referral. A consumer does not buy firm i's product if $v - p_i - hx < v - p - h\tilde{x}$ or $v - p_i - hx < v - p - h(x_R^{**} - \tilde{x})$, that is, if $\tilde{x} < x + \frac{p_i - p}{h}$ or $\tilde{x} > x_R^{**} - x - \frac{p_i - p}{h}$.

One additional round of search results in consumer utility no less than $v - p - \frac{1}{2}hx_R^{**}$. First, assume that the consumer utility from firm i's product is at least as high as that under the worst realization of an additional search : $v - p_i - hx \geq v - p - \frac{1}{2}hx_R^{**}$ (or $x + \frac{p_i - p}{h} \leq \frac{1}{2}x_R^{**}$, or $p_i \leq p + \frac{1}{2}hx_R^{**} - hx$). In this case, the consumer may recall firm i's product under some realizations of $\tilde{x}$. For notational simplicity, we focus on the case of $\tilde{x} > 0$. If $\tilde{x} > x_R^{**}$, she receives firm j's referral and obtains utility $v - p$. If $x + \frac{p_i - p}{h} \leq \tilde{x} \leq x_R^{**} - x - \frac{p_i - p}{h}$, she recalls firm i's product. Finally, if $\tilde{x} < x + \frac{p_i - p}{h}$ or $x_R^{**} - x - \frac{p_i - p}{h} < \tilde{x} \leq x_R^{**}$ she buys at firm j or at the firm referred by j, which is located at $x_R^{**} + \varepsilon$. Therefore, a customer's gain from engaging in an additional search when firm i charges p_i and other firms charge p is as follows:

$$
\begin{aligned}
& (1-2x_R^{**})(v-p)+4\left(\frac{1}{2}x_R^{**}-x-\frac{p_i-p}{h}\right)(v-p_i-hx) \qquad \text{(A10)}\\
& +4\int_0^{x+\frac{p_i-p}{h}}(v-p-h\tilde{x})d\tilde{x}-c-(v-p_i-hx)\\
= \; & (1-2x_R^{**})(p_i-p+hx)+4\left(x+\frac{p_i-p}{h}\right)(p_i-p+hx)-2h\left(x+\frac{p_i-p}{h}\right)^2-c\\
= \; & h\sqrt{1-8(c/h)}\left(x+\frac{p_i-p}{h}\right)+2h\left(x+\frac{p_i-p}{h}\right)^2-h\times\frac{c}{h}\\
= \; & 2h\left(x+\frac{p_i-p}{h}-\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{8c}{h}}\right)\left(x+\frac{p_i-p}{h}+\frac{1}{4}+\frac{1}{4}\sqrt{1-\frac{8c}{h}}\right).
\end{aligned}
$$

By setting the above gain to zero, we obtain the critical distance $x_B(p_i,p)$ for firm i: $x_B(p_i,p)=\frac{1}{2}x_R^{**}+\frac{p-p_i}{h}$, where $x_R^{**}=\frac{1}{4}-\frac{1}{4}\sqrt{1-8\,(c/h)}$. Note that the condition we assumed $v-p_i-hx\geq v-p-\frac{1}{2}hx_R^{**}$ is satisfied for $x\leq x_B(p_i,p)$.

Second, suppose that $v-p_i-hx<v-p-\frac{1}{2}hx_R^{**}$ (or $x>x_B(p_i,p)$). In that case, the consumer leaves firm i to search further. Thus, we obtain the threshold distance $x_B(p_i,p)$ for a consumer's the optimal stopping rule (when visiting firm i).■

Proof of Proposition 6. Since we are analyzing a symmetric referral equilibrium, we assume that consumers expect that every firm charges price p, and that x_R^{**} describes the common equilibrium referral rule for $c\leq\frac{h}{8}$. Consider a consumer whose willingness-to-pay is v. She has the following expected utility from the initial search (given the optimal stopping rule characterized by x_R^{**}):

$$
\begin{aligned}
EU_R(v,p) &= 4\int_0^{\frac{x_R^{**}}{2}}(v-hx-p)dx+(1-2x_R^{**})(v-p)-c \qquad \text{(A11)}\\
&= (v-p)-\frac{1}{2}hx_R^{**2}-c\\
&= v-p-\frac{1}{2}hx_R^{**},
\end{aligned}
$$

since $c=\frac{1}{2}hx_R^{**}-\frac{1}{2}hx_R^{**2}$ from the definition of x_R^{**} in Lemma 3. That is, a consumer whose willingness-to-pay v satisfies $v-p-\frac{1}{2}hx_R^{**}\geq 0$ obtains a nonnegative expected utility, $EU_R(v,p)\geq 0$. On the other hand, if a consumer's willingness-to-pay v satisfies $v-p-\frac{1}{2}hx_R^{**}<0$, it is easy to see that $EU_R(v,p)<0$ for any stopping rule. Thus, given that p

is a prevailing symmetric price, a consumer engages in the initial search if and only if her willingness-to-pay v is not less than $v_R^{**}(p) = p + \frac{1}{2}hx_R^{**}$.

We will next show that there is a symmetric equilibrium when $v_R^{**} \leq 1$. Suppose that firm i's price is p_i and that other firms choose a symmetric price p and make referrals to customers located farther than x_R^{**}. We know from Lemma 3 that for $c \leq \frac{h}{8}$, $x_B(p_i, p) = \frac{1}{2}x_R^{**} + \frac{p-p_i}{h} = \frac{1}{4} - \frac{1}{4}\sqrt{1-8(c/h)} + \frac{p-p_i}{h}$ characterizes the optimal stopping rule for a consumer visiting firm i. We can use this rule to calculate the demand function of firm i. As before, we can restrict our attention to $p_i \in \left[p + hx_R^{**} - \frac{1}{2}h, p + hx_R^{**} + \frac{1}{2}hx_R^{**2}\right]$ because it is strictly dominated for firm i to price outside this interval.

For all $p_i \in \left[p + hx_R^{**} - \frac{1}{2}h, p + hx_R^{**} + \frac{1}{2}hx_R^{**2}\right]$, consumers located at $x \leq x_B(p_i, p) \in [0, \frac{1}{4}]$ buy from firm i, and imperfect-match consumers referred by other firms also buy from firm i for sure, since the price of firm i is cheaper than expected. Thus, if $p_i > p$, then some consumers who visit firm i following an imperfect referral by other firms may decide to conduct a random search instead of buying firm i's product at an unexpectedly high price $p_i > p$. These consumers do not go back to the firm that issued the imperfect referral since the fact that they received a referral implies that they would prefer a random search to purchasing at the firm. Perfect referral consumers still purchase the good since $p_i \in \left[p + hx_R^{**} - \frac{1}{2}h, p + hx_R^{**} + \frac{1}{2}hx_R^{**2}\right]$. Thus, firm i has random search demand $2\left(\frac{1}{n}\right)x_B(p_i, p)$, imperfect referral demand $2 \times \left(\frac{n-1}{n}\right) \times \frac{1}{n-1}x_B(p_i, p) = 2\left(\frac{1}{n}\right)x_B(p_i, p)$, and perfect referral demand $\left(\frac{n-1}{n}\right) \times \frac{1}{n-1}(1 - 2x_R^{**})$. Hence, we have

$$\begin{aligned}\widetilde{D}_{Ri}^{n+}(p_i, p) &= \underbrace{\left(\frac{1}{n}\right)2x_B(p_i, p)}_{\text{new consumers}} + \underbrace{\left(\frac{1}{n}\right)2x_B(p_i, p)}_{\text{imperfect referral}} + \underbrace{\left(\frac{n-1}{n}\right) \times \frac{1}{n-1}(1 - 2x_R^{**})}_{\text{perfect referral}} \quad \text{(A13)} \\ &= \frac{1}{n}\left(1 + \frac{4(p - p_i)}{h}\right),\end{aligned}$$

where $x_R^{**} = \frac{1}{2} - \frac{1}{2}\sqrt{1-8(c/h)}$ and n is very large. On the other hand, if $p_i \leq p$ then imperfect referral consumers are simply happy to purchase since the price os the referrerd firm (firm i) is unexpectedly cheap. Thus, firm i's demand per searcher is

$$\widetilde{D}^{n-}_{Ri}(p_i,p) = \underbrace{\left(\frac{1}{n}\right) 2x_B(p_i,p)}_{\text{new consumers}} + \underbrace{\left(\frac{n-1}{n}\right) \times \frac{1}{n-1} 2x^{**}_R}_{\text{imperfect referral}} + \underbrace{\left(\frac{n-1}{n}\right) \times \frac{1}{n-1}(1-2x^{**}_R)}_{\text{perfect referral}} \tag{A12}$$
$$= \frac{1}{n}\left(1+\frac{2(p-p_i)}{h}\right).$$

Let v^{**}_R denote the critical valuation for market participation, which is the valuation of the consumer who is indifferent between initiating sequential search and staying outside the market in the referral equilibrium. The total measure of participating consumers is then $n(1-v^{**}_R)$, and firm i's demand function is

$$D^n_{Ri}(p_i,p) = \begin{cases} (1-v^{**}_R)\left(1+\frac{4(p-p_i)}{h}\right) & \text{if } p_i \le p \\ (1-v^{**}_R)\left(1+\frac{2(p-p_i)}{h}\right) & \text{if } p_i > p \end{cases} \tag{A14}$$

Hence, when firm i sets a price p_i and all firms are expected to charge p and use referral rule x^{**}_R, firm i's profit function in the limit as $n \to \infty$ is

$$\pi_{Ri}(p_i,p) = \begin{cases} (1-v^{**}_R)p_i\left(1+\frac{2(p-p_i)}{h}\right) & \text{if } p_i \le p \\ (1-v^{**}_R)p_i\left(1+\frac{4(p-p_i)}{h}\right) & \text{if } p_i > p \end{cases}. \tag{A15}$$

The first-order conditions are then

$$\frac{\partial \pi_{Ri}(p_i,p)}{\partial p_i} = 0 = \begin{cases} (1-v^{**}_R)\left(1+\frac{2p-4p_i}{h}\right) & \text{if } p_i \le p \\ (1-v^{**}_R)\left(1+\frac{4p-8p_i}{h}\right) & \text{if } p_i \ge p \end{cases} \tag{A16}$$

This implies that at $p_i = p$, we have $\left.\frac{\partial \pi_{Ri}(p_i,p)}{\partial p_i}\right|_{p_i \uparrow p} = (1-v^{**}_R)\left(\frac{h-2p}{h}\right)$ and $\left.\frac{\partial \pi_{Ri}(p_i,p)}{\partial p_i}\right|_{p_i \downarrow p} = (1-v^{**}_R)\left(\frac{h-4p}{h}\right)$, where $\left.\frac{\partial \pi_{Ri}(p_i,p)}{\partial p_i}\right|_{p_i \uparrow p}$ and $\left.\frac{\partial \pi_{Ri}(p_i,p)}{\partial p_i}\right|_{p_i \downarrow p}$ denote the left- and right-derivative at p. Thus, if $p \le \frac{h}{2}$ then firms have no price-cutting incentive at the common price p, while if $p \ge \frac{h}{4}$ then there is no price-raising incentive at the common price p. This implies that there is a continuum of symmetric equilibria for all prices $p^{**}_R \in [\frac{h}{4}, \frac{h}{2}]$.

Since $v^{**}_R(p) = p + \frac{1}{2}hx^{**}_R$ is an increasing function in p, when equilibrium price is $p^{**}_R = \frac{h}{2}$, v^{**}_R is the greatest and the least market participation occurs. Substituting $p^{**}_R = \frac{h}{2}$ and $x^{**}_R = \frac{1}{2} - \frac{1}{2}\sqrt{1-8\,(c/h)}$ into $v^{**}_R(p) = p + \frac{1}{2}hx^{**}_R$, we obtain $v^{**}_R(\frac{h}{2}) = \frac{1}{4}h\left(3-\sqrt{1-8\,(c/h)}\right)$. Since $v^{**}_R(\frac{h}{2}) \le 1$ implies $v^{**}_R(p^{**}_R) \le 1$ for all $p^{**}_R \in [\frac{h}{4}, \frac{h}{2}]$, we can say that if $v^{**}_R(\frac{h}{2}) = \frac{1}{4}h\left(3-\sqrt{1-8\,(c/h)}\right) \le 1$, there is a symmetric equilibrium for all $p^{**}_R \in [\frac{h}{4}, \frac{h}{2}]$. This condition is equivalent to $h \le \frac{4}{3}$ or $c \le \frac{h}{8}\left(1-\left(3-\frac{4}{h}\right)^2\right)$.

The total referral intensity that includes both perfect and imperfect referrals is $(1 - x_R^{**}) = \frac{1}{2}\sqrt{1 - 8\,(c/h)} + \frac{1}{2}$. The referral intensity is reduced due to consumer misrepresentation because $1 - x_R^{**} < 1 - 2x_R^*$. To see this, note that the inequality is equivalent to $\frac{1}{2}\sqrt{1 - 8\,(c/h)} + \frac{1}{2} < \sqrt{1 - 4\,(c/h)}$, which holds for any $c \le \frac{h}{8}$.■

Proof of Proposition 7. Under price discrimination, consumers can misrepresent their position in two ways: to obtain a referral and to obtain the lowest price from the referred firm. Let p_R^{**} be the lowest price a firm charges any consumer reporting to be located in region $-x_R^* \le x \le x_R^*$. By misrepresenting her location, a consumer is able to always purchase at the lowest price p_R^{**}. Thus, the situation is exactly the same as in the uniform pricing case, and the referral equilibria (p_R^{**}, x_R^{**}) are as in Proposition 6.

When consumers cannot misrepresent their position, the situation is very different. We will consider a random search model. Note that if there is no random search equilibrium, then there is no referral equilibrium in this case as well. This is because in a referral equilibrium, referrals are given only to consumers who are willing to conduct additional random search. The proof of nonexistence of random search equilibrium is by contradiction. Suppose there were a symmetric pure-strategy equilibrium to the random search model in which each firm sets a common price function $p(x)$.

To calculate the equilibrium price function, assume that all firms except firm i choose price function $p(x)$, while firm i sets an arbitrary price function $p_i(x)$. Assuming that consumers continue to hold equilibrium beliefs about price functions, even after observing $p_i(x) \neq p(x)$, a consumer will optimally stop searching at firm i located at distance x if firm i's product yields a net utility of at least $u^*(v)$: $v - hx - p_i(x) \ge u^*(v)$. Then, to keep the consumer from leaving, firm i should charge $p_i(x) = p - hx$, where $p = v - u^*(v)$.[20]

This implies that the price function $p_i(x)$ of any firm i provides a full compensation for the disutility of product mismatch, hx, and in the symmetric equilibrium, the symmetric

[20] We need to show that $v - u^*(v)$ does not depend on v. A consumer's utility (net of price) from each random draw is an i.i.d. random variable. Suppose that a consumer with valuation $v = 1$ has a random utility $\mathbf{u}(1) = \mathbf{u}$ where $\mathbf{u}$ follows a density function $f(u)$. Then, a consumer whose valuation is v has a random utility $\mathbf{u}(v) = \mathbf{u} - (1 - v)$. Following Wolinsky (1986), we know that the optimal stopping rule for the random search problem is described by the cut-off utility w^* (defined implicitly by $\int_{w^*}^{1} (v - w^*) f(v) dv = c$) such that (i) if $u \ge w^*$ then stop searching, and (ii) if $u < w^*$ then continue to search. Thus, a consumer with valuation v has a reservation utility $u^*(v) = w^* - (1 - v)$. It follows that $(v - u^*(v)) = w^* - 1$ does not depend on v.

price function is $p(x) = p - hx$, where $p = v - u^*(v)$. When all firms set the same price function $p(x) = p - hx$, consumers stop searching immediately because they derive the net utility of $v - hx - p(x) = v - p$ from any product. However, if this is the case, firm i would rather deviate by charging a higher price, e.g. $p_i(x) = p(x) + c$, because all consumers would continue buying at firm i even at the higher price. This proves that there is no symmetric pure-strategy equilibrium with consumer participation.■