Objective Subjective Probabilities

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Abstract

This note shows that if the space of events is sufficiently rich and the subjective probability function of each individual is non-atomic, then there is a σ-algebra of events over which everyone will have the same probability function, and moreover, the range of these probabilities is the whole $[0, 1]$ segment.

1 Introduction

An important assumption in the formal analysis of social choice theory is the existence of social lotteries, that is, lotteries whose outcomes are social policies.[1] Such lotteries can increase the fairness of the social allocation mechanism or solve disputes in a cheap, efficient manner. For a social lottery to work, it must be considered fair by all individuals in society. In particular, if society finds it optimal to randomize over the k pure social policies $s_1, \ldots, s_k$ by using the probability vector $p = (p_1, \ldots, p_k)$, then everyone in society must agree that the mechanism is indeed using these probabilities.[2]

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[1]See Harsanyi [9], and more recently, Epstein and Segal [6], Broome [3], Kamm [11], or Karni and Safra [10].

[2]Unlike the cake division problem, where participants want to ensure receiving *at least* their fair share, in social choice individuals often wish also not to receive *more* than their fair share.

But do such mechanisms exist? Diamond [4] thought that when probabilities are subjective, the answer is no. Even in the model of Anscombe and Aumann [2], where each decision maker is assumed to face subjective horse lotteries and objective roulettes, it does not follow that all decision makers agree on what is objective. An Italian-speaking person, facing a die whose sides are marked *Uno, Tre, Cinque, Sette, Otto, Dieci* will consider the event "the die will show an odd number" to be objective, while a non Italian-speaking person will consider it subjective (or even ambiguous). Nothing in the assumptions and structure of the Anscombe–Aumann model implies agreement on what is a roulette lottery. The issue is even more critical in Savage's [13] framework, where all events are assumed to be subjective.

Recently, Ghirardato et. al. [7] showed that even if probabilities don't exist (that is, beliefs are ambiguous), it is still possible, under some assumptions, to obtain mixture-like operators over random variables. But these procedures are subjective, and cannot be jointly used. Machina [12], on the other hand, assumes that preferences are smooth and proves that for each $r \in [0,1]$ there is a sequence of events E_n such that for each i, $\mu_i(E_n) \to r$. Unfortunately, as noted by Machina, the limits of these sequences of events don't necessarily exist.[3] Moreover, from the social point of view it may be important for everyone to agree that an event has probability exactly $\frac{1}{n}$, not approximately $\frac{1}{n}$.

In this note we show that if the space of events is sufficiently rich and the subjective probability function of each individual is non-atomic, then there is a σ-algebra of events over which everyone will have the same probability function, and moreover, the range of these probabilities is the whole $[0,1]$ segment. In other words, even in a fully subjective world (for example, Savage's), there is a rich set of events that can be used for joint randomizations. We prove existence, but we don't know how to construct specific such σ-algebras. This does not annul the contribution of this note. Randomization in social choice theory plays an important theoretical role, but it doesn't necessarily follow that policy makers do actually randomize. Our aim is to close a theoretical gap that exists in the literature—if commonly accepted devices do not exist, then models using randomization to enhance fairness

[3]For example, let the event A_n be "the n-th digit to the right of the decimal point of the temperature tomorrow will be odd." Then as $n \to \infty$, all individual beliefs regarding these events will converge to $\frac{1}{2}$. But there is no sense of limit for which $\lim A_n$ exists as an *event* of probability $\frac{1}{2}$.

would become void. Theorem 1 shows that there are enough events over which decision makers agree.

2 A Theorem

Theorem 1 *Let $\mu_1, \ldots, \mu_n$ be nonatomic, countably additive probability measures on a measurable space (S, Σ). Then there is a sub-σ-algebra $\hat{\Sigma}$ of Σ on which all the measures agree, which is rich in the sense that for every real number $r \in [0, 1]$, it contains a set of (unanimous) measure r.*

Proof: We start by using a well known result of Dubins and Spanier, which is restated in the Appendix. According to their theorem, it is possible to partition S into two sets E_0 and $E_1 = E_0^c$ belonging to Σ such that $\mu_i(E_0) = \mu_i(E_1) = 1/2$, for all $i = 1, \ldots, n$. Let A_1 denote the σ-algebra generated by this partition, namely $A_1 = \{\varnothing, E_0, E_1, S\} \subset \Sigma$.

Repeating this operation, we can partition E_0 into two disjoint sets E_{00} and E_{01}, and also partition E_1 into two disjoint sets E_{10} and E_{11}, so that $\mu_i(E_{b_1 b_2}) = 1/4$ for all $i = 1, \ldots, n$ and $b_1 = 0, 1$ and $b_2 = 0, 1$. Let A_2 denote the σ-algebra generated by $\{E_{00}, E_{01}, E_{10}, E_{11}\}$. Note that $A_1 \subset A_2 \subset \Sigma$.

Proceeding in this fashion, for each m, partition S into 2^m pairwise disjoint sets $E_{b_1 \cdots b_m}$, where each $b_j \in \{0, 1\}$, satisfying

$$E_{b_1 \cdots b_{m-1} b_m} \subset E_{b_1 \cdots b_{m-1}},$$

and

$$\mu_i(E_{b_1 \cdots b_m}) = 1/2^m$$

for all $i = 1, \ldots, n$. Letting A_m denote the σ-algebra generated by this partition, we have $A_{m-1} \subset A_m \subset \Sigma$.

Set $A = \bigcup_{m=1}^{\infty} A_m \subset \Sigma$. Then it is easy to verify that A is an algebra, but not a σ-algebra, and that all the measures $\mu_1, \ldots, \mu_n$ agree on A. Let μ denote the common restriction of each μ_i to A. Then, for any dyadic rational $q = k/2^m$ in the unit interval there is a set E in $A_m \subset A$ with $\mu(E) = q$.

Let $\hat{\Sigma} = \sigma(A) \subset \Sigma$, the σ-algebra generated by A. By the Carathéodory Extension Theorem (see Appendix), μ has a unique extension to $\hat{\Sigma}$, which we again denote by μ. Since this extension is unique, each μ_i agrees with μ on $\hat{\Sigma}$.

Moreover the range of μ is all of $[0,1]$. To see this, let r belong to the unit interval. Then r has a binary expansion $r = \sum_{m=1}^{\infty} b_m/2^m$, where each b_m is a binary digit (bit), 0 or 1. For each m, choose the set $F_m \in A_m$ by

$$F_m = \begin{cases} \varnothing & \text{if } b_m = 0 \\ E_{\underbrace{0\cdots0}_{m-1}1} & \text{if } b_m = 1. \end{cases}$$

Note that $\mu(F_m) = b_m/2^m$. By construction, the sets F_m are pairwise disjoint. (To see this suppose F_k and F_m are nonempty with $k < m$. Then $F_k = E_{\underbrace{0\cdots0}_{k-1}1}$ and F_m is a subset of $E_{\underbrace{0\cdots0}_{k-1}0}$, which is disjoint from F_k.) Thus the set $F = \bigcup_{m=1}^{\infty} F_m$ belongs to $\hat{\Sigma}$ and satisfies $\mu(F) = r$. ■

3 An Example

The following example shows that we cannot extend our result to more than a finite number of individuals. For consider the countably infinite set $\{0,1,2,\ldots\}$. Let $S = [0,1]$ equipped with its Borel σ-algebra, and let $\{q_i : i = 0,1,2,\ldots\}$ be an enumeration of the rationals in $[0,1)$ where $q_0 = 0$. Person i's subjective probability P_i has the density f_i given by

$$f_i(t) = \begin{cases} \frac{1}{2} & t < q_i \\ \frac{1-\frac{q_i}{2}}{1-q_i} & t \geqslant q_i \end{cases}$$

Note that P_0 is just Lebesgue measure.

We now want to show that there is no event A such that $P_i(A) = \frac{1}{2}$ for all $i = 0,1,\ldots$. Suppose A is such an event. Computing the probabilities according to person 0 and person $i > 0$, we obtain

1. $\lambda(A \cap [0,q_i)) + \lambda(A \cap [q_i,1]) = \frac{1}{2}$.
2. $\frac{1}{2}\lambda(A \cap [0,q_i)) + \frac{2-q_i}{2-2q_i}\lambda(A \cap [q_i,1]) = \frac{1}{2}$,

where λ is ordinary Lebesgue measure. The solution of this system is $\lambda(A \cap [0,q_i)) = q_i/2$ and $\lambda(A \cap [q_i,1]) = (1-q_i)/2$. For all rationals a and b it now follows, by taking $q_i > b$, that $\lambda(A \cap [a,b]) = (b-a)/2$.

It is well known (see for instance, Halmos [8, Theorem A in §16, p. 68]) that there is no Lebesgue measurable set (and so no Borel set) satisfying this property.

Appendix

The following well known result is due to Dubins and Spanier [5]. It may also be found in Aliprantis and Border [1, Theorem 12.34, p. 445].

Dubins–Spanier Theorem Let $\mu_1, \ldots, \mu_n$ be nonatomic probability measures on a measurable space (S, Σ). Given $\alpha_1, \ldots, \alpha_m \geq 0$ with $\sum_{j=1}^m \alpha_j = 1$, there is a partition $\{E_1, \ldots, E_m\}$ of S satisfying $\mu_i(E_j) = \alpha_j$ for all $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

Dubins and Spanier also include a lesser known result, which is a slight sharpening of part of the Lyapunov Convexity Theorem. It shows that the family of events on which the measures agree is rich, but it does not show that it includes a rich σ-algebra.

Theorem 2 (Dubins–Spanier [5, Lemma 5.3]) *Let $\mu_1, \ldots, \mu_n$ be nonatomic (countably additive) probability measures on a measurable space (S, Σ). Then there is a subfamily $\{E_\alpha : \alpha \in [0,1]\}$ of Σ satisfying*

$$\mu_i(E_\alpha) = \alpha \quad \text{for all } i = 1, \ldots, n,$$

and

$$\alpha < \beta \Longrightarrow E_\alpha \subset E_\beta.$$

A complete statement of the Carathéodory Extension Theorem may be found in Aliprantis and Border [1, Theorem 9.22, p. 343]. For our purposes, we need only the following special case.

Carathéodory Extension Theorem Let $\mathcal{A}$ be an algebra of subsets of X and let μ be a probability measure on $\mathcal{A}$. Then μ has a unique extension to $\sigma(\mathcal{A})$, the σ-algebra generated by $\mathcal{A}$.

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