Are Universal Preferences Possible? Calibration Results for Non-Expected Utility Theories*

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December 5, 2005

Abstract

Rabin [37] proved that a low level of risk aversion with respect to small gambles leads to a high, and absurd, level of risk aversion with respect to large gambles. Rabin's arguments strongly depend on expected utility theory, but we show that similar arguments apply to almost all non-expected utility theories and even to theories dealing with uncertainty. The set of restrictions needed in order to avoid such absurd behavior may suggest that the assumption of universality of preferences over final wealth is too strong.

1 Introduction

One of the fundamental hypotheses about decision makers' behavior in risky environments is that they evaluate actions by considering possible *final* wealth levels. Throughout the last fifty years the final-wealth hypothesis has been widely used in the classical theory of expected utility as well as in its

*We thank Eddie Dekel, Larry Epstein, Simon Grant, Eran Hanany, Edi Karni, Mark Machina, Matthew Rabin, Ariel Rubinstein, and Shunming Zhang for their suggestions. Zvi Safra thanks the Israel Institute of Business research for its financial support.

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applications. Moreover, many of the new alternatives to expected utility, alternatives that were developed during the last twenty five years in order to overcome the limited descriptive power of expected utility, are also based on the hypothesis that only final wealth levels matter.

The final-wealth hypothesis is analytically tractable as it requires decision makers to behave according to a unique, universal preference relation over final-wealth distributions. Suggested deviations from this hypothesis require much more elaborate and complex analysis. For example, postulating that decision makers ignore final wealth levels and, instead, care about possible gains and losses, may require using many preference relations and necessitates the need for a mechanism that defines the appropriate reference points. However, the lack of descriptiveness of some of the final-wealth models and, in particular, of the final-wealth expected utility models, have increased the popularity of gain-losses model (such as prospect theory and its offsprings).

Recently, Rabin [37] offered a strong theoretical argument against final-wealth expected utility theory: Seemingly innocuous levels of risk aversion with respect to small gambles lead to enormous levels of risk aversion with respect to large gambles. For example, decision makers who at all wealth levels reject an even chance of winning \$110 or losing \$100 will also reject an even chance of losing \$1000, *regardless of the potential gain*. This latter rejection is obviously absurd. Rabin offered even more stunning versions of this result: Even if the decision maker only rejects the small gamble at all wealth levels in a bounded range between 0 and 300,000, then at a wealth level of \$290,000 he rejects an even chance of loosing \$2000 and gaining \$12,000,000. [1]

A possible response to Rabin's argument is that its basic assumption is actually wrong: No reasonable expected utility decision maker will reject the small gamble at *all* wealth levels (see Palacios-Huerta and Serrano [35] and LeRoy [26]). Palacios-Huerta and Serrano claim that even if expected utility decision makers are risk averse, moderate observed levels of relative risk aversion imply that their levels of absolute risk aversion go down to zero as wealth increases. Hence, at sufficiently large wealth levels, decision makers will accept the small gamble. This claim also applies to the bounded version of Rabin's result, as the size of the interval [0, 300,000] is quite large.

[1] For an earlier claim that a low level of risk aversion in the small implies huge risk aversion at the large, although without detailed numerical estimates, see Hansson [21] and Epstein [13].

Our analysis rejects this defence of expected utility theory. As we show in section 2, Rabin's calibration results can be strengthened by restricting the length of the intervals to less than forty thousands. Over such intervals, the claim of Palacios-Huerta and Serrano is less compelling. [2]

A natural conclusion from Rabin's argument and from the results of section 2 is that final-wealth expected utility should be replaced with more general final-wealth theories. [3] Several alternatives to expected utility theory were introduced into the literature in the last twenty five years. These include rank-dependent utility (Weymark [48] and Quiggin [36]), betweenness (Chew [4, 5], Fishburn [16], and Dekel [11]), quadratic utility (Machina [27] and Chew, Epstein, and Segal [7]), disappointment aversion (Gul [19]), the more general differentiable non-expected utility model (Machina [27]), and the general class of theories satisfying first order risk aversion (Segal and Spivak [45] and Epstein and Zin [15]). These theories seem to be immune to Rabin's argument since their local behavior (evaluating small gambles) is usually separable from their global behavior (evaluating large gambles). Indeed, it can easily be seen that rank-dependent with linear utility (Yaari [50]) is capable of exhibiting both a relatively strong aversion to small gambles and a sensible degree of risk aversion with respect to large gambles.

Nevertheless, if we extend the variety of small lotteries rejected by the decision maker, for example, by assuming that they are rejected even when added to an existing risk, then Rabin's fatal conclusions are not limited to final-wealth expected utility theory only. Our main result is that simple extensions of Rabin's argument apply to many known models of final-wealth non-expected utility theories (sections 4 and 5. All proofs appear in the appendix). Moreover, the analysis of the case of objective probabilities (i.e., under risk) is carried over to the case of subjective probabilities and uncertainty (section 6) and to some theories that are based on gain and losses,

[2] Palacios-Huerta and Serrano [35] show that assuming constant relative risk aversion, that is, vNM utilities of the form $u(x) = \frac{x^{1-\gamma}}{1-\gamma}$, rejections of $(-100, \frac{1}{2}; 110, \frac{1}{2})$ over a range of 300,000 imply extremely high values of γ, the measure of relative risk aversion (see Table II in [35]). It turns out, however, that when the range of the wealth levels goes down to 30,000, the values of γ, although still high, become a lot more reasonable. In other words, one cannot reject Rabin's arguments by claiming that the "if" part of his analysis (the rejection of small lotteries) is empirically invalid.

[3] Rabin [37, p. 1288] speculates that a possible solution may be to use theories of first order risk aversion [45], where for small risks the risk premium is of the same order of magnitude as the risk itself. Sections 4.3.2 and 5.3.1 extend (variants of) Rabin's results to functionals satisfying first order risk aversion.

rather than final wealth (section 7.1).

To sum, during the last twenty five years, and especially since the works of Machina [27] and Schmeidler [42], a large part of modern decision theory has been based on the assumption that expected utility theory can be replaced by theories of non-linear preferences that are still complete, transitive, and universal. Our results cast doubt on the possibility of this approach. We do not prove its impossibility and we cannot offer one theorem that covers all non-expected utility preferences. However, we show that almost all known models are vulnerable to arguments very similar to the stronger calibration results of section 2. The set of restrictions our paper imposes on such preferences may suggest that either transitivity, or more likely, universality, of preferences should be dropped.

2 New Calibration Results for Expected Utility Theory

Rabin's results were directed at showing that rejections of small, favorable, even bets must lead to the rejection of enormously favorable even bets. In this section we first show that Rabin's results hold even if the range at which the small lotteries are rejected is significantly smaller than the one suggested by Rabin. We then show that rejections of small even bets must also lead to the rejection of extremely profitable low risk investment opportunities.

Following Rabin [37], consider a risk averse expected utility maximizer with a concave vNM utility function u, who, for $\ell < g$, is rejecting the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels x in a given interval $[a, b]$ (a and b can take infinite values; when they are finite we assume, for simplicity, that $b - a = k(\ell + g)$ for some integer k).

Rejecting the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at $a + \ell$ implies

$$u(a + \ell) > \tfrac{1}{2}u(a) + \tfrac{1}{2}u(a + \ell + g)$$

Assume, without loss of generality, $u(a) = 0$ and $u(a + \ell) = \ell$, and obtain

$$u(a + \ell + g) < 2\ell$$

By concavity, $u'(a) \geqslant 1$ and

$$u'(a + \ell + g) < \frac{\ell}{g} \leqslant \frac{\ell}{g}u'(a) \tag{1}$$

Similarly, we get

$$u'(b) < u'(a)\left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \tag{2}$$

Concavity implies that for every c, $u(c+\ell+g) \leqslant u(c) + (\ell+g)u'(c)$, hence

$$u(b) \leqslant u(a) + (\ell+g)u'(a)\sum_{i=1}^{\frac{b-a}{\ell+g}}\left(\frac{\ell}{g}\right)^{i-1} \tag{3}$$

Likewise, for every c, $u(c-\ell-g) \leqslant u(c) - (\ell+g)u'(c)$, hence

$$u(a) \leqslant u(b) - (\ell+g)u'(b)\sum_{i=1}^{\frac{b-a}{\ell+g}}\left(\frac{g}{\ell}\right)^{i-1} \tag{4}$$

Normalizing $u(a) = 0$ and $u'(a) = 1$ we obtain from eqs. (2) and (3)

$$u'(b) \leqslant \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \quad \text{and} \quad u(b) \leqslant (\ell+g)\frac{1-\left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}}}{1-\frac{\ell}{g}} \tag{5}$$

For concave u we now obtain that for every $x \notin [a,b]$

$$u(x) \leqslant \begin{cases} -(a-x) & x < a \\ u(b) + (x-b)\left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} & x > b \end{cases} \tag{6}$$

Alternatively, a normalization with $u(b) = 0$ and $u'(b) = 1$ gives (by eqs. (2) and (4))

$$u'(a) \geqslant \left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} \quad \text{and} \quad u(a) \leqslant -(\ell+g)\frac{1-\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}}}{1-\frac{g}{\ell}} \tag{7}$$

and hence, for every $x \notin [a,b]$

$$u(x) \leqslant \begin{cases} u(a) - (a-x)\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} & x < a \\ x - b & x > b \end{cases} \tag{8}$$

Inequalities (5) and (6) imply that if for all wealth levels x between a and b the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$, then when his wealth level is a, he will also reject any lottery of the form $(-L, p; G, 1-p)$, $G > b-a$, provided that

$$L > \left[(\ell + g) \frac{1 - \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}}}{1 - \frac{\ell}{g}} + (G + a - b) \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \right] \frac{1-p}{p} \tag{9}$$

For $\ell = 100$, $g = 110$, $a = 100{,}000$, and $b = 142{,}000$, we obtain that the decision maker prefers the wealth level 100,000 to the lottery that with even chance leaves him with wealth level 97,690 or with wealth level 10,000,000. The following table offers the value of L for different levels of G, $b - a$, and g when $\ell = 100$ and $p = \frac{1}{2}$.

G	$b-a$	$g = 101$	$g = 105$	$g = 110$	$g = 125$
	10,000	123,740	21,491	4,316	1,134
200,000	20,000	79,637	5,810	2,330	1,125
	40,000	39,586	4,316	2,310	1,125
	10,000	611,376	95,531	12,867	1,174
1,000,000	20,000	376,873	12,662	2,421	1,125
	40,000	150,023	4,375	2,310	1,125
	10,000	6,097,288	928,479	109,064	1,617
10,000,000	20,000	3,720,787	89,752	3,450	1,125
	40,000	1,392,440	5,035	2,310	1,125

Table 1: If the decision maker rejects $(-100, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels between a and b, then at a he also rejects $(-L, \frac{1}{2}; G, \frac{1}{2})$, values of L entered in the table.

If $p \neq \frac{1}{2}$, the values of Table 1 should be multiplied by $\frac{1-p}{p}$ (see eq. (9)). For example, if the decision maker rejects $(-100, \frac{1}{2}; 110, \frac{1}{2})$ on a range of 20,000, then he also rejects the lotteries $(-383, \frac{9}{10}; 10{,}000{,}000, \frac{1}{10})$ and $(-3.45, \frac{999}{1000}; 10{,}000{,}000, \frac{1}{1000})$. This decision maker will even refuse to pay four cents for a 1:100,000 chance of winning 10 million dollars!

Inequalities (7) and (8) imply that if for all wealth levels x between a and b the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$, then he will also reject,

at b, any lottery of the form $(-(b-a), p; G, 1-p)$, provided that

$$p(\ell+g)\frac{\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}}-1}{\frac{g}{\ell}-1} > (1-p)G \tag{10}$$

For $\ell = 100$, $g = 110$, $b = 100{,}000$, $a = 68{,}500$, and $G = 900{,}000$, we obtain that the decision maker will prefer the wealth level 100,000 to the lottery that with probability 0.00026 leaves him with wealth level 68,500 and with probability 0.99974 leaves him with wealth level 1,000,000.

In Table 2 below $\ell = 100$ and the wealth level is b. The table presents, for different combinations of $L = b-a$, G, and g, values of p such that a rejection of $(-100, \frac{1}{2}; g, \frac{1}{2})$ at all $x \in [b-L, b]$ leads to a rejection of $(-L, p; G, 1-p)$ at b.

L	G	$g = 101$	$g = 105$	$g = 110$	$g = 125$
	100,000	0.8859	0.7132	0.3397	0.0054
10,000	1,000,000	0.9872	0.9613	0.8373	0.0519
	100,000,000	0.9998	0.9996	0.9981	0.8456
	100,000	0.7462	0.1740	0.0054	$2.7 \cdot 10^{-7}$
20,000	1,000,000	0.9671	0.6781	0.0516	$2.7 \cdot 10^{-6}$
	100,000,000	0.9996	0.9952	0.8447	$2.7 \cdot 10^{-4}$
	100,000	0.5929	0.0189	$5.8 \cdot 10^{-5}$	$1.3 \cdot 10^{-11}$
30,000	1,000,000	0.9357	0.1621	$5.8 \cdot 10^{-4}$	$1.3 \cdot 10^{-10}$
	100,000,000	0.9993	0.9508	0.0550	$1.3 \cdot 10^{-8}$
	100,000	0.3137	$1.7 \cdot 10^{-4}$	$6.6 \cdot 10^{-9}$	$3.2 \cdot 10^{-20}$
50,000	1,000,000	0.8205	0.0017	$6.6 \cdot 10^{-8}$	$3.2 \cdot 10^{-19}$
	100,000,000	0.9978	0.1421	$6.6 \cdot 10^{-6}$	$3.2 \cdot 10^{-17}$
	100,000	0.1107	$4.3 \cdot 10^{-7}$	$7.8 \cdot 10^{-14}$	$5.5 \cdot 10^{-31}$
75,000	1,000,000	0.5545	$4.3 \cdot 10^{-6}$	$7.8 \cdot 10^{-13}$	$5.5 \cdot 10^{-30}$
	100,000,000	0.9920	$4.3 \cdot 10^{-4}$	$7.8 \cdot 10^{-11}$	$5.5 \cdot 10^{-28}$

Table 2: If the decision maker rejects $(-100, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels in $[b-L, b]$, then at b he also rejects $(-L, p; G, 1-p)$, values of p entered in the Table.

We believe that the values of Table 2 are even more disturbing than those of Rabin [37], as many of the entries represent an almost sure gain of huge amounts of money, where with a very small probability less than 100,000 may be lost. In the next section we show that similar tables can be constructed for many non-expected utility models. The numbers may be less stunning than those of Tables 1 and 2, but usually not by much.

3 Definitions and Assumptions

Having Tables 1 and 2 in mind, we begin with the following definition

Definition 1 The vector (ℓ, g, L, G, c) is an upper calibration quintuple if a risk averse expected utility decision maker who is rejecting $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels in $[w, w+c]$ will also reject $(-L, \frac{1}{2}; G, \frac{1}{2})$ at the wealth level w. The vector $(\ell, g, L, G, \varepsilon)$ is a lower calibration quintuple if a risk averse expected utility decision maker who is rejecting $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels in $[w-L, w]$ will also reject $(-L, \varepsilon; G, 1-\varepsilon)$ at the wealth level w.

For example, $(100, 101, 123{,}740, 200{,}000, 10{,}000)$ is an upper calibration quintuple (see Table 1) and $(100, 101, 10{,}000, 100{,}000, 0.8859)$ is a lower calibration quintuple (see Table 2). Observe that upper and lower quintuples do not depend on the values of w.

We assume throughout that preference relations over distributions are risk averse, monotonically increasing, continuous, and that they display a modest degree of smoothness:

Definition 2 The functional V belongs to the set $\mathcal{V}$ if

1. V is monotonically increasing with respect to first order stochastic dominance;
2. V is continuous with respect to the topology of weak convergence;
3. V exhibits aversion to mean preserving spreads;
4. At every distribution F, the indifference set of V through F has a unique tangent hyperplane.

Observe that the hyperplane of part (4) is an indifference set of an expected utility functional with a unique (up to positive affine transformations) vNM utility function $u(\cdot; F)$, which is called the local utility at F (see Machina [27]).[4] According to the context, utility functionals are defined over lotteries (of the form $X = (x_1, p_1; \ldots; x_n, p_n)$) or over cumulative distribution functions (denoted F, H). Degenerate cumulative distribution functions are denoted δ_x.

Machina [27] introduced the following assumptions.

[4]The existence of such hyperplanes does not require Fréchet differentiability (as in Machina [27]) or even Gâteaux differentiability (as in Chew, Karni, and Safra [8]—see Dekel [11]). We use these notions of differentiability in section 5 below.

Definition 3 The functional $V \in \mathcal{V}$ satisfies Hypothesis 1 (H1) if for any distribution F, $-\frac{u''(x;F)}{u'(x;F)}$ is a nonincreasing function of x.

The functional $V \in \mathcal{V}$ satisfies Hypothesis 2 (H2) if for all F and H such that F dominates H by first order stochastic dominance and for all x

$$-\frac{u''(x;F)}{u'(x;F)} \geqslant -\frac{u''(x;H)}{u'(x;H)}$$

In the sequel, we will also use the opposite of these assumptions, denoted ¬H1 and ¬H2. ¬H1 says that for any distribution F, $-\frac{u''(x;F)}{u'(x;F)}$ is a nondecreasing function of x while ¬H2 says that if F dominates H by first order stochastic dominance, then for all x

$$-\frac{u''(x;F)}{u'(x;F)} \leqslant -\frac{u''(x;H)}{u'(x;H)}$$

Also, we will say that H1 (or ¬H1) is satisfied in $\mathcal{S}$ if it satisfied at all $F \in \mathcal{S}$. Similarly, H2 (or ¬H2) is satisfied in $\mathcal{S}$ if it applies to all $F, H \in \mathcal{S}$.[5]

Machina [27, 28] shows that H1 and H2 conform with many violations of expected utility, like the Allais paradox, the common ratio effect [1] and the mutual purchase of both insurance policies and public lottery tickets. Hypothesis H2 implies that indifference curves in the set $\{(x, p; y, 1-p-q; z, q) : p + q \leqslant 1\}$ (where $x > y > z$ are fixed) become steeper as one moves from δ_z to δ_x. That is, indifference curves in the (q, p) triangle fan out. The experimental evidence concerning H2 is inconclusive. Battalio, Kagel, and Jiranyakul [2] and Conlisk [9] suggest that indifference curves become less steep as one moves closer to either δ_x or δ_z. However, Conlisk's experiment does not prove a violation of H2 near δ_x.[6] Battalio et al. did find some violations of this assumption, but as most of their subjects were consistent with expected utility theory, only a small minority of them violated this hypothesis. For further citations of violations of H2, see Starmer [46, Sec. 5.1.1].

[5] Preferences may satisfy neither H2 nor ¬H2, see section 5.3.1 below.

[6] In this part of the experiment, subjects were asked to rank $B = (5, 0.88; 1, 0.11; 0, 0.01)$ and $B^* = (5, 0.98; 0, 0.02)$. Most ranked B^* higher than B. But this pair does not dominate the pair $(1, 1)$ and $(5, 0.1; 1, 0.89; 0, 0.01)$ of the Allais paradox, and therefore the fact that most decision makers prefer $(1, 1)$ to $(5, 0.1; 1, 0.89; 0, 0.01)$ does not prove a violation of H2. Moreover, we suspect that most decision makers would prefer $(5, 0.89; 1, 0.11)$ to $(5, 0.99; 0, 0.01)$, which is consistent with H2.

The next definition relates to decision makers who reject the noise $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ which is added to all outcomes of a lottery. Some of our results will refer to such decision makers. for a lottery $X = (x_1, p_1; \ldots; x_n, p_n)$, define $(X - \ell, \frac{1}{2}; X + g, \frac{1}{2}) = (x_1 - \ell, \frac{p_1}{2}; x_1 + g, \frac{p_1}{2}; \ldots; x_n - \ell, \frac{p_n}{2}; x_n + g, \frac{p_n}{2})$.

Definition 4 Let $x_1, \ldots, x_n \in [a, b]$. The functional V satisfies (ℓ, g) random risk aversion in $[a, b]$ if for all X, $V(X) > V(X - \ell, \frac{1}{2}; X + g, \frac{1}{2})$.

The lottery X in the above definition serves as background risk—risk to which the binary lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is added. Paiella and Guiso [34] provide some data showing that decision makers are more likely to reject a given lottery in the presence of background risk.[7] Accordingly, if rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is likely when added to non-stochastic wealth, it is even more likely when added to a lottery. In other words, (ℓ, g) random risk aversion is more behaviorally acceptable than the behavioral assumption that decision makers reject the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ used by Rabin.

4 Calibration Results for Betweenness Functionals

We start by analyzing the set of betweenness functionals (Chew [4, 5], Fishburn [16], and Dekel [11]). Indifference sets of such preferences are hyperplanes: If F and H are in the indifference set $\mathcal{I}$, then for all $\alpha \in (0, 1)$, so is $\alpha F + (1 - \alpha)H$. Formally:

Definition 5 V satisfies betweenness if for all F and H satisfying $V(F) \geqslant V(H)$ and for all $\alpha \in (0, 1)$, $V(F) \geqslant V(\alpha F + (1 - \alpha)H) \geqslant V(H)$.

The fact that indifference sets of the betweenness functional are hyperplanes implies that for all F, the indifference set through F can also be viewed as an indifference set of an expected utility functional with vNM utility $u(\cdot; F)$.

[7] Rabin and Thaler [38], on the other hand, seem to claim that a rejection of a small lottery is likely only when the decision maker is unaware of the fact that he is exposed to many other risks. See section 7 for a further discussion.

4.1 Betweenness and Hypotheses 1 & 2

Our first result implies that betweenness functionals are susceptible to Rabin-type criticism whenever they satisfy Hypotheses 1 or 2, or their opposites, $\neg$H1 or $\neg$H2.

Theorem 1 *Let $V \in \mathcal{V}$ be a betweenness functional and let $(\ell, g, L, G, \varepsilon)$ and $(\ell, g, \bar{L}, \bar{G}, c)$ be lower and upper calibration quintuples, respectively. Define $\mathcal{S}_1 = \{\delta_x : x \in [a, b]\}$ for some a and b satisfying $b - a = \max\{L, c\}$ and assume that, for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$.*

1. *If V satisfies H1 or H2 on $\mathcal{S}_1$, then $V(b, 1) > V(b - L, \varepsilon; b + G, 1 - \varepsilon)$.*
2. *If V satisfies $\neg$H1 or $\neg$H2 on $\mathcal{S}_1$, then $V(a, 1) > V(a - \bar{L}, \frac{1}{2}; a + \bar{G}, \frac{1}{2})$.*

In other words, Tables 1 and 2 apply not only to expected utility theory, but also to betweenness functions satisfying Hypotheses 1, 2, or their opposites.

Remark 1 If V is expected utility and if preferences satisfy H1, then a rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at one point implies the rejection of big lotteries as in Table 2.

4.2 Betweenness and Random Risk Aversion

The second result related to betweenness functionals is similar to the first, except that Hypotheses 1 and 2 and their opposites are replaced by the property of random risk aversion with respect to binary lotteries (that is, lotteries with two outcomes at most).

Theorem 2 *Let $V \in \mathcal{V}$ be a betweenness functional satisfying (ℓ, g) random risk aversion with respect to binary lotteries in $[w - L, w + c]$ and let $(\ell, g, L, G, \varepsilon)$ and $(\ell, g, \bar{L}, \bar{G}, c)$ be lower and upper calibration quintuples, respectively. Then either*

1. *$V(w, 1) > V(w - L, \varepsilon; w + G, 1 - \varepsilon)$; or*
2. *$V(w, 1) > V(w - \bar{L}, \frac{1}{2}; w + c + \bar{G}, \frac{1}{2})$.*

In other words, a betweenness functional satisfying (ℓ, g) random risk aversion is bound to reject lotteries from at least one of the two tables of section 2. The fact that these results are weaker than those of expected utility (where lotteries from both tables are rejected) is besides the point. Theorems 1 and 2 mimic Rabin's claim against expected utility for betweenness functionals: Seemingly reasonable behavior in the small leads to absurd behavior in the large.

4.3 Some Betweenness Functionals

In this section we analyze two specific betweenness functionals: Chew's [4] weighted utility and Gul's [19] disappointment aversion theories. Both are supported by simple sets of axioms.

4.3.1 Weighted Utility

We show first that the expected utility analysis of section 2 can be extended to the case of weighted utility without the need to assume either H1, H2, ¬H1, ¬H2, or random risk aversion. This functional, suggested by Chew [4], is given by

$$V(F) = \frac{\int v dF}{\int h dF} \tag{11}$$

for some functions $v, h : \Re \to \Re$. The local utility of V at δ_w is a function for which the indifference set through δ_w coincides with the corresponding indifference set of V, that is

$$\begin{aligned} \int u(x;\delta_w)dF(x) = u(w;\delta_w) \quad &\Longleftrightarrow \quad \frac{\int v(x)dF(x)}{\int h(x)dF(x)} = \frac{v(w)}{h(w)} \\ &\Longleftrightarrow \quad \int \left[v(x) - \frac{v(w)}{h(w)} h(x) \right] dF(x) = 0 \end{aligned}$$

As vNM functions are unique up to positive linear transformations, we can assume, without loss of generality, that

$$u(x;\delta_w) = v(x) - \frac{v(w)}{h(w)} h(x) \tag{12}$$

Suppose that for every $x \in [a, b]$, $\delta_x \succ (x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$ and that $h(x) \neq \frac{1}{2}h(x-\ell) + \frac{1}{2}h(x+g)$. As $(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$ is below the indifference curve of

$\succeq$ through δ_x, and as this is also an indifference curve of the expected utility preferences with the vNM function $u(\cdot;\delta_x)$, it follows that $u(x;\delta_x) - [\frac{1}{2}u(x-\ell;\delta_x) + \frac{1}{2}u(x+g;\delta_x)] > 0$. Replacing u by the expression on the right hand side of Eq. (12) implies that at $w = x$

$$v(x) - \frac{1}{2}[v(x-\ell) + v(x+g)] - \frac{v(w)}{h(w)}\left[h(x) - \frac{1}{2}[h(x-\ell) + h(x+g)]\right] > 0$$

Differentiating the left hand side with respect to w we obtain

$$-\frac{v'(w)h(w) - v(w)h'(w)}{h^2(w)}\left[h(x) - \frac{1}{2}[h(x-\ell) + h(x+g)]\right] \tag{13}$$

Monotonicity with respect to first order stochastic dominance implies that $v(w)h'(w) - v'(w)h(w)$ does not change sign [4, Corollary 5].[8] The sign of the derivative of the expression in (13) with respect to w thus depends on the sign of $h(x) - \frac{1}{2}[h(x-\ell) + h(x+g)]$, which was assumed to be different from zero. In other words, for a given x, the inequality $u(x;\delta_w) > \frac{1}{2}[u(x-\ell;\delta_w) + u(x+g;\delta_w)]$ is satisfied either for all $w \in [x,b]$ (denote the set of x having this property K_1) or for all $w \in [a,x]$ (likewise, denote the set of x having this property K_2).

For every $x \in [a,b]$, either $x \in K_1$, or $x \in K_2$. At least half of the points $a, \ldots, a + i(\ell+g), \ldots, b$ therefore belong to the same set, K_1 or K_2. In order to obtain calibration results, we need enough wealth levels at which $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is rejected, and moreover, the distance between any two such points should be at least $\ell + g$. In section 2 we just divided the segment $[a,b]$ into $\frac{b-a}{\ell+g}$ segments to obtain Tables 1 and 2. Here we can get only $\frac{b-a}{2(\ell+g)}$ such points, hence we need to observe rejections of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ on twice the ranges needed for Tables 1 and 2.

If K_1 contains these points, take $w = b$ and apply Table 2. If K_2 contains these points, take $w = a$ and apply Table 1.[9]

[8] In Chew's [4] notation, $V(F) = \int \alpha\phi dF / \int \alpha dF$. To obtain the representation (11), let $h = \alpha$ and $\phi = v/h$. Monotonicity in x of $\alpha(x)(\phi(x) - \phi(s))$ for all s is equivalent to monotonicity of $v(x) - h(x)v(s)/h(s)$, that is, to $v'(x) - h'(x)v(s)/h(s) \neq 0$.

[9] Risk aversion implies the concavity of $u(\cdot;\delta_w)$; see the proof of Theorem 1.

4.3.2 Disappointment Aversion

Gul's [19] disappointment aversion is a special case of betweenness, according to which the value of the lottery $p = (x_1, p_1; \ldots; x_n, p_n)$ is given by

$$V(p) = \gamma(\alpha) \sum_x q(x)u(x) + [1 - \gamma(\alpha)] \sum_x r(x)u(x)$$

where $p = \alpha q + (1-\alpha)r$ for a lottery q with positive probabilities on outcomes that are better than p and a lottery r with positive probabilities on outcomes that are worse than p, and where $\gamma(\alpha) = \alpha/[1 + (1-\alpha)\beta]$ for some $\beta \in (-1, \infty)$. Risk aversion behavior is obtained whenever u is concave and $\beta \geqslant 0$ [19, Th. 3].

We show that this functional satisfies (ℓ, g) random risk aversion with respect to binary lotteries in $[r, s]$ if, and only if, an expected utility maximizer with the vNM utility function u rejects the lotteries $(-\ell, \frac{1}{2}; g, \frac{1}{2})$.

Claim 1 *Let $V \in \mathcal{V}$ be a disappointment aversion functional with the associated utility function u. Then V satisfies (ℓ, g) random risk aversion with respect to binary random lotteries in $[r, s]$ if, and only if, for every $x \in [r, s]$, $u(x) > \frac{1}{2}u(x - \ell) + \frac{1}{2}u(x + g)$.*

By claim 1, disappointment aversion theory is compatible with (ℓ, g) random risk aversion. Moreover, to a certain extent, the requirements for this form of risk aversion are similar to Rabin's requirement under expected utility theory. By Theorem 2, disappointment aversion is vulnerable to both Tables 1 and 2.

4.3.3 Chew & Epstein on Samuelson

Samuelson [41] proved that if a lottery X is rejected by an expected utility maximizer at all wealth levels, then the decision maker also rejects an offer to play the lottery n times, for all values of n. This result has similar flavor to Rabin's calibration analysis, as positive expected value of X implies that the probability of losing money after playing the lottery many times goes to zero.

Chew and Epstein [6] showed, by means of an example, that this result does not necessarily hold for non-expected utility preferences. They suggested a special form of a betweenness functional V, given implicitly by

$$\int \varphi(x - V(F))dF = 0$$

and proved that under some conditions, a lottery with a positive expected payoff may be rejected when played once, but many repetitions of it may still be attractive. They similarly identified rank dependent functionals with the same property.

We don't find these results to contradict ours. We do not claim that *all* large lotteries with high expected return will be rejected. Tables 1 and 2 identify *some* very attractive large lotteries that under some conditions are rejected. In the same way in which one cannot dismiss the Allais paradox by claiming that there are many other decision problems that are consistent with expected utility theory, one cannot claim here that the fact that non-expected utility theories are not vulnerable to Samuelson's criticism implies that they are also immune to Rabin's. Moreover, if the number of times one needs to play a lottery before it becomes attractive is absurdly high (say, one million), Samuelson's identification of unreasonable behavior is still valid, as is Rabin's.

5 General Functionals

In this section we analyze preferences that do not satisfy the betweenness axiom, hence we need to make sure that the functionals are sufficiently smooth. This is achieved by requiring them to be Gâteaux differentiable (see below). This differentiability is weaker than Fréchet differentiability, used by Machina [27], and it is consistent with the family of rank dependent utility functionals (see Chew, Karni and Safra [8]).

Definition 6 (Zeidler [51, p. 191]) The functional V is Gâteaux differentiable at F if for every H,

$$\delta V(F, H - F) := \frac{\partial}{\partial t} V((1-t)F + tH)\Big|_{t=0}$$

exists and is a continuous linear function of $H - F$. V is Gâteaux differentiable if it is Gâteaux differentiable at all F.

If the functional V is Gâteaux differentiable at F, then there exists a function $u(\cdot; F)$ such that for every H and t,

$$V((1-t)F + tH) - V(F) = t \int u(x; F) d(H - F)(x) + o(t) \qquad (14)$$

(see [51]). Note that although all indifference sets of betweenness functionals are hyperplanes, these functionals are not necessarily Gâteaux differentiable (see Dekel [11]).

The following lemma is needed since the calibration results for expected utility rely on the concavity of the vNM utility function. The relation between risk aversion and concavity of local utilities is known to hold for Fréchet differentiable functionals (Machina [27]), for rank dependent functionals (Chew, Karni and Safra [8]), and for betweenness functionals (Chew [4]) but (as far as we know) not for the class of all Gâteaux differentiable functionals.

Lemma 1 *If $V \in \mathcal{V}$ is Gâteaux differentiable then all its local utilities are concave.*

5.1 Quasi-concave Functionals

Definition 7 The functional V is quasi concave if for all F and H satisfying $V(F) \geqslant V(H)$ and for all $\alpha \in (0,1)$, $V(\alpha F + (1-\alpha)H) \geqslant V(H)$. V is quasi convex if for all such F, H, and α, $V(\alpha F + (1-\alpha)H) \leqslant V(F)$.

The next result is similar to Theorem 1. Like betweenness functionals, quasi concave functionals which satisfy the H1 or H2 assumptions (or their inverse—see below) on slightly bigger sets are susceptible to Rabin-type criticism. Unlike Theorem 1, we need to assume Gâteaux differentiability.

Theorem 3 *Let $V \in \mathcal{V}$ be a quasi concave, Gâteaux differentiable functional and let $(\ell, g, L, G, \varepsilon)$ and $(\ell, g, \bar{L}, \bar{G}, c)$ be lower and upper calibration quintuples, respectively. Define $\mathcal{S}_3 = \{X : \mathrm{supp}(X) \subseteq [a,b]\}$ for some a and b satisfying $b - a = \max\{L + g, c + \ell\}$ and assume that, for all $x \in [a,b]$, $V(x,1) > V(x-\ell, \frac{1}{2}; x+g, \frac{1}{2})$.*

1. *If V satisfies* H1 *or* H2 *on $\mathcal{S}_3$ then*

$$V(b,1) > V(b-g-L, \varepsilon; b-g+G, 1-\varepsilon).$$

2. *If V satisfies* ¬H1 *or* ¬H2 *on $\mathcal{S}_3$ then*

$$V(a,1) > V(a-\bar{L}, \tfrac{3}{4}; a+\bar{G}, \tfrac{1}{4}).$$

In other words, adjustments that are not larger than g are made in the outcomes of the large lotteries of Table 2 and the probabilities of Table 1 are changed to $\frac{3}{4}$ and $\frac{1}{4}$. Obviously, the rejection of the new big lotteries is still absurd. [10]

5.2 General Functionals

Proving a calibration result for general non-expected utility functionals requires slightly stronger assumptions than that of Theorem 3.

Definition 8 The functional V satisfies approximate risk aversion with respect to $(\ell, g, \varepsilon, a, b)$ if for all $x \in [a, b]$ and for all $\bar{\varepsilon} \leqslant \varepsilon$,

$$V(a, \bar{\varepsilon}; x, 1 - \bar{\varepsilon}) > V(a, \bar{\varepsilon}; x - \ell, \tfrac{1-\bar{\varepsilon}}{2}; x + g, \tfrac{1-\bar{\varepsilon}}{2})$$

In other words, when some small probability $\bar{\varepsilon}$ is set aside for the outcome a, the decision maker prefers the rest of the probability to yield x rather than an even (residual) chance for $x - \ell$ and $x + g$. Note that approximate risk aversion implies the rejection of the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all $x \in [a, b]$.

Theorem 4 *Let $V \in \mathcal{V}$ be a Gâteaux differentiable functional and let $(\ell, g, L, G, \varepsilon)$ be a lower calibration quintuple. Define $\mathcal{S}_4 = \{X : \mathrm{supp}(X) \subseteq [b - g - L, b]\}$ for some b and assume that V is approximately risk averse with respect to $(\ell, g, \varepsilon, b - g - L, b)$. If V satisfies* H2 *on $\mathcal{S}_4$ then*

$$V(b, 1) > V(b - g - L, \varepsilon; b - g + G, 1 - \varepsilon)$$

Here too, the decision maker rejects small variations of the very attractive lotteries of Table 2. [11]

5.3 Some Functional Forms

In this section we analyze two families of non-betweenness functional, the rank dependent and the quadratic utilities.

[10] Assuming ¬H2 we actually prove that $V(a, 1) > V(a - \bar{L}, \frac{1}{2} + \frac{\ell}{L}; a + \bar{G}, \frac{1}{2} - \frac{\ell}{L})$. See line (29) at the end of the proof of Theorem 3.

[11] Similar results hold if hypothesis H1 is assumed, provided that the relevant interval is increased to a length of $L + G$ and that quasi convexity is assumed.

5.3.1 Rank Dependent Functionals

Rank dependent theory was first suggested by Weymark [48] and Quiggin [36]. Its general form is given by

$$V(F) = \int v(x) df(F(x))$$

For $X = (x_1, p_1; \ldots; x_n, p_n)$ with $x_1 \leqslant \ldots \leqslant x_n$, the finite version of this functional is given by

$$V(X) = v(x_1)f(p_1) + \sum_{i=2}^{n} v(x_i) \left[f\left(\sum_{j=1}^{i} p_j\right) - f\left(\sum_{j=1}^{i-1} p_j\right) \right]$$

This functional is Gâteaux differentiable iff f is differentiable. We assume throughout that v and f are strictly increasing, that v is differentiable, that f is continuously differentiable, and that f' is bounded, and bounded away from zero. The local utility of the rank dependent functional is given by

$$u(x; F) = \int^{x} v'(y) df(F(y)) \tag{15}$$

Assuming risk aversion, both v and f are concave. The concavity of f implies that the functional is quasi convex. (For these results, see Chew, Karni, and Safra [8]). Moreover, if f is not linear, the rank dependent functional satisfies neither H1 nor H2. Assumption H1 is not satisfied since, at every F, the local utility function $u(\cdot; F)$ is not differentiable at points at which F is discontinuous. Therefore, at such points, the local utility displays an extreme level of risk aversion. As for H2, consider lotteries of the form $(a, q; b, 1-p-q; c, p)$ where $a < b < c$ and assume $u(b) = 0$. Then $V(a, q; b, 1-p-q; c, p) = u(a)f(q) + u(c)(1 - f(1-p))$. The slope of the indifference curve (in the probability triangle) through (q, p) is $\frac{-u(a)f'(q)}{u(c)f'(1-p)}$ hence it is constant whenever $p + q = 1$. Each such lottery where $p + q < 1$ therefore both dominates, and is dominated by, lotteries of the form $(a, 1-p; c, p)$, and the slopes of indifference curves must be constant, hence expected utility. We cannot therefore utilize theorem 4 as it stands and will need to modify it.

Let $X = (x_1, p_1; \ldots; x_n, p_n)$ with the cumulative distribution function F. By eq. (15), at $x \notin \{x_1, \ldots, x_n\}$,

$$-\frac{u''(x; F)}{u'(x; F)} = -\frac{v''(x)}{v'(x)}$$

We therefore replace H1 ($\neg$H1) with the assumption H1* ($\neg$H1*), that $-\frac{v''(x)}{v'(x)}$ is weakly decreasing (increasing). Clearly, these assumptions are equivalent to the assumptions that for $w' > w$ ($w' < w$), if a lottery X is accepted at w, then it is also accepted at w'. The next theorem extends the calibration results to the class of rank dependent functionals.

Theorem 5 *Let $V \in \mathcal{V}$ be a rank dependent functional.*

1. *Suppose V satisfies H1* and let $(\ell, g, L, G, \varepsilon)$ be a lower calibration quintuple. There exist $d = d(\eta)$ and $G(\eta) \rightarrow_{\eta\downarrow 0} G$ such that if V is (ℓ, g) random risk averse on $[a, a+d]$ then $V(a, 1) > V(a-L, \varepsilon; a+G(\eta), 1-\varepsilon)$.*

2. *Suppose V satisfies $\neg$H1* and let (ℓ, g, L, G, c) be an upper calibration quintuple. There exist $d = d(\eta)$, $L(\eta) \rightarrow_{\eta\downarrow 0} L$, and $G(\eta)_{\eta\downarrow 0} G$ such that if V is (ℓ, g) random risk averse on $[a-d, a]$ then $V(a, 1) > V(a - L(\eta), \frac{1}{2}; a + G(\eta), \frac{1}{2})$.*

Risk aversion implies that both u and f are concave. Yaari [49, 50] discussed two kinds of risk aversion: Rejection of mean preserving spreads, and preferences for the expected value of a lottery over the lottery itself. In expected utility theory both notions are equivalent to each other (Rothschild and Stiglitz [39]). But under the rank dependent analysis, the former requires concave u and f, while the latter requires concave u but only $f(p) \geqslant p$ for all p. The proof of theorem 5 requires something even weaker, that $f(\frac{1}{2}) \geqslant \frac{1}{2}$ and $f(\varepsilon) \geqslant \varepsilon$.[12] In other words, this theorem applies not only to decision makers who satisfy the strong definition of risk aversion, but also to those who satisfy its weaker form.[13]

The analysis of theorem 5 extends to cumulative prospect theory (Tversky and Kahneman [47]), where a lottery is evaluated by a rank-dependent functional for outcomes above a reference point w, and with another rank dependent functional for outcomes below this point. The extension to this theory holds if all outcomes are better than w and $\neg$H1* is assumed. If potential losses are involved, the utility function becomes convex, and the "if" part of theorem 5 is not satisfied.

[12] Empirical and theoretical studies support this assumption. See Starmer [46] and Karni and Safra [23].

[13] There is another sense in which the statement of the theorem is stronger than required. As is clear from the proof, one needs to require H1* (and $\neg$H1*) only on $[a-L, a+d]$.

Remark 2 Neilson [33] observed that Rabin's results extend to rank dependent functionals and lotteries of the form $(-\ell, p; g, 1-p)$ for $p = f^{-1}(\frac{1}{2})$. For example, if $f(0.6) = \frac{1}{2}$, then a rejection of $(-100, 0.6; 110, 0.4)$ at all wealth levels will lead to a rejection of any lottery where there is a 60% of losing \$500.[14] But as values of f are not observable, in our framework the "if" part reads only as "the decision maker rejects $(-100, 0.6; 110, 0.4)$ at all wealth levels." This certainly does not lead to any level of risk aversion with respect to large lotteries, as it is consistent with maximizing expected value. Moreover, empirical evidence suggests that $f^{-1}(\frac{1}{2}) < \frac{1}{2}$ (see Starmer [46]).

5.3.2 Dual Theory

Yaari's [50] dual theory is a special case of the rank dependent model, where $u(x) = x$. According to this theory, the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels w iff

$$w > (w-\ell) f\left(\tfrac{1}{2}\right) + (w+g)\left[1 - f\left(\tfrac{1}{2}\right)\right] \Longleftrightarrow f\left(\tfrac{1}{2}\right) > \frac{g}{\ell + g}$$

But this does not imply any outstanding risk aversion for large lotteries. For example, for $f(\frac{1}{2}) = \frac{110}{210}$, the decision maker will accept any lottery of the form $(-L, \frac{1}{2}; G, \frac{1}{2})$ for all L and G such that $G > 1.1L$. As the results of Theorem 5 are not satisfied even though H1 trivially holds, it must follow that the dual theory violates (ℓ, g) random risk aversion. By the proof of Theorem 5, it follows that for a sufficiently large number of equally probable outcomes n, the decision maker prefers $(X - \ell, \frac{1}{2}; X + g, \frac{1}{2})$ to X. Whether this is how decision makers behave is an empirical question.

5.3.3 Quadratic Utility

The quadratic functional was suggested by Machina [27, fnt. 45] and by Chew, Epstein, and Segal [7], and is given by

$$V(F) = \int\int \varphi(x, y) dF(x) dF(y)$$

for some symmetric, continuous, and monotonic function φ. For finite lotteries V is given by $V(x_1, p_1; \ldots; x_n, p_n) = \sum_i \sum_j \varphi(x_i, x_j) p_i p_j$. The local

[14] For a related argument, see Cox and Sadiraj [10].

utility is

$$u(x;F) = \int \varphi(x,y)dF(y)$$

To check H2, we want to show that

$$-\frac{\int \varphi_{xx}(x,y)dF(y)}{\int \varphi_x(x,y)dF(y)} \tag{16}$$

is increasing as we replace F by H such that H dominates F by first order stochastic dominance. Obviously it is sufficient to check this requirement only for finite F where only one of the outcomes is replaced by a higher outcome. In other words, if F is the distribution of $(x_1, p_1; \ldots; x_n, p_n)$, then we want to show that the derivative of the expression in (16) with respect to x_i is positive. In other words, H2 requires $\varphi_{xxy}\varphi_x - \varphi_{xx}\varphi_{xy} < 0$. Likewise, $\neg$H2 requires $\varphi_{xxy}\varphi_x - \varphi_{xx}\varphi_{xy} > 0$.

6 Uncertainty

Uncertainty, defined as a situation where the decision maker is not sure about the probabilities of different events, seems to be immune to Rabin-type criticism, which crucially depends on attitudes towards risk, the lack of knowledge which outcome will obtain. In fact, even the terminology of this paper's theorems doesn't suit the world of uncertainty, where equally probable events don't necessarily exist.

Recent developments in the literature enable us, however, to extend our results to decision under uncertainty. In fact, some of these results hardly need any modification at all. Our claims in this section rely heavily on Machina's [30] recent results, showing that under some assumptions, uncertain beliefs must contain a probabilistic kernel.

Consider, for simplicity, the case where the set of states of the world is the segment $\mathcal{T} = [a, b] \subset \Re$. An act is a finite-valued, Lebesgue-measurable function $f : \mathcal{T} \to \Re$, given by $(x_1; E_1; \ldots; x_n, E_n)$, where $E_1, \ldots, E_n$ is a Lebesgue-measurable partition of $\mathcal{T}$. The decision maker has a preference relation over acts which is representable by a function W. This function is smooth in the events if it is differentiable with respect to the events $E_1, \ldots, E_n$ (for details, see Machina [30]. See also Epstein [14]). Machina observed that regardless of the underlying preferences over acts, as long as W

is event-differentiable, the decision maker will consider the event "the n-th decimal point of $x \in \mathcal{T}$ is even" as an almost probabilistic event of probability $\frac{1}{2}$, becoming more so as $n \to \infty$. Unfortunately, the limit of these events does not exist, therefore Machina [30] obtains only *almost* objective probabilities on such events. For every $p \in [0,1]$ there is a sequence of almost objective events with p as the limit probability.

We discuss next three models of decision making under uncertainty, and show how all three are vulnerable to the calibration results of our paper.

6.1 Non Additive Probabilities

Choquet preferences over uncertain acts $(x_1; E_1; \ldots; x_n, E_n)$ where $x_1 \leqslant \ldots \leqslant x_n$ are representable by

$$V(x_1; E_1; \ldots; x_n, E_n) = u(x_1) + \sum_{i=2}^{n} u(x_i) \left[\nu\left(\cup_{j=1}^{i} E_j\right) - \nu\left(\cup_{j=1}^{i-1} E_i\right)\right]$$

where $u : \Re \to \Re$ is increasing, $\nu(\varnothing) = 0$, $\nu(\mathcal{T}) = 1$, and $E \subset E' \Longrightarrow \nu(E) \leqslant \nu(E')$ (see Schmeidler [42] and Gilboa [17]). According to Machina [29, 31], on the almost objective events these preferences converge to rank dependent. Since the range of the almost objective probabilities is $[0,1]$, we can use the results of section 5.3.1 to obtain that if the almost objective preferences satisfy the requirement of Theorem 5, then there must be some calibration quintuples to which these preferences are vulnerable.

6.2 MaxMin

Gilboa and Schmeidler [18] suggested the following theory for evaluation of uncertain prospects. The decision maker has a family $\mathcal{P}$ of possible distributions over the set of events, and a certain utility function u. The expected utility of each uncertain prospect is evaluated with respect to this utility function and each of the possible distributions in $\mathcal{P}$. The value of the prospect is the minimum of these evaluations.

Machina [29, 30] shows that if $\mathcal{P}$ is finite, then on the probabilistic kernel such preferences converge to expected utility. Tables 1 and 2 thus apply directly to this theory.

6.3 Beliefs over Probabilities

A third group of theories explaining decision under uncertainty assumes the existence of a set $\mathcal{P}$ of possible distributions over the set of events. The decision maker has beliefs over the likelihood of each of these distributions. Uncertain prospects are thus modeled as two stage lotteries. These lotteries are evaluated by using backward induction [15] where at each stage the decision maker is using the same non-expected utility functional (Segal [43]), or different expected utility functions (Klibanoff, Marinacci, and Mukerji [24] and Ergin and Gul [12]). A related theory is suggested by Halevy and Feltkamp [20], where the decision maker has beliefs over $\mathcal{P}$, and is interested in the average outcome each possible element of $\mathcal{P}$ yields if played repeatedly. In all cases, if the reduction of compound lotteries axiom is assumed, such lotteries are equivalent to simple lotteries.

Suppose the set $\mathcal{P}$ is finite, and each of its members is a non-atomic, countably additive probability measure on Σ, the Borel σ-algebra of $\mathcal{T}$. Border, Ghirardato, and Segal [3] show that under these assumptions there is a sub-σ-algebra $\hat{\Sigma}$ of Σ on which all the measures agree, and which is rich in the sense that for every real number $r \in [0, 1]$, it contains a set of (unanimous) measure r.

Let $E_1, \ldots, E_n \in \hat{\Sigma}$ be a partition of $\mathcal{T}$. As all the possible distributions in $\mathcal{P}$ agree on these events and give them the probabilities $p_1, \ldots, p_n$, the two stage lottery discussed above collapses to a simple lottery with these probabilities. Therefore, the preferences over uncertain prospects where all events are in $\mathcal{P}$ become either expected utility (as is the case with Klibanoff, Marinacci, and Mukerji [24], Ergin and Gul [12], and Halevy and Feltkamp [20]), or the underlying non-expected utility preferences (Segal [43]). The present paper's results apply to these probabilistic sub-preferences.

7 Some Alternative Explanations

This paper makes two arguments. *1.* Rabin's [37] analysis requires much smaller domains over which small lotteries are rejected, and *2.* his analysis can be extended to many non-expected utility theories. The second result is troublesome. It is one thing to hammer another nail into the coffin of a

[15] Without the reduction of compound lotteries axiom. See Kreps and Porteus [25] and Segal [44].

battered theory. Yet having a general argument against (almost) all known alternatives to expected utility has not yet been done in the literature. How should we react to such results?

7.1 Final Wealth or Gains and Loses?

Cox and Sadiraj [10] and Rubinstein [40] suggest that expected utility theory can still explain Rabin's results, provided decision makers' utilities are defined on gains and losses rather than on final wealth levels (Markowitz [32]). Rejecting small gambles at *all* wealth levels reflects a certain degree of concavity at a given point (the zero point) of the utility function defined on gains and losses. As concavity level at one given point does not reflect on concavity levels at other points, this rejection of small gambles can coexist with the acceptance of pre-specified large gambles, even when lotteries' evaluation is done in accordance with expected utility theory (applied to gains and losses). Note that although this explanation is in the spirit of prospect theory (Kahneman and Tversky [22]), the use of a non-linear probability transformation function is not required.

Section 4 seems to challenge this analysis. Suppose the reference point with respect to which the decision maker evaluates a lottery is its certainty equivalent For each wealth level w the decision maker has a vNM utility function $u_w(\cdot)$ with respect to which he evaluates lotteries on the indifference curve through δ_w. So $X \succeq (w, 1)$ if and only if $\mathrm{E}[u_w(X)] \geqslant u_w(w)$, which implies that $X \sim Y \sim (w, 1)$ iff $\mathrm{E}[u_w(X)] = \mathrm{E}[u_w(Y)] = u_w(w)$. Clearly, the preference relation over lotteries is representable by a betweenness functional, and the "utility from changes from w" is the local utility $u_w(\cdot)$ (see also Machina [27, p. 307]). As such, it is vulnerable to the analysis of theorems 1 and 2.

A more sophisticated model of reference point may require the functional used to compare X and Y to depend on X and Y. Such theories are very likely to lead to violations of transitivity. Of course, such violations are inconsistent with all the theories we discussed in this paper.

7.2 Loss Aversion

Rabin and Thaler [38] suggest two reasons for risk aversion in the small. The first is loss aversion (rather than diminishing marginal utility), the second is decision makers' tendency to think about problems in isolation, not realizing

that the small risk they consider is added to a lot of existing risk. Decision makers therefore behave myopically and pay too much to avoid some risks.

Loss aversion seems to be associated with kinks in the utility function at the current holding, which is linked to first order risk aversion (see Segal and Spivak [45]). But the disappointment aversion (section 4.3.2) and rank dependent (section 5.3.1) theories represent first order risk aversion, and yet are both vulnerable to Rabin-type objections.

We agree with Rabin and Thaler that paying \$15/day for loss damage waiver on Hyundai Sonata (price: \$15,999) is wrong, as this rate is equivalent to an annual rate of \$5475 just for a protection against theft and damage to the car. But these numbers are irrelevant to our analysis as the risk aversion we discuss (following Rabin [37]) is very modest—a rejection of an even chance to win 110 or lose 100.

8 Summary

Rabin [37] proved that if a risk averse expected utility maximizer is modestly risk averse in the small, then he must be absurdly risk averse in the large. This paper shows that this criticism is not limited to expected utility theory—similar arguments can be made against (almost) all its known transitive alternatives, both under risk and under uncertainty. For these extensions we need one of two types of assumptions: Random risk aversion, that relates to noise added to an existing lottery, and H1 or H2 (or their opposites), that relate to changes in attitudes towards risk that result from moving from one lottery to another.

Given risk aversion, random risk aversion seems a natural assumption. As $(X-\ell, \frac{1}{2}; X+\ell, \frac{1}{2})$ is a mean preserving spread of X, risk aversion implies that $X \succ (X-\ell, \frac{1}{2}; X+g, \frac{1}{2})$ for $g = \ell$. Continuity implies such preferences for some $g > \ell$, and uniform continuity implies $X \succ (X-\ell, \frac{1}{2}; X+g, \frac{1}{2})$ on any given compact set of lotteries. The size of g (given $\ell = 100$) is an empirical question, as is the question whether subjects prefer the wealth level w to the simple lottery $(w-\ell, \frac{1}{2}; w+g, \frac{1}{2})$.

It shouldn't be too hard to find examples for preferences that on no set $\mathcal{S} = \{X : \delta_{a+20,000} \succeq X \succeq \delta_a\}$ satisfy H1, H2, or their opposites. But are such preferences reasonable? There are after all good reasons why these assumptions were made (for example, in order to explain the Allais paradox). As we've shown in this paper, none of the existing (transitive) theories can

escape the extended forms of our calibration results. Finding a relation that violates H1, H2, and their opposites is a theoretical possibility. But is tailoring such a functional the right thing to do? How complicated must a theory be before we decide that it is impractical to work with it? Our analysis suggests that the transitive alternatives to expected utility theory may have reached this point.

Appendix

Proof of Theorem 1 Let $\mathcal{I}$ be the indifference set of V through δ_w and let $u(\cdot;\delta_w)$ be one of the increasing vNM utilities obtained from $\mathcal{I}$ (note that all these utilities are related by positive affine transformations). Obviously, $\mathcal{I}$ is also an indifference set of the expected utility functional defined by $U(F) = \int u(x;\delta_w)dF(x)$. By monotonicity, V and U increase in the same direction relative to the indifference set $\mathcal{I}$.

We show first that $u(\cdot;\delta_w)$ is concave. Suppose not. Then there exist z and H such that $z = \mathrm{E}[H]$ and $U(H) > u(z;\delta_w)$. By monotonicity and continuity, there exist p and z' such that $(z,p;z',1-p) \in \mathcal{I}$, hence $(H,p;z',1-p)$ is above $\mathcal{I}$ with respect to the expected utility functional U. Therefore, $(H,p;z',1-p)$ is also above $\mathcal{I}$ with respect to V, which is a violation of risk aversion, since $(H,p;z',1-p)$ is a mean preserving spread of $(z,p;z',1-p)$.

Case 1: Assume H1 or H2. We show that for every $x \in [a,b]$, $u(x;\delta_b) > \frac{1}{2}u(x-\ell;\delta_b) + \frac{1}{2}u(x+g;\delta_b)$. By betweenness, $V(x,1) > V(x-\ell,\frac{1}{2};x+g,\frac{1}{2})$ implies that, for all $\varepsilon > 0$, $V(x,1) > V(x,1-\varepsilon;x-\ell,\frac{\varepsilon}{2};x+g,\frac{\varepsilon}{2})$. Using the local utility at δ_x we obtain

$$u(x;\delta_x) > \tfrac{1}{2}u(x-\ell;\delta_x) + \tfrac{1}{2}u(x+g;\delta_x) \tag{17}$$

Eq. (17) holds in particular for $x = b$, and therefore H1 implies that for $x \leqslant b$,

$$u(x;\delta_b) > \tfrac{1}{2}u(x-\ell;\delta_b) + \tfrac{1}{2}u(x+g;\delta_b) \tag{18}$$

Also, since $x \leqslant b$, H2 implies that the local utility $u(\cdot;\delta_b)$ is more concave than the local utility $u(\cdot;\delta_x)$. Hence starting at eq. (17) and moving from δ_x to δ_b, H2 implies eq. (18).

By the expected utility calibration result of section 2, it now follows that the expected utility decision maker with the vNM function $u(\cdot;\delta_b)$ satisfies

$u(b;\delta_b) > \varepsilon u(b-L;\delta_b)+(1-\varepsilon)u(b+G;\delta_b)$ and the lottery $(b-L,\varepsilon;b+G,1-\varepsilon)$ lies below the indifference set $\mathcal{I}$. Therefore,

$$V(b,1) > V(b-L,\varepsilon;b+G,1-\varepsilon)$$

Case 2: Assume ¬H1 or ¬H2. Here we show that for every $x \in [a,b]$, $u(x;\delta_a) > \frac{1}{2}u(x-\ell;\delta_a) + \frac{1}{2}u(x+g;\delta_a)$. As before, eq. (17) holds for $x = a$, and ¬H1 implies

$$u(x;\delta_a) > \tfrac{1}{2}u(x-\ell;\delta_a) + \tfrac{1}{2}u(x+g;\delta_a) \tag{19}$$

Also, since $x \geqslant a$, ¬H2 implies that the local utility $u(\cdot;\delta_a)$ is more concave than the local utility $u(\cdot;\delta_x)$ and hence eq. (19).

By the expected utility calibration result of section 2, it now follows that the expected utility decision maker with the vNM function $u(\cdot;\delta_a)$ satisfies $u(a;\delta_a) > \frac{1}{2}u(a-\bar{L};\delta_a) + \frac{1}{2}u(a+\bar{G};\delta_a)$ and the lottery $(a-\bar{L},\frac{1}{2};a+\bar{G},\frac{1}{2})$ lies below the indifference set $\mathcal{I}$. Therefore,

$$V(a,1) > V(a-\bar{L},\tfrac{1}{2};a+\bar{G},\tfrac{1}{2})$$

■

Proof of Theorem 2 For every x and z such that $w+c \geqslant x > w > z \geqslant w-L$ there is a probability p such that $X = (x,p;z,1-p)$ satisfies $V(X) = V(w,1)$. By random risk aversion,

$$V(x,p;z,1-p) > V(x-\ell,\tfrac{p}{2};x+g,\tfrac{p}{2};z-\ell,\tfrac{1-p}{2};z+g,\tfrac{1-p}{2}) \tag{20}$$

Denote the distribution functions of the two lotteries in eq. (20) by F and G, respectively. Betweenness implies that the local utilities at δ_w and at F are the same (up to a positive linear transformation). Hence, by using the local utility $u(\cdot;\delta_w)$, we obtain,

$$\begin{aligned} &p[u(x;\delta_w) - \tfrac{1}{2}(u(x-\ell;\delta_w) + u(x+g;\delta_w))]+ \\ &\quad (1-p)[u(z;\delta_w) - \tfrac{1}{2}(u(z-\ell;\delta_w) + u(z+g;\delta_w))] > 0 \end{aligned} \tag{21}$$

Suppose there are $x^* > w > z^*$ such that $u(x^*;\delta_w) \leqslant \frac{1}{2}[u(x^*-\ell;\delta_w)+u(x^*+g;\delta_w)]$ and $u(z^*;\delta_w) \leqslant \frac{1}{2}[u(z^*-\ell;\delta_w)+u(z^*+g;\delta_w)]$. Then inequality (21) is reversed, and by betweenness, the inequality at (20) is reversed; a contradiction to random risk aversion. Therefore, at least one of the following holds. Either for all $x \in [w-L,w]$,

$$u(x;\delta_w) > \tfrac{1}{2}[u(x-\ell;\delta_w) + u(x+g;\delta_w)]$$

and, using an argument similar to that of the former proof, betweenness implies $V(w,1) > V(w-L,\varepsilon;w+G,1-\varepsilon)$ and first claim of the theorem is satisfied; or, for all $x \in [w, w+c]$,

$$u(x;\delta_w) > \tfrac{1}{2}[u(x-\ell;\delta_w) + u(x+g;\delta_w)]$$

and, similarly to the proof of Theorem 1, betweenness implies that $V(w,1) > V(w-\bar{L},\frac{1}{2};w+c+\bar{G},\frac{1}{2})$ and the second claim of the theorem is satisfied. ■

Proof of Claim 1 Suppose V satisfies (ℓ, g) random risk aversion with respect to binary lotteries. The local utility of V at δ_w is

$$u(x;\delta_w) = \begin{cases} u(x) & x \leqslant w \\ \frac{u(x)+\beta u(w)}{1+\beta} & x > w \end{cases}$$

By the proof of Theorem 2, for every $w \in [r,s]$, either for all $x \leqslant w$ or for all $x \geqslant w$,

$$u(x;\delta_w) > \tfrac{1}{2}u(x-\ell;\delta_w) + \tfrac{1}{2}u(x+g;\delta_w) \tag{22}$$

Let $w^* = \sup\{w : \forall x \leqslant w \text{ eq. (22) is satisfied}\}$. By the definition of the local utility, for all $x \leqslant w^*$, $u(x) > \frac{1}{2}u(x-\ell) + \frac{1}{2}u(x+g)$. Also, by the definition of w^* and by the proof of Theorem 2, for all $x > w > w^*$, $v(x) > \frac{1}{2}v(x-\ell)+\frac{1}{2}v(x+g)$, where $v(x) = \frac{u(x)+\beta u(w)}{1+\beta}$. As v is an affine transformation of u, u too satisfies this property.

Suppose now that for all x, $u(x) > \frac{1}{2}u(x-\ell) + \frac{1}{2}u(x+g)$. Then for every w and x, eq. (22) is satisfied. The local utility $u(\cdot;\delta_w)$ therefore rejects $(x-\ell,\frac{1}{2};x+g,\frac{1}{2})$ at all x. Starting from $X = (x_1, p; x_2, 1-p) \sim \delta_w$, we observe that by betweenness the local utility at X is $u(\cdot;\delta_w)$, and that this local utility rejects the (ℓ, g) randomness that is added to W. By betweenness, this randomness is also rejected by the functional V. ■

Proof of Lemma 1 Suppose $u(\cdot;F)$ is not concave. Then there exist H^* and H such that H is a mean preserving spread of H^*, but

$$\int u(x;F)dH^*(x) < \int u(x;F)dH(x) \tag{23}$$

For every ε, $(1-\varepsilon)F + \varepsilon H$ is a mean preserving spread of $(1-\varepsilon)F + \varepsilon H^*$, hence, by risk aversion, $V((1-\varepsilon)F + \varepsilon H) \leqslant V((1-\varepsilon)F + \varepsilon H^*)$. As this inequality holds for all ε, it follows that

$$\frac{\partial}{\partial \varepsilon} V((1-\varepsilon)F + \varepsilon H)\Big|_{\varepsilon=0} \leqslant \frac{\partial}{\partial \varepsilon} V((1-\varepsilon)F + \varepsilon H^*)\Big|_{\varepsilon=0}$$

Hence, by equation (14),

$$\int u(x;F)dH(x) \leqslant \int u(x;F)dH^*(x)$$

A contradiction to inequality (23). ■

Proof of Theorem 3

Case 1: Assume H1. As $V(b,1) > V(b-\ell, \frac{1}{2}; b+g, \frac{1}{2})$, it follows by continuity that there are p and $\zeta > 0$ such that for all $p' \in (p, p+\zeta)$ [16]

$$V(b,1) \geqslant V(b, 1-p; b-\ell, \tfrac{p}{2}; b+g, \tfrac{p}{2}) > V(b, 1-p'; b-\ell, \tfrac{p'}{2}; b+g, \tfrac{p'}{2}) \quad (24)$$

Denote by F the distribution of $(b, 1-p; b-\ell, \frac{p}{2}; b+g, \frac{p}{2})$. By differentiability, obtain that for a sufficiently close $p', p'' \in [p, p+\zeta]$, $p' < p''$,

$$\begin{aligned}(1-p')u(b;F) + \tfrac{p'}{2}u(b-\ell;F) + \tfrac{p'}{2}u(b+g;F) > \\ (1-p'')u(b;F) + \tfrac{p''}{2}u(b-\ell;F) + \tfrac{p''}{2}u(b+g;F)\end{aligned}$$

Hence at $x = b$,

$$u(x;F) > \tfrac{1}{2}u(x-\ell;F) + \tfrac{1}{2}u(x+g;F) \quad (25)$$

By H1 the inequality holds for all $x \leqslant b$. Taking $b-\ell$ to be the wealth level, Table 2 implies

$$u(b-\ell;F) > \varepsilon u(b-\ell-L;F) + (1-\varepsilon)u(b-\ell+G;F) \quad (26)$$

[16] Of course, quasi concavity implies inequality (24) for all $p' > p$ for some $p \in [0,1]$. But as we will use this part of the proof in the proof of Theorem 4 below, where quasi concavity is not assumed, we prefer to rely here on continuity.

Hence,

$$\int u(x;F)dF(x) =$$
$$\tfrac{p}{2}u(b-\ell;F) + (1-p)u(b;F) + \tfrac{p}{2}u(b+g;F) >$$
$$u(b-\ell;F) >$$
$$\varepsilon u(b-\ell-L;F) + (1-\varepsilon)u(b-\ell+G;F)$$

which implies that, according to the local utility at F, F is preferred to the lottery $(b-\ell-L,\varepsilon;b-\ell+G,1-\varepsilon)$. To conclude this case assume, by way of negation, that $V(F) \leqslant V(b-\ell-L,\varepsilon;\, b-\ell+G,1-\varepsilon)$. Then, by quasi concavity, all the lotteries in the interval connecting F and $(b-\ell-L,\varepsilon;$ $b-\ell+G,1-\varepsilon)$ are weakly preferred to F. By Gâteaux differentiability, this implies $u(b;F) \leqslant \varepsilon u(b-\ell-L;F) + (1-\varepsilon)u(b-\ell+G;F)$, a contradiction to eq. (26). Hence,

$$V(b,1) \geqslant V(F) > V(b-\ell-L,\varepsilon;b-\ell+G,1-\varepsilon)$$

where the first inequality follows by eq. (24). By monotonicity,

$$V(b,1) > V(b-g-L,\varepsilon;b-g+G,1-\varepsilon)$$

Case 2: Assume H2 and consider $x \in [a,b-g]$. As in Case 1, the preference $V(x,1) > V(x-\ell,\frac{1}{2};x+g,\frac{1}{2})$ implies the existence of p such that, at F, the distribution of $(x,1-p;x-\ell,\frac{p}{2};x+g,\frac{p}{2})$,

$$u(x;F) > \tfrac{1}{2}u(x-\ell;F) + \tfrac{1}{2}u(x+g;F)$$

Obviously, δ_b dominates F by first order stochastic dominance whenever $b > x+g$. Hence, by H2, $u(x;\delta_b) > \frac{1}{2}u(x-\ell;\delta_b) + \frac{1}{2}u(x+g;\delta_b)$ for all $x \in [a,b-g]$. Table 2 now implies

$$\begin{aligned} u(b;\delta_b) > u(b-g;\delta_b) > \\ \varepsilon u(b-g-L;\delta_b) + (1-\varepsilon)u(b-g+G;\delta_b) \end{aligned} \tag{27}$$

This inequality means that as we start moving from δ_b in the direction of $(b-g-L,\varepsilon;b-g+G,1-\varepsilon)$, the utility V decreases. By quasi concavity,

$$V(b,1) > V(b-g-L,\varepsilon;b-g+G,1-\varepsilon)$$

Case 3: Assume ¬H1. Similarly to Case 1, the preference $V(a,1) > V(a-\ell,\frac{1}{2};a+g,\frac{1}{2})$ implies the existence of p such that at F, the distribution of $(a,1-p;a-\ell,\frac{p}{2};a+g,\frac{p}{2})$, the inequality

$$u(a;F) > \tfrac{1}{2}u(a-\ell;F) + \tfrac{1}{2}u(a+g;F)$$

is satisfied and

$$V(a,1) \geqslant V(F) \tag{28}$$

By ¬H1, the inequality holds for all $x \geqslant a$ and, by Table 1,

$$u(a;F) > \tfrac{1}{2}u(a-\bar{L};F) + \tfrac{1}{2}u(a+\bar{G};F)$$

Set $u(a;F)=0$ and use the concavity of $u(\cdot;F)$ (Lemma 1) to obtain

$$u(a-\ell;F) > \tfrac{\ell}{\bar{L}}u(a-\bar{L};F)$$

Using the last two inequalities we obtain

$$\begin{aligned}
&\int u(x;F)dF(x) = \\
&\tfrac{p}{2}u(a-\ell;F) + (1-p)u(a;F) + \tfrac{p}{2}u(a+g;F) > \\
&\tfrac{p}{2}u(a-\ell;F) + \left(1-\tfrac{p}{2}\right)u(a;F) > \\
&\tfrac{p}{2}\tfrac{\ell}{\bar{L}}u(a-\bar{L};F) + \left(1-\tfrac{p}{2}\right)\left[\tfrac{1}{2}u(a-\bar{L};F) + \tfrac{1}{2}u(a+\bar{G};F)\right] > \\
&(\tfrac{\ell}{2\bar{L}})u(a-\bar{L};F) + \tfrac{1}{2}\left[\tfrac{1}{2}u(a-\bar{L};F) + \tfrac{1}{2}u(a+\bar{G};F)\right] > \\
&\tfrac{3}{4}u(a-\bar{L};F) + \tfrac{1}{4}u(a+\bar{G};F)
\end{aligned}$$

where the last inequality holds since $\frac{\ell}{2\bar{L}} < \frac{1}{2}$ and $u(a-\bar{L};F) < 0$. As before, $\int u(x;F)dF(x) > \frac{3}{4}u(a-\bar{L};F) + \frac{1}{4}u(a+\bar{G};F)$ means that V decreases as we start moving from F in the direction of $(a-\bar{L};\frac{3}{4};a+\bar{G},\frac{1}{4})$. Finally, eq. (28) and quasi concavity imply

$$V(a,1) \geqslant V(F) > V(a-\bar{L},\tfrac{3}{4};a+\bar{G},\tfrac{1}{4})$$

Case 4: Assume ¬H2 and obtain, similarly to Case 2, that $u(x;\delta_a) > \frac{1}{2}u(x-\ell;\delta_a) + \frac{1}{2}u(x+g;\delta_a)$ for all $x \in [a+\ell,b]$. Now, by Table 1,

$$u(a+\ell;\delta_a) > \tfrac{1}{2}u(a+\ell-\bar{L};\delta_a) + \tfrac{1}{2}u(a+\ell+\bar{G};\delta_a)$$

Set $u(a+\ell;\delta_a)=0$ and use the concavity of $u(\cdot;\delta_a)$ to obtain

$$u(a;\delta_a) > \tfrac{\ell}{\bar{L}} u(a+\ell-\bar{L};\delta_a)$$

Summing the last two inequalities yields (recall that $u(a+\ell;\delta_a)=0$)

$$\begin{aligned} u(a;\delta_a) &> \left(\tfrac{1}{2}+\tfrac{\ell}{\bar{L}}\right) u(a+\ell-\bar{L};\delta_a) + \tfrac{1}{2}u(a+\ell+\bar{G};\delta_a) \\ &> \left(\tfrac{1}{2}+\tfrac{\ell}{\bar{L}}\right) u(a+\ell-\bar{L};\delta_a) + \left(\tfrac{1}{2}-\tfrac{\ell}{\bar{L}}\right) u(a+\ell+\bar{G};\delta_a) \qquad (29) \\ &> \tfrac{3}{4}u(a+\ell-\bar{L};\delta_a) + \tfrac{1}{4}u(a+\ell+\bar{G};\delta_a) \end{aligned}$$

Where the last inequality holds since $\frac{\ell}{\bar{L}} < \frac{1}{4}$. Finally, quasi concavity and monotonicity imply

$$\begin{aligned} V(a,1) &> V(a+\ell-\bar{L},\tfrac{3}{4};a+\ell+\bar{G},\tfrac{1}{4}) \\ &> V(a-\bar{L},\tfrac{3}{4};a+\bar{G},\tfrac{1}{4}) \end{aligned} \qquad \blacksquare$$

Proof of Theorem 4 Follow the proof of Theorem 3 Case 2 up to the conclusion that an expected utility decision maker with the vNM utility $u(\cdot;\delta_b)$ satisfies $u(b;\delta_b) > \varepsilon u(b-g-L;\delta_b) + (1-\varepsilon)u(b-g+G;\delta_b)$ (see eq. (27)). By Gâteaux differentiability, this implies that for sufficiently small μ, the decision maker with wealth level b prefers not to participate in the lottery $(b-g-L,\mu\varepsilon;b,1-\mu;b-g+G,\mu(1-\varepsilon))$. We now show that $\mu=1$, which is the claim of the theorem.

Let $\bar{\mu} = \max\{\mu : V(b,1) \geqslant V(b-g-L,\mu\varepsilon;b,1-\mu;b-g+G,\mu(1-\varepsilon))\}$ and suppose that $\bar{\mu}<1$. Denote by $\bar{F}$ the distribution of the lottery $(b-g-L,\bar{\mu}\varepsilon;b,1-\bar{\mu};b-g+G,\bar{\mu}(1-\varepsilon))$. We want to show that for all $x\in[b-g-L,b-g]$,

$$u(x;\bar{F}) > \tfrac{1}{2}u(x-\ell;\bar{F}) + \tfrac{1}{2}u(x+g;\bar{F}) \qquad (30)$$

By approximate risk aversion,

$$\begin{aligned} \hat{X} &:= (b-g-L,\bar{\mu}\varepsilon;x,1-\bar{\mu}\varepsilon) \succ \\ \tilde{X} &:= \left(b-g-L,\bar{\mu}\varepsilon;x-\ell,\tfrac{1-\bar{\mu}\varepsilon}{2};x+g,\tfrac{1-\bar{\mu}\varepsilon}{2}\right) \end{aligned}$$

Let $\hat{F}$ and $\tilde{F}$ denote the distributions of $\hat{X}$ and $\tilde{X}$, respectively. Similarly to the derivation of eq. (25), it follows by Gâteaux differentiability that there exists F in the line segment connecting $\hat{F}$ and $\tilde{F}$ for which

$$u(x;F) > \tfrac{1}{2}u(x-\ell;F) + \tfrac{1}{2}u(x+g;F)$$

As $\bar{F}$ dominates both $\hat{F}$ and $\tilde{F}$ by first order stochastic dominance it dominates F as well and eq. (30) follows by H2.

Following an analysis similar to that of the proof of Theorem 3, the local utility at $\bar{F}$ satisfies

$$u(b;\bar{F}) > \varepsilon u(b-g-L;\bar{F}) + (1-\varepsilon)u(b-g+G;\bar{F}) \tag{31}$$

Let H denote the cumulative distribution function of $(b-g-L,\varepsilon;b-g+G,1-\varepsilon)$. Then, by Gâteaux differentiability and eq. (31),

$$\begin{aligned}&\frac{\partial}{\partial t}V((1-t)\bar{F}+tH) = \\ &\quad (1-\bar{\mu})[\varepsilon u(b-g-L;\bar{F}) + (1-\varepsilon)u(b-g+G;\bar{F}) - u(b;\bar{F})] < 0\end{aligned}$$

But this means that we can go beyond $\bar{\mu}$ (following the analysis we had at the beginning of Case 2 of the proof of Theorem 3); a contradiction. Hence $\bar{\mu} = 1$ and

$$V(b,1) > V(b-g-L,\varepsilon;b-g+G,1-\varepsilon) \qquad \blacksquare$$

Proof of Theorem 5 Assume H1* (the analysis of the case where ¬H1* is satisfied is similar). Let $\ell(\eta) = \ell(1+2\eta)$ and $g(\eta) = g(1-2\eta)$. Choose $\eta > 0$ such that $\ell(\eta) < g(\eta)$ and $\theta = 2m\eta$, where $m \leqslant \inf f'$. Continuous differentiability and boundedness of f' imply the existence of n such that $|p-q| \leqslant \frac{1}{n} \Rightarrow |f'(p) - f'(q)| < \theta$. Fix n. By the mean value theorem, for every i there are points $r \in [\frac{i-1}{n}, \frac{2i-1}{2n}]$, $s \in [\frac{2i-1}{2n}, \frac{i}{n}]$, and $t \in [\frac{i-1}{n}, \frac{i}{n}]$ such that

$$\begin{aligned}f'(r) &= \frac{f\left(\frac{2i-1}{2n}\right) - f\left(\frac{i-1}{n}\right)}{\frac{1}{2n}} \\ f'(s) &= \frac{f\left(\frac{i}{n}\right) - f\left(\frac{2i-1}{2n}\right)}{\frac{1}{2n}} \\ f'(t) &= \frac{f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)}{\frac{1}{n}}\end{aligned}$$

As $|r-t|, |s-t| \leqslant \frac{1}{n}$, we obtain $|f'(r) - f'(t)|, |f'(s) - f'(t)| < \theta$, hence

$$\left|\frac{f\left(\frac{2i-1}{2n}\right) - f\left(\frac{i-1}{n}\right)}{f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)} - \frac{1}{2}\right| < \eta \text{ and } \left|\frac{f\left(\frac{i}{n}\right) - f\left(\frac{2i-1}{2n}\right)}{f\left(\frac{i}{n}\right) - f\left(\frac{i-1}{n}\right)} - \frac{1}{2}\right| < \eta \tag{32}$$

To see why, observe that

$$\frac{f\left(\frac{2i-1}{2n}\right)-f\left(\frac{i-1}{n}\right)}{f\left(\frac{i}{n}\right)-f\left(\frac{i-1}{n}\right)} = \frac{f'(r)}{2f'(t)} \in \left(\frac{f'(t)\pm\theta}{2f'(t)}\right) = \left(\frac{1}{2} \pm \frac{\theta}{2f'(t)}\right) \subset \left(\frac{1}{2} \pm \frac{\theta}{2m}\right) = \left(\frac{1}{2} \pm \eta\right)$$

Denote $d = d(\eta) := n(\ell + g)$ and consider the lottery $X = (a + (\ell + g)i, \frac{1}{n})_{i=1}^{n}$ satisfying $\text{supp}(X) \subseteq [a, a+d]$. By random risk aversion on $[a, a+d]$, X is superior to the lottery $(X - \ell, \frac{1}{2}; X + g, \frac{1}{2}) = (a + (\ell + g)i - \ell, \frac{1}{2n}; a + (\ell + g)i + g, \frac{1}{2n})_{i=1}^{n}$. Using the rank dependent form we obtain that

$$\begin{aligned}\textstyle\sum_{i=1}^{n} \Big\{ & v(a + (\ell + g)i) \left[f\left(\tfrac{i}{n}\right) - f\left(\tfrac{i-1}{n}\right)\right] - \\ & \Big(v(a + (\ell + g)i - \ell) \left[f\left(\tfrac{2i-1}{2n}\right) - f\left(\tfrac{i-1}{n}\right)\right] + \\ & v(a + (\ell + g)i + g) \left[f\left(\tfrac{i}{n}\right) - f\left(\tfrac{2i-1}{2n}\right)\right]\Big)\Big\} > 0\end{aligned}$$

There is therefore j, such that

$$\begin{aligned} & v(a + (\ell + g)j) \left[f\left(\tfrac{j}{n}\right) - f\left(\tfrac{j-1}{n}\right)\right] > \\ & \quad v(a + (\ell + g)j - \ell) \left[f\left(\tfrac{2j-1}{2n}\right) - f\left(\tfrac{j-1}{n}\right)\right] + \\ & \quad v(a + (\ell + g)j + g) \left[f\left(\tfrac{j}{n}\right) - f\left(\tfrac{2j-1}{2n}\right)\right]\end{aligned}$$

Denote $x^* = a + (\ell + g)j$. By H1*, for all $x \leqslant x^*$ (and in particular, for $x = a$),

$$\begin{aligned} & v(x) \left[f\left(\tfrac{j}{n}\right) - f\left(\tfrac{j-1}{n}\right)\right] > \\ & \quad v(x - \ell) \left[f\left(\tfrac{2j-1}{2n}\right) - f\left(\tfrac{j-1}{n}\right)\right] + v(x + g) \left[f\left(\tfrac{j}{n}\right) - f\left(\tfrac{2j-1}{2n}\right)\right]\end{aligned}$$

Following the analysis of section 2, assume, without loss of generality, that $u(x - \ell) = 0$ and $u(x) = \ell$. By the last inequality and inequality (32),

$$v(x + g) < \ell \, \frac{\left[f\left(\frac{j}{n}\right)-f\left(\frac{j-1}{n}\right)\right]}{\left[f\left(\frac{j}{n}\right)-f\left(\frac{2j-1}{2n}\right)\right]} < \frac{\ell}{\frac{1}{2}-\eta} = \frac{2\ell}{1-2\eta}$$

Obviously, $v'(x - \ell) \geqslant 1$ and

$$v'(x + g) < \frac{\frac{2\ell}{1-2\eta}-\ell}{g} = \frac{\ell}{g}\frac{(1+2\eta)}{(1-2\eta)} \leqslant \frac{\ell}{g}\frac{(1+2\eta)}{(1-2\eta)} v'(x - \ell) \tag{33}$$

As $\ell(\eta) < g(\eta)$, inequality (33) is similar to inequality (1). Similarly to inequality (10), an expected utility decision maker with the vNM utility function v rejects all lotteries $(L, \varepsilon; G(\eta), 1 - \varepsilon)$ satisfying

$$p(\ell(\eta) + g(\eta)) \frac{\left(\frac{g(\eta)}{\ell(\eta)}\right)^{\frac{b-a}{\ell+g}} - 1}{\frac{g(\eta)}{\ell(\eta)} - 1} > (1 - p)G(\eta)$$

Obviously, $G(\eta) \to_{\eta \downarrow 0} G$. Risk aversion implies $f(p) \geqslant p$, therefore at a, the decision maker rejects the lottery $(-L, \varepsilon; G(\eta), 1-\varepsilon)$. ■

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