Kidney Exchange with Good Samaritan Donors: A Characterization

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Abstract

We analyze mechanisms to kidney exchange with good samaritan donors where exchange is feasible not only among donor-patient pairs but also among such pairs and non-directed alturistic donors. We show that *you request my donor-I get your turn* mechanism (Abdulkadiroğlu and Sönmez [1999]) is the only mechanism that is *Pareto efficient, individually rational, strategy-proof, weakly neutral* and *consistent*.

Keywords: Kidney exchange, matching, strategy-proofness, consistency.

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1 Introduction

Transplantation is the preferred treatment for the most serious forms of kidney disease. Unfortunately there is a considerable shortage of deceased-donor kidneys, compared to demand. Because healthy people have two kidneys and can remain healthy on one, it is also possible for a kidney patient to receive a live-donor transplant. There were 6,086 live-donor transplants in the U.S. in 2004. However, a willing, healthy donor may not always be able to donate to her intended patient, because of either blood-type or immunological incompatibilities. Rapaport [1986] is the first to propose *kidney exchange* between two such incompatible pairs in case each donor can feasibly donate a kidney to the patient of the other pair. Ross et. al [1997] reinforced this idea and in 2000 transplantation community issued a consensus statement declaring kidney exchange to be ethically acceptable (Abecassis et. al [2000]). In the period 2000-2004 feasible exchanges were sought in an unorganized way in parts of the U.S., and a relatively small number of them has been carried out. Roth, Sönmez, and Ünver [2004] (henceforth RSÜ [2004]) observed that the impact of this idea can be significantly increased if exchange is organized and modeled kidney exchange as a mechanism design problem. Since then, centralized clearinghouses for kidney exchange has been established in New England (see Roth, Sönmez, and Ünver [2005]), Ohio and Maryland.[1]

The two main sources of kidneys for transplantation is deceased-donor kidneys and live-donations from family and friends. U.S. congress views deceased-donor kidneys offered for transplantation as a national resource, and the National Organ Transplant Act of 1984 established the Organ Procurement and Transplantation Network (OPTN). Run by the United Network for Organ Sharing (UNOS), OPTN has developed a centralized priority mechanism for allocation of deceased-donor kidneys. In addition to direct exchange between incompatible pairs, another form of exchange considered in the transplantation literature is an *indirect exchange* (Ross and Woodle [2000]). In this kind of exchange, the patient of the incompatible pair receives an upgrade in the deceased-donor priority list in exchange for donor's kidney. Unlike in direct exchange between incompatible pairs, certain patient groups may suffer a loss under indirect exchange (see Zenios, Woodle and Ross [2001]) and the transplantation community does not have a uniform view on its implementation. RSÜ [2004] considered both direct and indirect exchanges as well as their more elaborate nested versions. Currently indirect exchanges are considered by the New England Program for Kidney Exchange (see www.nepke.org) whereas only two pair direct exchanges are considered

[1] The efforts in New England and Ohio are both collaboration of several transplant centers whereas in Maryland, Johns Hopkins has a single-center kidney exchange program.

by the Ohio Solid Organ Transplantation Consortium. While indirect exchanges are also avoided by the Paired Kidney Exchange Transplant Program of Johns Hopkins, they have recently pursued a closely related idea: In May 2005, surgeons at Johns Hopkins performed an exchange between an altruistic, non-directed living donor (also known as a *Good Samaritan donor*), two incompatible patient-donor pairs, and a patient on the deceased-donor priority list. Unlike the deceased-donor kidneys, donations from Good Samaritan donors (henceforth GS-donors) are not regulated by the law. Nevertheless, rare donations from GS-donors have been mostly treated similar to deceased-donor kidneys and allocated through the centralized priority mechanism. In this way each GS-donor gives a gift of life to a stranger on the priority list. In the recent exchange at Johns Hopkins, however,

- the kidney from the GS-donor is transplanted to the patient of the first incompatible pair,
- the kidney from the first incompatible pair is transplanted to the patient of the second incompatible pair, and
- the kidney from the second incompatible pair is transplanted to the highest priority patient on the deceased-donor priority list.

In this way, not only the GS-donor gave a gift of life to a stranger, but she also facilitated two others which otherwise would not be possible. Organization of such exchanges is the theme of this paper and we analyze mechanisms for kidney exchange with GS-donors.

As we have already mentioned, the allocation of deceased-donor kidneys is regulated and there is some resistance in the transplantation community against integrating kidney exchange with deceased-donor priority lists through indirect exchange. In contrast, currently, the allocation of kidneys from GS-donors is not regulated and there is flexibility on its allocation. As the Johns Hopkins example illustrates, there are potential gains from integrating allocation of GS-donations with kidney exchange. We consider a model where kidneys from GS-donors are initially offered to the kidney exchange pool, and only if they are unassigned they are sent to the deceased-donor priority list. We analyze mechanisms that integrate donations from GS-donors with kidney exchange, and the interaction with the deceased-donor list is implicit in our paper. Observe that for any patient in the exchange pool who receives a kidney from a GS-donor, there exists a (not necessarily distinct) patient whose incompatible donor remains unassigned. While not explicitly modeled in the paper, we interpret these donors to be sent to the deceased-donor priority list (just as the unassigned GS-donors).

So the idea is, the kidney exchange pool "owes" a live-donor kidney to the deceased-donor priority list for each kidney transplanted to a patient in the pool from a GS-donor.

In our model each problem consists of a set of incompatible patient-donor pairs, a set of GS-donors whose kidneys are not "attached" to any particular patient, and a list of strict patient preferences on all donors. Given fixed sets of patients and donors, an allocation is a *matching* of patients and donors so that each patient is assigned one donor and no donor is assigned to more than one patient. A *mechanism* is a systematic procedure that selects a matching for each problem. This model is a special case of *house allocation with existing tenants* (Abdulkadiroğlu and Sönmez [1999]) model where there are a number of existing tenants each with an initial house, a number of vacant houses, and a number of newcomers none of whom has initial claims on any house. Patients and their incompatible donors in our model correspond to existing tenants and their initial houses, and GS-donors who are not attached to any particular patient correspond to vacant houses.

Abdulkadiroğlu and Sönmez [1999] introduce the following mechanism which we refer as *You Request My Donor-I Get Your Turn (YRMD-IGYT)* in the present context: Patients are prioritized in a queue and they are assigned their top choice donor among still unassigned donors in priority order. This continues until a patient "requests" the incompatible donor of a patient who has not been assigned a donor yet. In this case this request is put on hold; the patient whose incompatible donor is requested is moved to the top of the queue, directly in front of the requester and the process continues with the modified queue. This is repeated any time there is a request for the incompatible donor of a patient whose assignment is yet to be finalized. If a cycle of requests is formed, each patient in the cycle is assigned the donor she requested and removed from the system together with their assignments.

Abdulkadiroğlu and Sönmez [1999] showed that the YRMD-IGYT mechanism is *Pareto efficient, individually rational* (in the sense that each patient is guaranteed a donor that is no worse than her paired-donor) and *strategy-proof*. In this paper we present a full characterization of the YRMD-IGYT mechanism based on these three axioms together with *weak neutrality* and *consistency*. *Weak neutrality* requires the outcome of a mechanism to be independent of the names of the GS-donors. The formulation of *consistency* is less obvious in the present context. The traditional *consistency* axiom compares any pair of economies where one economy is obtained from the other by removal of a group of agents together with their assignments under the mechanism for which *consistency* is tested, and it requires this mechanism to insist on the same assignment as in the original economy for each remaining agent.[2] If a mechanism is *consistent*, then it eliminates incentives to renegotiate upon departure any group of agents with their assignments. This is very plausible in the present context for distinct exchanges in a matching are often conducted weeks, even months apart (although all transplants in a given exchange are conducted simultaneously due to incentives reasons). The difficulty, however, is that upon the removal of a group of patients with their assignments, what remains may not always be a well-defined problem. For example if two patients are assigned each others' paired-donors and if one of them leaves with her assignment, in what remains there is a patient with no paired-donor. A natural formulation here would be requiring *consistency* whenever the reduced economy is well-defined, but as it turns out this version is not strong enough for the full characterization of the YRMD-IGYT mechanism.[3] For full characterization we also need a mechanism to insist on its outcome if a set of *unassigned* donors are removed (provided that what remains is a well-defined economy). The *consistency* axiom we present in the paper is the following: When a group of patients are removed from a problem together with their assignments under a mechanism ϕ and possibly together with some unassigned donors under ϕ, what remains may not be a well-defined problem. But if it is, then the assignments of the remaining patients under mechanism ϕ should not be affected by this departure. So if some of the exchanges are finalized while the others are pending, and even if some unassigned GS-donors have a change of heart and they are no longer willing for alturistic donation, the remaining patients should still have no reason to request another run of the mechanism.

1.1 Related Literature

As we have already indicated, *kidney exchange* as an application of economic theory is recently brought to the attention of economists by RSÜ [2004]. Roth, Sönmez and Ünver [2005b] considers a related model where each patient is indifferent between all compatible kidneys and no exchange can involve more than two pairs. Our modeling choice is closer to the first of these two papers: Just as RSÜ [2004], our model is a generalization of *housing markets* (Shapley and Scarf [1974]); but unlike RSÜ [2004], our model is a special case of *house allocation with existing tenants* model. Although we are unaware of any characterization result for the latter model, there is a rich axiomatic literature on *allocation of indivisible goods* in general including in housing markets and in *house allocation problems* (Hylland and

[2] See Thomson [1996] for a comprehensive survey.

[3] The mechanism in Example 5 satisfies all four other axioms and this version of *consistency* but not the stronger version we present in the paper.

Zeckhauser [1977]).

YRMD-IGYT mechanism is a generalization of both *Gale's Top Trading Cycles* mechanism for housing markets and the *simple serial dictatorship* for house allocation. When preferences are strict, *Gale's Top Trading Cycles* mechanism gives the unique core outcome of a housing market (Roth and Postlewaite 1977) and it is *strategy-proof* (Roth 1982). Indeed it is the only mechanism that is *Pareto efficient, individually rational,* and *strategy-proof* (Ma 1994). In the context of housing markets, Svensson [1999] shows that the *simple serial dictatorship* is the only mechanism that is *strategy-proof, nonbossy* and *neutral* while Ergin [2000] shows it is the only mechanism that is *Pareto efficient, consistent* and *neutral*. Our characterization is a natural generalization of each of Ma [1994], Svensson [1999], and Ergin [2000] results.[4]

2 Kidney Exchange with Good Samaritan Donors

Let $\mathcal{I}$ be a finite set of patients and $\mathcal{D}$ be a finite set of donors such that $|\mathcal{D}| \geq |\mathcal{I}|$. Each patient $i \in \mathcal{I}$ has a distinct **paired-donor** $d_i \in \mathcal{D}$ and has strict preferences P_i on all donors in $\mathcal{D}$. Let R_i denote the weak preference relation induced by R_i and for any subset of donors $D \subset \mathcal{D}$, let $\mathcal{R}(D)$ denote the set of all strict preferences over D.

A **kidney exchange problem with good samaritan donors**, or simply a **problem**, is a triple $\langle I, D, R\rangle$ where:

- $I \subseteq \mathcal{I}$ is any set of patients,
- $D \subseteq \mathcal{D}$ is any set of donors such that $d_i \in D$ for any $i \in I$, and,
- $R = (R_i)_{i\in I} \in [\mathcal{R}(D)]^{|I|}$ is a preference profile.

Given a problem $\langle I, D, R\rangle$, the set of "unattached" donors $D\setminus\{d_i\}_{i\in I}$ is referred as **Good Samaritan donors** (or in short **GS-donors**). Observe that the paired-donor d_j of a patient j is formally a GS-donor in a problem $\langle I, D, R\rangle$ if $d_j \in D$ although $j \notin I$.[5]

Since the information on patients and donors are embedded in a preference profile, whenever convenient we will denote a problem with simply a preference profile.

[4] Other axiomatic studies in housing markets and house allocation include Chambers [2004], Ehlers [2002], Ehlers and Klaus [2005], Ehlers, Klaus and Papai [2002], Kesten [2004], Miyagawa [2002], Papai [2000, 2004].

[5] This observation will be useful when we formalize the consistency axiom later on. We will be considering such situations as patient j being assigned a GS-donor and leaving the problem. The paired-donor d_j of patient j is no longer attached to any patient in the reduced problem, and hence treated as a GS-donor.

Given $I \subseteq \mathcal{I}$ and $D \subseteq \mathcal{D}$, a **matching** is a mapping $\mu : I \to D$ such that

$$\mu(i) = c \text{ and } \mu(j) = d \quad \Rightarrow \quad c \neq d \qquad \text{for any distinct } i, j \in I.$$

We refer $\mu(i)$ as the **assignment** of patient i. A matching is simply an assignment of donors to patients such that each patient is assigned one donor and no donor is assigned to more than one patient. Let $\mathcal{M}(I, D)$ denote the set of matchings for given I, D.

A **mechanism** is a systematic procedure that assigns a matching for each problem R. The outcome of mechanism ϕ for problem R is denoted by $\phi[R]$ and the assignment of patient i under ϕ for problem R is denoted by $\phi[R](i)$. For any $J \subseteq I$, let $\phi[R](J) = \{\phi[R](j)\}_{j \in J}$ be the set of donors assigned to patients in J.

3 The Axioms

3.1 Individual Rationality, Pareto Efficiency and Strategy-Proofness

Throughout this section, we fix $I \subseteq \mathcal{I}$ and $D \subseteq \mathcal{D}$.

A matching is ***individually rational*** if no patient is assigned a donor worse than her paired-donor. Formally, a matching $\mu \in \mathcal{M}$ is *individually rational* if, $\mu(i) R_i d_i$ for any $i \in I$. A mechanism is ***individually rational*** if it always selects an individually rational matching.

A matching is ***Pareto efficient*** if there is no other matching that makes every patient weakly better off and some patient strictly better off. Formally, a matching $\mu \in \mathcal{M}$ is *Pareto efficient* if there is no matching $\nu \in \mathcal{M}$ such that $\nu(i) R_i \mu(i)$ for all $i \in I$ and $\nu(j) P_j \mu(j)$ for some $j \in I$. A mechanism is ***Pareto efficient*** if it always selects a Pareto efficient matching.

A mechanism is ***strategy-proof*** if no patient can ever benefit by misrepresenting her preferences. Formally a mechanism ϕ is *strategy-proof* if for any problem $R \in [\mathcal{R}(D)]^{|I|}$, any patient $i \in I$, and any potential misrepresentation $R_i^* \in \mathcal{R}(D)$, we have $\phi[R_i, R_{-i}](i) R_i \phi[R_i^*, R_{-i}](i)$.

3.2 Weak Neutrality and Consistency

Each of the three axioms we introduced so far is defined for fixed sets of patients and donors. In contrast, our next axiom *weak neutrality* relates problems with possibly different sets of donors and final axiom *consistency* relates problems with different sets of patients and donors.

A mechanism is ***weakly neutral*** if labeling of GS-donors has no affect on the outcome of the mechanism.

We need additional notation to introduce our final axiom.

Fix I, D. For any patient $i \in I$, preference relation $R_i \in \mathcal{R}(D)$, and set of donors $C \subset D$, let R_i^C be the **restriction of preference R_i to donors in C**.[6] That is,

$$cR_i^C d \quad \Longleftrightarrow \quad cR_i d \qquad \text{for any } c, d \in C.$$

For any $J \subset I$, let $R_J = (R_i)_{i \in J}$ be the **restriction of profile R to patients in J**.[7] For any $J \subset I$ and $C \subset D$, let $R_J^C = (R_i^C)_{i \in J}$ be the **restriction of profile R to patients in J and donors in C.**

Given a problem $\langle I, D, R \rangle$, a set of patients $J \subset I$, and a set of donors $C \subset D$, we refer $\left\langle J, C, R_J^C \right\rangle$ as the **restriction of problem $\langle I, D, R \rangle$ to patients in J and donors in C.** The triple $\left\langle J, C, R_J^C \right\rangle$ itself is a **well-defined reduced problem** if $d_j \in C$ for any $j \in J$.

Given a problem $\langle I, D, R \rangle$, a set of donors $C \subset D$ is **unassigned** under mechanism ϕ if $\phi[R](I) \cap C = \emptyset$.

Given a problem $\langle I, D, R \rangle$, the removal of a set of patients $J \subset I$ together with their assignments $\phi[R](J)$ under ϕ and a set of unassigned donors $C \subset D$ under ϕ results in a well-defined reduced problem $\left\langle I \setminus J,\ D \setminus (\phi[R](J) \cup C),\ R_{-J}^{-\phi[R](J) \cup C} \right\rangle$ if

$$(\phi[\,R\,](\,J\,) \cup C) \cap \{d_i\}_{i \in I \setminus J} = \emptyset.$$

A mechanism ϕ is ***consistent*** if for any problem $\langle I, D, R \rangle$, whenever the removal of a set of patients $J \subset I$ together with their assignments $\phi[R](J)$ and a (possibly empty) set of unassigned donors $C \subset D$ results in a well-defined reduced problem,

$$\phi[R_{-J}^{-\phi[R](J) \cup C}]\,(i) = \phi\,[R]\,(i) \qquad \text{for any } i \in I \setminus J.$$

So under a *consistent* mechanism, the removal of

- a set of patients,
- their assignments, and

[6] Given $R_i \in \mathcal{R}(D)$, we will often denote $R_i^{D \setminus C}$ by R_i^{-C}.

[7] Given fixed $D \in \mathcal{D}$ and $R \in [\mathcal{R}(D)]^{|I|}$, we will often denote $R_{I \setminus J}$ by R_{-J}.

- some unassigned donors

does not affect the assignments of remaining patients provided that the removal results in a well-defined reduced problem. As we have argued in the introduction, distinct kidney exchanges are often performed months apart and *consistency* removes the incentives to request a new run of the mechanism upon completion of part of the exchanges.

4 You Request My Donor-I Get Your Turn Mechanism

You Request My Donor-I Get Your Turn mechanism (or *YRMD-IGYT* mechanism in short) is introduced by Abdulkadiroğlu and Sönmez [1999] in the context of *house allocation with existing tenants* and further studied by Chen and Sönmez [2002] and Sönmez and Ünver [2005].[8] In order to define this mechanism we need the following additional notation:

A **(priority) ordering** is a one-to-one and onto function $f : \{1, 2, \dots, |\mathcal{I}|\} \to \mathcal{I}$. Here $f(1)$ indicates the patient with the highest priority in $\mathcal{I}$, $f(2)$ indicates the patient with the second highest priority in $\mathcal{I}$, and so on. Let $\mathcal{F}$ be the set of all orderings. Given a set of patients $J \in \mathcal{I}$, patient $j \in J$ is the **highest priority patient in** J **under** f if $f^{-1}(j) \leq f^{-1}(i)$ for any $i \in J$. Given a set of patients $J \in \mathcal{I}$, the **restriction of** f **to** J is an ordering f_J of the patients in J which orders them as they are ordered in f. Formally $f_J : \{1, 2, \dots, |J|\} \longrightarrow J$ is a one-to-one and onto function such that for any $i, j \in J$,

$$f_J^{-1}(i) \leq f_J^{-1}(j) \Longleftrightarrow f^{-1}(i) \leq f^{-1}(j).$$

Each ordering $f \in \mathcal{F}$ defines a YRMD-IGYT mechanism. Let ψ^f denote the YRMD-IGYT mechanism induced by ordering $f \in \mathcal{F}$. For any set of patients $I \subset \mathcal{I}$ and set of donors $C \subset \mathcal{D}$, let $\psi^f[R_J^C]$ denote the outcome of the YRMD-IGYT mechanism induced by ordering f_J for problem $\left\langle J, C, R_J^C \right\rangle$.

For any problem $\langle I, D, R \rangle$, matching $\psi^f[R]$ is obtained with the following YRMD-IGYT algorithm in several rounds.

Round 1(a): Construct a graph in which each patient and each donor is a node. In this graph:

[8] YRMD-IGYT mechanism is a generalization of both *Gale's Top Trading Cycles* mechanism (for *housing markets* (Shapley and Scarf 1974)), and *serial dictatorship* (for *house allocation problems* (Hylland and Zeckhauser 1977)). Abdulkadiroğlu and Sönmez [1999] provided two algorithms, *You Request My House-I Get Your Turn (YRMH-IGYT)* algorithm and the *Top Trading Cycles (TTC)* algorithm, to implement this mechanism. The description we provide below is based on the description that utilizes the TTC algorithm.

- each patient "points to" her top choice donor (i.e. there is a directed link from each patient to her top choice donor),
- each paired-donor $d_i \in D$ points to her paired-patient i in case $i \in I$, and to the highest priority patient in I otherwise,
- and each GS-donor points to the patient with the highest priority in I.

Since there is a finite number of patients and donors, there is at least one cycle. (A **cycle** is an ordered list $(c_1, j_1, \ldots, c_k, j_k)$ of donors and patients where donor c_1 points to patient j_1, patient j_1 points to donor c_2, donor c_2 points to patient j_2, ..., donor c_k points to patient j_k, and patient j_k points to donor c_1.) If there is no cycle *without a GS-donor* then skip to Round 1(b). Otherwise consider each cycle without a GS-donor. (Observe that if there is more than one such cycle, they do not intersect.) Assign each patient in such a cycle the donor she points to and remove each such cycle from the graph. Construct a new graph with the remaining patients and donors such that

- each remaining patient points to her first choice among the remaining donors,
- each remaining paired-donor $d_i \in D$ points to her paired-patient i in case her paired patient i remains in the problem, and to the highest priority remaining patient otherwise,
- and each GS-donor points to the highest priority remaining patient.

There is a cycle. If there is no cycle without a GS-donor then skip to Round 1(b); otherwise carry out the implied exchange in each such cycle and proceed similarly until either no patient is left or there exists no cycle without a GS-donor.
Round 1(b): There is a unique cycle in the graph, and it includes both the highest priority patient among remaining patients and a GS-donor.[9] Assign each patient in such a cycle the donor she points to and remove each such cycle from the graph. Proceed with Round 2.

In general, at
Round t(a): Construct a new graph with the remaining patients and donors such that

- each remaining patient points to her first choice among the remaining donors,

[9]That is because each GS-donor points to the highest priority patient among remaining patients.

- each remaining paired-donor $d_i \in D$ points to her paired-patient i in case her paired patient i remains in the problem, and to the highest priority remaining patient otherwise,

- and each remaining GS-donor points to the highest priority remaining patient.

There is a cycle. If the only remaining cycle includes either a GS-donor or a paired-donor whose paired-patient has left, then skip to Round t(b); otherwise carry out the implied exchange in each such cycle and proceed similarly until either no patient is left or the only remaining cycle includes either a GS-donor or a paired-donor whose paired-patient has left.
Round t(b): There is a unique cycle in the graph, and it includes the highest priority patient among remaining patients and either a GS-donor or a paired-donor whose paired-patient has left. Assign each patient in such a cycle the donor she points to and remove each such cycle from the graph. Proceed with Round t+1.

The algorithm terminates when there is no patient left in the graph.

5 Characterization of the YRMD-IGYT Mechanisms

Our main result is a characterization of the YRMD-IGYT mechanism:

Theorem 1: A mechanism is *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral*, and *consistent* if and only if it is a YRMD-IGYT mechanism.

We present our main result through two propositions:

Proposition 1: For any ordering $f \in \mathcal{F}$, the induced YRMD-IGYT mechanism ψ^f is *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral* and *consistent*.

Proof of Proposition 1: Let $f \in \mathcal{F}$. *Pareto efficiency*, *individual rationality* and *strategy-proofness* of ψ^f follows from Abdulkadiroğlu and Sönmez [1999]. *Weak neutrality* of ψ^f directly follows from the description of the YRMD-IGYT algorithm (i.e., under the relabeled economy, the relabeled version of the same sequence of cycles will form).

We next prove that ψ^f is *consistent*. Fix a problem $\langle I, D, R\rangle$. Let $C \subset D$ be such that $\psi^f[R](I) \cap C = \emptyset$ and $J \subset I$ be such that $(\psi^f[R](J) \cup C) \cap \{d_i\}_{i \in I \setminus J} = \emptyset$ so that the reduced problem $\left\langle I \setminus J,\ D \setminus (\psi^f[R](J) \cup C),\ R_{-J}^{-\psi^f[R](J) \cup C}\right\rangle$ is well-defined. Consider the execution of the YRMD-IGYT algorithm to obtain matching $\psi^f[R]$ and suppose it terminates after round t^*. For any $t \in \{1, 2, \ldots, t^*\}$, let A^t be the set of patients who formed cycles and received their assignments in Round t(a) and, let B^t be the set of patients who formed a cycle received their assignments in Round t(b). Since no patient in J is assigned the paired-donor of a patient in $I \setminus J$, set J can be partitioned as $\{I^t, J^t\}_{t\in\{1,2,\ldots,t^*\}}$ where

- $I^t \subseteq A^t$ is a set of patients who form one or more cycles in Round t(a) of YRMD-IGYT algorithm, and
- $J^t \subseteq B^t$ is a set of patients $\{j_1, j_2, \ldots j_k\}$ such that

  1. $\psi^f[R](j_\ell) = d_{j_{\ell+1}}$ for any $\ell \in \{1, 2, \ldots, k-1\}$, and
  2. $\psi^f[R](j_k)$ is a GS-donor or the paired-donor of a patient in $\cup_{s=1}^{t-1} J^s$.

Consider, the reduced problem $R_{-J}^{-\psi^f[R](J)\cup C}$, and the execution of YRMD-IGYT algorithm to obtain $\psi^f\left[R_{-J}^{-\psi^f[R](J)\cup C}\right]$.

Round 1(a): In Round 1(a), having removed the patients in J has no affect on any remaining cycles and all patients in $A^1 \backslash I^1$ forms the same cycles as in the original problem. Since some of the donors in the original problem are removed in the reduced problem, cycles that form in subsequent rounds in the original problem may form earlier in Round 1(a) in the reduced problem. A cycle that is not removed remains a cycle in subsequent rounds until removed. Keep any cycle involving patients in $I \backslash (A^1 \cup B^1)$ until the round it formed under the original problem and skip to Round 1(b).

Round 1(b): If $J^1 = \emptyset$, then exact same cycle forms in Round 1(b) as before and each patient in B^1 receives the same assignment as before. If $J^1 = B^1$ then this round is skipped. Let $J^1 \subset B^1$ be such that $J^1 \neq \emptyset$. Let $(d_g, i_1, d_{i_2}, i_2, \ldots, d_{i_k}, i_k)$ be the cycle formed in Round 1(b) of the original problem where i_1 is the highest priority patient in $I \backslash A^1$ under ordering f and d_g is a GS-donor. We have $J^1 = \{i_\ell, i_{\ell+1}, \ldots, i_k\}$ for some $\ell \in \{2, \ldots, k\}$ for otherwise someone in J_1 would have been assigned the paired-donor of a patient who has been removed (and thus the reduced problem would not have been well-defined). Having been the highest priority patient in a larger set, patient i_1 is still the highest priority patient among the remaining patients. Moreover since patient i_ℓ has been removed, donor d_{i_ℓ} is a GS-donor in the reduced problem. Hence donor d_{i_ℓ} points to i_1 in Round 1(b). In addition patient i_1 points to d_{i_2} (as before), donor d_{i_2} points to patient i_2 (as before), $\ldots$,patient $i_{\ell-1}$ points to d_{i_ℓ} (as before). Hence $\left(d_{i_\ell}, i_1, d_{i_2}, \ldots d_{i_{\ell-1}}, i_{\ell-1}\right)$ is a cycle in Round 1(b). Therefore each patients in $B^1 \setminus J^1$ receives the same assignment in the reduced problem as before. We remove this cycle from the reduced problem and proceed with Round 2.

We similarly continue with Round 2, and so on.[10] Therefore, each patient in $I \backslash J$ is assigned the same donor as under $\psi^f[R]$, completing the proof. $\diamondsuit$

Proposition 2: Let ϕ be a *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral,* and *consistent* mechanism. Then $\phi = \psi^f$ for some $f \in \mathcal{F}$.

Proof of Proposition 2: Let ϕ be a *Pareto efficient*, *individually rational*, *strategy-proof*, *weakly neutral,* and *consistent* mechanism. Let $d^* \in D \setminus \{d_i\}_{i \in \mathcal{I}}$ be a good-Samaritan donor. We will recursively construct an ordering $f \in \mathcal{F}$ as follows:

- We determine $f(1)$ as follows: Let $R^1 \in [\mathcal{R}(\mathcal{D})]^{|\mathcal{I}|}$ be such that for any $i \in \mathcal{I}$,

  $$d^* R_i^1 d_i R_i^1 d \quad \text{for any } d \in \mathcal{D} \backslash \{d^*\}.$$

  By *Pareto efficiency* of ϕ, there exists some $h_1 \in \mathcal{I}$ such that $\phi[R^1](h_1) = d^*$. Let $f(1) = h_1$.

- For any $t > 1$, upon determining patients $f(1), f(2), \ldots, f(t-1)$, we determine $f(t)$ as follows: Let $R^t \in [\mathcal{R}(\mathcal{D})]^{|\mathcal{I}|}$ be such that

  * $R_i^t = R_i^1$ for any $i \in \mathcal{I} \backslash \{f(1), f(2), \ldots, f(t-1)\}$, and
  * $d_i R_i^t d$ for any $i \in \{f(1), f(2), \ldots, f(t-1)\}$ and $d \in \mathcal{D}$.

  By *individual rationality* of ϕ, $\phi[R^t](i) = d_i$ for all $i \in \{f(1), f(2), \ldots, f(t-1)\}$ and by *Pareto efficiency* of ϕ, we have $\phi[R^t](h_t) = d^*$ for some $h_t \in \mathcal{I} \backslash \{f(1), f(2), \ldots, f(t-1)\}$. Let $f(t) = h_t$.

This uniquely defines an ordering $f \in \mathcal{F}$. We will prove that $\phi = \psi^f$.

Fix a problem $\langle I, D, R \rangle$. We construct matching $\psi^f[R]$ by using the YRMD-IGYT algorithm. For each round t of the algorithm let A^t be the set of patients removed in Round t(a) of the algorithm and let B^t be the set of patients removed in Round t(b) of the algorithm.

We next construct a preference profile $R' \in \mathcal{R}^{|I|}$ that will play a key role in our proof. Consider a patient $i \in I$ and let t be such that $i \in A^t \cup B^t$. Two cases are possible:

[10] The only difference in the argument in the following rounds is that, in Round t(b) for $t \in \{1, 2, \ldots, t^*\}$, the patient referred as patient d_g in our argument could be either a GS-donor or the paired-donor of a patient in $\cup_{s=1}^{t-1} J^s$.

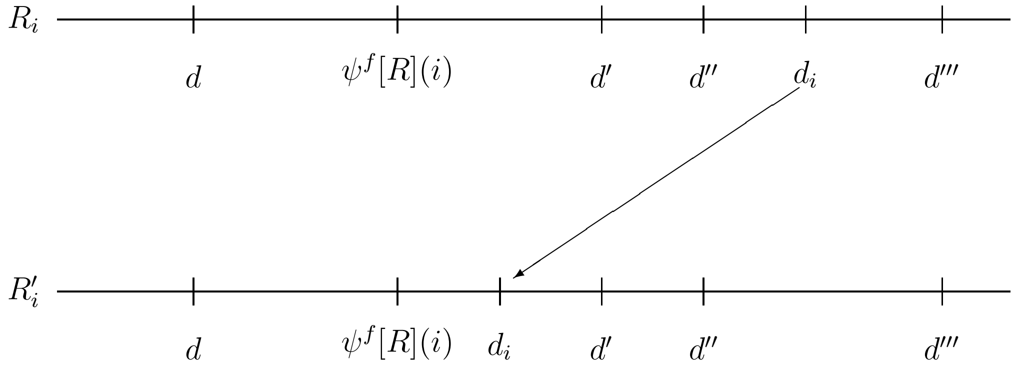

Figure 1: Construction of Preference R'_i for Case 1

Case 1: Either $i \in A^t$ or $i \in B^t$ although she is not the highest priority patient in B^t under ordering f: If $\psi^f\left[R\right](i) = d_i$ then $R'_i = R_i$. Otherwise let R'_i be such that

1. $cP'_i d \Longleftrightarrow cP_i d$ for any $c, d \in D \backslash \{d_i\}$.
2. $\psi^f\left[R\right](i)\, P'_i d_i P'_i d$ for any $d \in D \setminus \{d_i\}$ s.t. $\psi^f\left[R\right](i)\, P_i d$.

That is, R'_i is obtained from R_i by simply inserting donor d_i right after donor $\psi^f\left[R\right](i)$ and keeping the relative ranking of the rest of the donors as in R_i. (See Figure 1.)

Case 2: $i \in B^t$ and she is the highest priority patient in B^t under ordering f: Let $\psi^f\left[R\right](B^t)$ be the set of donors allocated in Round t(b) of the YRMD-IGYT algorithm. Note that $\psi^f\left[R\right](i)\, R_i c$ for any $c \in \psi^f\left[R\right](B^t) \cup \{d_i\}$. We construct R'_i as follows:

1. $cP'_i d \Longleftrightarrow cP_i d$ for any $c, d \in D \backslash \left(\psi^f\left[R\right](B^t \setminus \{i\}) \cup \{d_i\}\right)$.
2. $cP'_i d \Longleftrightarrow cP_i d$ for any $c, d \in \psi^f\left[R\right](B^t)$.
3. $\psi^f\left[R\right](i) P'_i c P'_i d_i P'_i d$
   for any $c \in \psi^f\left[R\right](B^t \backslash \{i\})$, and $d \in D \backslash \left(\psi^f\left[R\right](B^t) \cup \{d_i\}\right)$ s.t. $\psi^f[R](i) P_i d$.

That is, R'_i is obtained from R_i in Case 2 by inserting donors in $\psi^f\left[R\right](B^t \setminus \{i\})$ right after donor $\psi^f\left[R\right](i)$ without altering their relative ranking, inserting donor d_i right after that group, and keeping the relative ranking of the rest of the donors as in R_i. (See Figure 2).

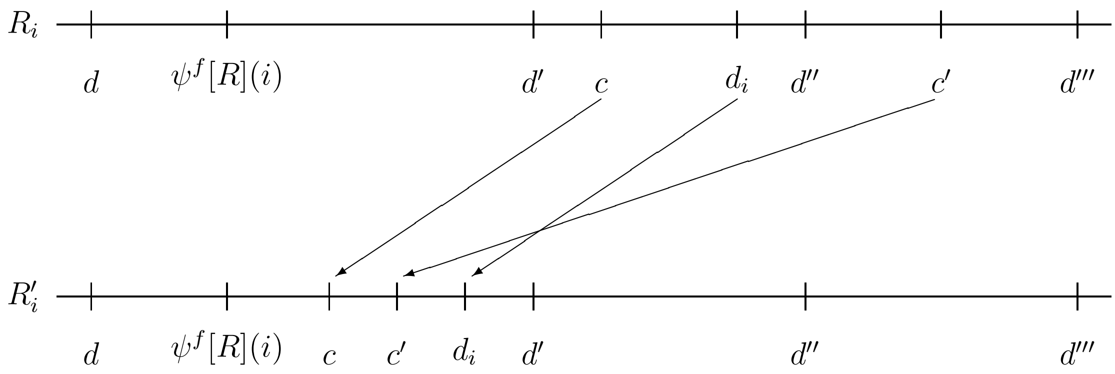

Figure 2: Construction of Preference R'_i for Case 2 when $\psi^f\left[R\right]\left(B^t\right) = \left\{\psi^f\left[R\right]\left(i\right), c, c'\right\}$

By construction, $\psi^f\left[R'\right] = \psi^f\left[R\right]$. We will prove four claims that will facilitate the proof of Proposition 2. We consider the patients in A^1 in the first two claims.

Claim 1: For any $\hat{R}_{-A^1} \in \mathcal{R}^{|I \setminus A^1|}$ and $i \in A^1$, we have $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f\left[R\right](i)$.

Proof of Claim 1: Fix $\hat{R}_{-A^1} \in \mathcal{R}^{|I \setminus A^1|}$. By induction, we will show that $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f\left[R\right](i)$ for all $i \in A^1$.

- Partition the patients in A^1 based on the cycle they belong. Let $A^1_1 \subseteq A^1$ be the set of patients encountered in the first cycle in Round 1(a) of the YRMD-IGYT algorithm. By *individual rationality* we have $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) \in \{\psi^f\left[R\right](i), d_i\}$ for any $i \in A^1_1$. Moreover $\psi^f\left[R\right](i)\, R'_i d_i$ for any $i \in A^1_1$, and $\psi^f\left[R\right](A^1_1) = \cup_{j \in A^1_1} d_j$. Hence by *Pareto efficiency*, $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f\left[R\right](i)$ for any $i \in A^1_1$.

- Let $A^1_t \subseteq A^1$ be the set of patients removed in t^{th} cycle in Round 1(a) of the YRMD-IGYT algorithm. In the inductive step, assume that for any patient j removed in the previous cycles, $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](j) = \psi^f\left[R\right](j)$. Given this, by *individual rationality* of ϕ, we have $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) \in \{\psi^f\left[R\right](i), d_i\}$ for any $i \in A^1_t$. Moreover $\psi^f\left[R\right](i)\, R'_i d_i$ for any $i \in A^1_t$, and $\psi^f\left[R\right](A^1_t) = \cup_{j \in A^1_t} d_j$. Hence by *Pareto efficiency*, $\phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f\left[R\right](i)$ for any $i \in A^1_t$.

$\diamond$

Note that the proof of Claim 1 is entirely driven by *Pareto efficiency* and *individual rationality* of ϕ. Therefore it directly implies the following corollary.

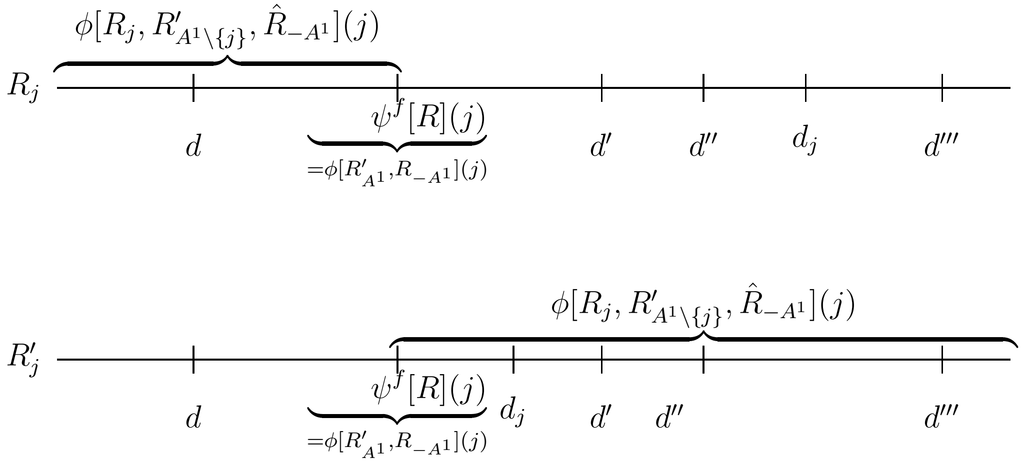

Figure 3: $\phi\left[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\right](j) = \phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](j) = \psi^f[R](j)$ by *strategy-proofness.*

Corollary 1: For any $\hat{R}_{-A^1} \in \mathcal{R}^{|I\setminus A^1|}$, any *Pareto efficient* and *individually rational* matching μ for problem $\left(R'_{A^1}, \hat{R}_{-A^1}\right)$, and any $i \in A^1$, we have $\mu(i) = \psi^f[R](i)$.

Claim 2: For any $\hat{R}_{-A^1} \in \mathcal{R}^{|I\setminus A^1|}$, and any $i \in A^1$, we have $\phi\left[R_{A^1}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i)$.

Proof of Claim 2: Fix $\hat{R}_{-A^1} \in \mathcal{R}^{|I\setminus A^1|}$. For any $J \subseteq A^1$, we will prove that $\phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i)$ for all $i \in A^1$ by induction on the size of J.

- Let $J = \{j\} \subseteq A^1$. By *strategy-proofness* of ϕ,

  $$\phi\left[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\right](j)\, R_j \phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](j) \text{ and } \phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](j)\, R'_j \phi\left[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\right](j).$$

  The above relation, construction of R'_j, and Claim 1 imply that (see Figure 3)

  $$\phi\left[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\right](j) = \phi\left[R'_{A^1}, \hat{R}_{-A^1}\right](j) = \psi^f[R](j).$$

  Therefore while problems $(R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1})$ and $(R'_{A^1}, \hat{R}_{-A^1})$ differ in preferences of patient j, her assignment under ϕ does not differ in these two problems. Hence matching $\phi\left[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\right]$ not only has to be *Pareto efficient* and *individually rational* under $(R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1})$ but also under $(R'_{A^1}, \hat{R}_{-A^1})$ and therefore by Corollary 1

  $$\phi\left[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1.$$

- Fix $k \in \{1, \ldots, |A^1| - 1\}$. In the inductive step, assume that for any $J \subset A^1$ with $|J| \leq k$,

  $$\phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1. \tag{1}$$

Fix $J \subseteq A^1$ such that $|J| = k+1$. Fix $j \in J$. By *strategy-proofness* of ϕ, we have

$$\phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right](j)\, R_j \phi\left[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\right](j) \text{ and}$$
$$\phi\left[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\right](j)\, R'_j \phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right](j)$$

The above relation and the construction of R'_j imply that

$$\phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right](j) = \phi\left[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\right](j) = \psi^f[R](j), \qquad (2)$$

where the second equality follows from the inductive assumption Equation 1 (since $|J\setminus\{j\}| = k$). Since the choice of $j \in J$ is arbitrary, Equation 2 holds for all $j \in J$. Therefore while problems $(R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1})$ and $(R'_{A^1}, \hat{R}_{-A^1})$ differ in preferences of patients in J, their assignments under ϕ do not differ in these two problems. Hence matching $\phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right]$ not only has to be *Pareto efficient* and *individually rational* under $(R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1})$ but also under $(R'_{A^1}, \hat{R}_{-A^1})$, and therefore by Corollary 1

$$\phi\left[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1,$$

completing the induction and the proof of Claim 2. $\diamond$

Let $B^1 = \{i_1, \ldots, i_k\}$ and let $(d_g, i_1, d_{i_2}, i_2 \ldots, d_{i_k}, i_k)$ be the cycle removed in Round 1(b) of the YRMD-IGYT algorithm where patient i_1 is the highest priority patient in $I\setminus A^1$ under ordering f, and donor d_g is a GS-donor. In order to simplify the notation, let $d_{i_{k+1}} \equiv d_g$. We have

$$\psi^f[R](i_\ell) = d_{i_{\ell+1}} \qquad \text{for all } \ell \in \{1, \ldots, k\}.$$

We consider the patients in B^1 in the next two claims.

Claim 3: $\phi\left[R'_{B^1}, R_{-B^1}\right](i) = \psi^f[R](i)$ for all $i \in B^1$.

Proof of Claim 3: First of all, observe that $\phi\left[R'_{B^1}, R_{-B^1}\right](i) = \psi^f[R](i)$ for all $i \in A^1$ by Claim 2. We will prove the claim by contradiction. Suppose that there exists a patient $i_\ell \in B^1$ such that $\phi\left[R'_{B^1}, R_{-B^1}\right](i_\ell) \neq \psi^f[R](i_\ell) = \psi^f[R'](i_\ell) = d_{i_{\ell+1}}$. Pick the last such patient in the cycle. Then

$$\begin{aligned}
\phi[R'_{B^1}, R_{-B^1}](i_m) &= d_{i_{m+1}} \quad \text{for all } m \in \{\ell+1, \ldots, k\} && \text{by the choice of } \ell,\\
\phi[R'_{B^1}, R_{-B^1}](i_\ell) &= d_{i_\ell} && \text{by Claim 2 and \textit{individual rationality} of } \phi,\\
\phi[R'_{B^1}, R_{-B^1}](i_{\ell-1}) &= d_{i_{\ell-1}} && \text{by above relation, Claim 2 and \textit{individual rationality} of } \phi,\\
&\vdots\\
\phi[R'_{B^1}, R_{-B^1}](i_2) &= d_{i_2} && \text{by above relation, Claim 2 and \textit{individual rationality} of } \phi.
\end{aligned}$$

Since i_1 is the highest priority patient in $I \backslash A^1$, Case 2 applies in construction of R'_{i_1}. Therefore by Claim 2 and *individual rationality* of ϕ we have $\phi\left[R'_{B^1}, R_{-B^1}\right](i_1) \in \left\{d_{i_1}, \ldots, d_{i_{k+1}}\right\}$, and since all but donors d_1 and $d_{i_{\ell+1}}$ are assigned to other patients by above relations,

$$\phi\left[R'_{B^1}, R_{-B^1}\right](i_1) \in \left\{d_{i_1}, d_{i_{\ell+1}}\right\}.$$

But donor $d_{i_{\ell+1}}$ can neither be left unmatched nor be matched with patient i_1 under $\phi\left[R'_{B^1}, R_{-B^1}\right]$ for otherwise assigning donor $d_{i_{m+1}}$ to patient i_m for all $m \in \{1, \ldots, \ell\}$ (and keeping the other assignments the same) would result in a Pareto improvement under $\left(R'_{B^1}, R_{-B^1}\right)$. Therefore,

$$\phi\left[R'_{B^1}, R_{-B^1}\right](i_1) = d_{i_1} \quad \text{ and } \quad \phi\left[R'_{B^1}, R_{-B^1}\right](j_1) = d_{i_{\ell+1}} \text{ for some } j_1 \in I \backslash \left(A^1 \cup B^1\right).$$

Iteratively form set S as follows:

Step 1. Let $j_1 \in S$ (i.e., patient j_1 is the first patient to be included in set S). Recall that $\phi\left[R'_{B^1}, R_{-B^1}\right](j_1) = d_{i_{\ell+1}}$. Let preferences $R''_{j_1} \in \mathcal{R}$ be such that

$$\underbrace{\phi\left[R'_{B^1}, R_{-B^1}\right](j_1)}_{=d_{i_{\ell+1}}} R''_{j_1} d_{j_1} R''_{j_1} d \qquad \text{for all } d \in D \backslash \{d_{i_{\ell+1}}\}.$$

Consider the problem $\left(R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\right)$. By Claim 2,

$$\phi\left(R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\right)(i) = \psi^f[R](i) \qquad \text{for all } i \in A^1. \tag{3}$$

By *strategy-proofness* of ϕ,

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\right](j_1)\, R''_{j_1} \underbrace{\phi\left[R'_{B^1}, R_{-B^1}\right](j_1)}_{=d_{i_{\ell+1}}}$$

and since donor $d_{i_{\ell+1}}$ is the top choice under R''_{j_1}

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\right](j_1) = \phi\left[R'_{B^1}, R_{-B^1}\right](j_1) = d_{i_{\ell+1}}. \tag{4}$$

Therefore

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\right](i_{\ell+1}) = d_{i_{\ell+2}} \qquad \text{by Eqn 3, Eqn 4, and } \textit{individual rationality} \text{ of } \phi$$

$$\vdots$$

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\right](i_k) = d_{i_{k+1}} \qquad \text{by Eqn 3, above relation, and } \textit{individual rationality} \text{ of } \phi$$

and

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](i_\ell) = d_{i_\ell} \qquad \text{by Eqn 3, Eqn 4, and } \textit{individual rationality} \text{ of } \phi$$

$$\vdots$$

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](i_2) = d_{i_2} \qquad \text{by Eqn 3, above relation, and } \textit{individual rationality} \text{ of } \phi.$$

Equation 3, above relations, and *individual rationality* of ϕ imply

$$\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](i_1) = d_{i_1}.$$

Step 2. If there is no patient $j_2 \in I\setminus(A^1 \cup B^1)$ such that $\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](j_2) = d_{j_1}$ then terminate the construction of S. (Thus $S = \{j_1\}$.) Otherwise such patient j_2 is the second patient to include in set S and let preferences $R''_{j_2} \in \mathcal{R}$ be such that

$$\underbrace{\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](j_2)}_{=d_{j_1}} R''_{j_2} d_{j_2} R''_{j_2} d \qquad \text{for all } d \in D\setminus\{d_{j_1}\}.$$

By Claim 2,

$$\phi\left[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1. \tag{5}$$

By *strategy-proofness* of ϕ,

$$\phi\left[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\right](j_2)\, R''_{j_2} \underbrace{\phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](j_2)}_{=d_{j_1}}$$

which in turn implies

$$\phi\left[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\right](j_2) = \phi\left[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\right](j_2) = d_{j_1}.$$

Therefore by *individual rationality* of ϕ, Equation 5, and construction of R''_{j_2},

$$\phi\left[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\right](j_1) = d_{i_{\ell+1}},$$

which in turn implies (using a similar argument as in Step 1)

$$\phi\left[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\right](i_m) = d_{i_{m+1}} \qquad \text{for all } m \in \{\ell+1, \ldots, k\},$$
$$\phi\left[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\right](i_m) = d_{i_m} \qquad \text{for all } m \in \{1, \ldots, \ell\}.$$

We continue iteratively and form set $S = \{j_1, j_2, \ldots, j_s\} \subseteq I \setminus (A^1 \cup B^1)$ and preference profile $\left(R'_{B^1}, R''_S, R_{-B^1 \cup S}\right)$ such that

$$\begin{aligned}
&\phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](i) \neq d_{j_s} && \text{for any } i \in I,\\
&\phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](j_m) = d_{j_{m-1}} && \text{for all } m \in \{2, \ldots, s\},\\
&\phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](j_1) = d_{i_{\ell+1}}, && \text{and}\\
&\phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](i_1) = d_{i_1}.
\end{aligned}$$

Observe that there is no patient $i \in I$ such that $\phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](i) = d_{j_s}$ for otherwise patient i would also be included in set S. Therefore, upon removing patients in $T = I \setminus (\{i_1\} \cup S)$ and their assigned donors $C = \phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](T)$, the reduced problem $\left(R'^{-C}_{i_1}, R''^{-C}_S\right)$ is well-defined. Note that the set of remaining patients is $\{i_1\} \cup S$, and donor $d_{i_{\ell+1}}$ is a GS-donor in the reduced problem (possibly together with other GS-donors). By *consistency* of ϕ, we have

$$\phi\left[R'^{-C}_{i_1}, R''^{-C}_S\right](i) = \phi\left[R'_{B^1}, R''_S, R_{-B^1\cup S}\right](i) \qquad \text{for all } i \in \{i_1\} \cup S.$$

In the rest of the proof, for the sake of notation we set

$$d_{j_0} \equiv d_{i_{\ell+1}}.$$

Note that the preference relation profile $(R'^{-C}_{i_1}, R''^{-C}_S) \in [\mathcal{R}(D \setminus C)]^{|S|+1}$ is given as follows:

$$R'^{-C}_{i_1} : \; d_{j_0} P'^{-C}_{i_1} d_{i_1} P'^{-C}_{i_1} \ldots \qquad\qquad R''^{-C}_S : \left\{ \begin{array}{c} d_{j_0} P''^{-C}_{j_1} d_{j_1} P''^{-C}_{j_1} \ldots \\ d_{j_1} P''^{-C}_{j_2} d_{j_2} P''^{-C}_{j_2} \ldots \\ \vdots \\ d_{j_{s-1}} P''^{-C}_{j_s} d_{j_s} P''^{-C}_{j_s} \ldots \end{array} \right.$$

We consider the remaining donors $D \setminus C$ and construct the following preference relation $R''^{-C}_{i_1} \in \mathcal{R}(D \setminus C)$:

$$d_{j_0} P''^{-C}_{i_1} d_{j_1} P''^{-C}_{i_1} \cdots P''^{-C}_{i_1} d_{j_s} P''^{-C}_{i_1} d_{i_1} P''^{-C}_{i_1} d \qquad \text{for all } d \in D \setminus \left(C \cup \{d_{j_0}, \ldots, d_{j_s}, d_{i_1}\}\right).$$

By *strategy-proofness* of ϕ,

$$\underbrace{\phi\left[R'^{-C}_{i_1}, R''^{-C}_S\right](i_1)}_{=d_{i_1}} R'^{-C}_{i_1} \phi\left[R''^{-C}_{\{i_1\}\cup S}\right](i_1).$$

Since $d_{j_0} \in \psi^f[R](B^1)$, we have $d_{j_0} P'^{-C}_{i_1} d_{i_1}$ and therefore $\phi\left[R''^{-C}_{\{i_1\}\cup S}\right](i_1) \neq d_{j_0}$. Moreover $\phi\left[R''^{-C}_{\{i_1\}\cup S}\right](i_1) \neq d_{i_1}$. That is because, the allocation

$$\begin{pmatrix} i_1 & j_1 & j_2 & \cdots & j_s \\ d_{j_s} & d_{j_0} & d_{j_1} & \cdots & d_{j_{s-1}} \end{pmatrix}$$

Pareto dominates any such allocation under $R''^{-C}_{\{i_1\}\cup S}$ contradicting *Pareto efficiency* of ϕ. Hence

$$\phi\left[R''^{-C}_{\{i_1\}\cup S}\right](i_1) \in \{d_{j_1}, \ldots, d_{j_s}\}.$$

Let $\phi\left[R''^{-C}_{\{i_1\}\cup S}\right](i_1) = d_{j_m}$. Then by *individual rationality*

$$\begin{aligned} \phi\left[R''^{-C}_{\{i_1\}\cup S}\right](j_p) &= d_{j_p} \qquad \text{for all } p \in \{m+1, \ldots, s\}, \\ \phi\left[R''^{-C}_{\{i_1\}\cup S}\right](j_p) &= d_{j_{p-1}} \qquad \text{for all } p \in \{1, \ldots, m\}. \end{aligned}$$

Therefore upon removing all patients except $\{i_1, j_m\}$ and all donors except $C' = \{d_{i_1}, d_{j_m}, d_{j_{m-1}}\}$ from the reduced problem $R''^{-C}_{\{i_1\}\cup S}$, the further reduced problem $R''^{C'}_{\{i_1,j_m\}}$ is well-defined. (That is because, d_{i_1} is unmatched, d_{j_m} is matched to patient i_1, and $d_{j_{m-1}}$ is matched to patient j_m under $\phi\left[R''^{-C}_{\{i_1\}\cup S}\right]$). In this further reduced problem, donor $d_{j_{m-1}}$ is the unique GS-donor. By *consistency* of ϕ, we have

$$\begin{aligned} \phi\left[R''^{C'}_{\{i_1,j_m\}}\right](i_1) &= \phi\left[R''^{-C}_{\{i_1\}\cup S}\right](i_1) = d_{j_m}, \\ \phi\left[R''^{C'}_{\{i_1,j_m\}}\right](j_m) &= \phi\left[R''^{-C}_{\{i_1\}\cup S}\right](j_m) = d_{j_{m-1}}. \end{aligned}$$

Note that the preference profile $R''^{C'}_{\{i_1,j_m\}} \in [\mathcal{R}(C')]^2$ is given as follows:

$$R''^{C'}_{\{i_1,j_m\}} : \begin{cases} d_{j_{m-1}} P''^{C'}_{i_1} d_{j_m} P''^{C'}_{i_1} d_{i_1} \\ d_{j_{m-1}} P''^{C'}_{j_m} d_{j_m} P''^{C'}_{j_m} d_{i_1} \end{cases}$$

We consider the donors in C' and construct the following preference relation $\tilde{R}^{C'}_{i_1} \in \mathcal{R}(C')$:

$$d_{j_{m-1}} \tilde{P}^{C'}_{i_1} d_{i_1} \tilde{P}^{C'}_{i_1} d_{j_m}.$$

By *strategy-proofness* of ϕ,

$$\underbrace{\phi\left[R''^{C'}_{\{i_1,j_m\}}\right](i_1)}_{=d_{j_m}} R''^{C'}_{i_1} \phi\left[\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right](i_1)$$

Since $d_{j_{m-1}} P''^{C'}_{i_1} d_{j_m}$ by construction, $\phi\left[\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right](i_1) \neq d_{j_{m-1}}$. Therefore by *individual rationality* of ϕ,

$$\phi\left[\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right](i_1) = d_{i_1}$$

and this together with *Pareto-efficiency* of ϕ imply

$$\phi\left[\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right](j_m) = d_{j_{m-1}}.$$

Recall that i_1 is the highest priority patient in $I \backslash A^1$ under ordering f. In particular, i_1 has higher priority than j_m, since $j_m \in I \backslash (A^1 \cup B^1)$. Let $i_1 = f(t)$ for some t. Consider the profile R^t used in construction of f. Any patient i ordered before i_1 has d_i as her first choice under R^t, whereas any other patient i has the GS-donor d^* as her first choice and d_i as her second choice under R^t. We have $\phi[R^t](i_1) = d^*$ and $\phi[R^t](i) = d_i$ for all $i \in I \backslash \{i_1\}$ by construction of f and *individual rationality* of ϕ. Therefore, upon removing all patients except $\{i_1, j_m\}$ and all donors except $C^* = \{d_{i_1}, d_{j_m}, d^*\}$ from problem R^t, the reduced problem $R^{tC^*}_{\{i_1,j_m\}}$ is well-defined. (That is because, d_{i_1} is unmatched, d_{j_m} is matched to j_m, and d^* is matched to i_1 under $\phi[R^t]$.) By *consistency* of ϕ,

$$\phi\left[R^{tC^*}_{\{i_1,j_m\}}\right](i_1) = \phi\left[R^t\right](i_1) = d^* \quad \text{and} \quad \phi\left[R^{tC^*}_{\{i_1,j_m\}}\right](j_m) = \phi\left[R^t\right](j_m) = d_{j_m}.$$

Note that the preference profile $R^{tC^*}_{\{i_1,j_m\}} \in [\mathcal{R}(C^*)]^2$ is given as follows:

$$R^{tC^*}_{\{i_1,j_m\}} : \begin{cases} d^* R^{tC^*}_{i_1} d_{i_1} R^{tC^*}_{i_1} d_{j_m} \\ d^* R^{tC^*}_{j_m} d_{j_m} R^{tC^*}_{j_m} d_{i_1} \end{cases}$$

There is a single GS-donor in both reduced problems $\left(\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right)$ and $R^{tC^*}_{\{i_1,j_m\}}$ while the patients and the paired-donors are the same. Under the profile $\left(\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right)$ each patient ranks the GS-donor $d_{j_{m-1}}$ as the first choice, her paired-donor as the second choice, and the paired-donor of the other patient as the third choice. Similarly under profile $R^{tC^*}_{\{i_1,j_m\}}$ each patient ranks the GS-donor d^* as the first choice, her paired-donor as the second choice, and the paired-donor of the other patient as the third choice. However, patient j_m is assigned the top ranked GS-donor $d_{j_{m-1}}$ under $\phi\left[\tilde{R}^{C'}_{i_1}, R''^{C'}_{j_m}\right]$ whereas patient i_1 is assigned the top ranked GS-donor d^* under $\phi\left[R^{tC^*}_{\{i_1,j_m\}}\right]$, contradicting *weak neutrality* of ϕ. Therefore, we have $\phi[R'_{B^1}, R_{-B^1}](i) = \psi^f[R](i)$ for all $i \in B^1$ completing the proof of Claim 3. $\diamond$

Claim 4: $\phi[R](i) = \psi^f[R](i)$ for all $i \in B^1$.

Proof of Claim 4: We prove the claim by induction. Starting from preference profile $\left(R'_{B^1}, R_{-B^1}\right)$, we will replace R'_i with R_i for each patient in $B^1 = \{i_1, \ldots, i_k\}$ one at a

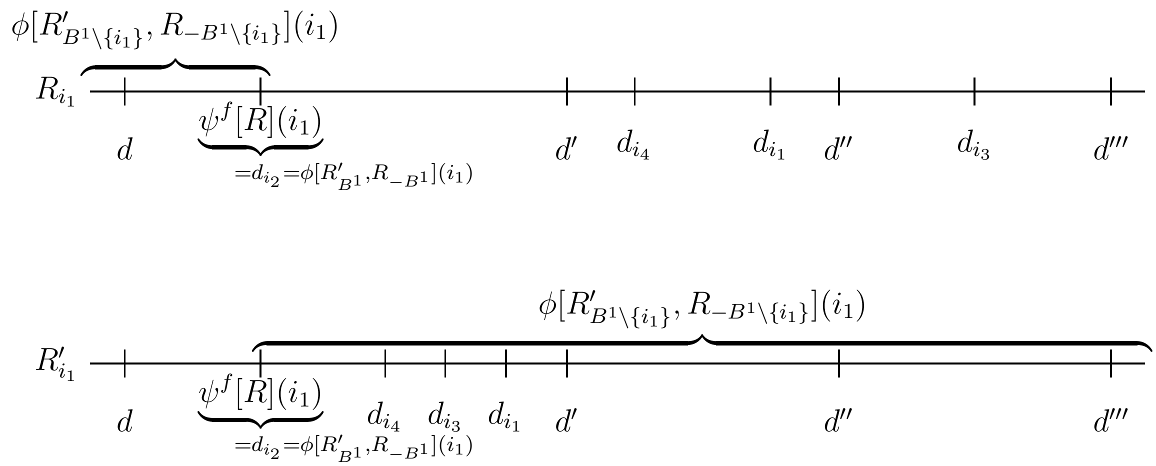

Figure 4: $\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_1) = \phi\left[R'_{B^1}, R_{-B^1}\right](i_1) = \psi^f[R](i_1) = d_{i_2}$ by *strategy-proofness* of ϕ for the case with $B^1 = \{i_1, i_2, i_3\}$ and $d_{i_4} \equiv d_g$ is a GS-donor.

time in order. Recall that $(d_g, i_1, d_{i_2}, i_2 \ldots, d_{i_k}, i_k)$ is the cycle removed in Round 1(b) of the YRMD-IGYT algorithm where patient i_1 is the highest priority patient in $I\setminus A^1$ under ordering f, and donor d_g is a GS-donor. Recall that $d_{i_{k+1}} \equiv d_g$. We have

$$\psi^f[R](i_\ell) = d_{i_{\ell+1}} \qquad \text{for all } \ell \in \{1, \ldots, k\}.$$

- Consider the preference profile $(R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}})$. By Claim 2,

$$\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1. \tag{6}$$

  By *strategy-proofness* of ϕ,

$$\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_1)\, R_{i_1} \underbrace{\phi\left[R'_{B^1}, R_{-B^1}\right](i_1)}_{=d_{i_2}} \qquad \text{and },$$

$$\underbrace{\phi\left[R'_{B^1}, R_{-B^1}\right](i_1)}_{=d_{i_2}} R'_{i_1} \phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_1).$$

  Recall that i_1 is the highest priority patient in B^1 under ordering f. Therefore, Case 2 applies to the construction of R'_{i_1} and the above relation together with construction of R'_{i_1} imply (see Figure 4)

$$\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_1) = \phi\left[R'_{B^1}, R_{-B^1}\right](i_1) = \psi^f[R](i_1) = d_{i_2}, \tag{7}$$

  where the second equality follows from Claim 3.

By *individual rationality* of ϕ, Equation 6, and construction of $R'_{B^1\setminus\{i_1\}}$ (for which Case 1 applies) we have

$$\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_\ell) \in \left\{d_{i_\ell}, d_{i_{\ell+1}}\right\} \qquad \text{for all } \ell \in \{2,\ldots,k\}. \tag{8}$$

Then,

$$\begin{aligned}
\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_2) &= d_{i_3} \quad \text{by Eqn 7 and Eqn 8,}\\
&\vdots\\
\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i_k) &= d_{i_{k+1}} \quad \text{by above relation and Eqn 8.}
\end{aligned}$$

We showed that

$$\phi\left[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\right](i) = \phi\left[R'_{B^1}, R_{-B^1}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in B^1.$$

- Let $\ell \in \{2,\ldots,k\}$ and $J = \{i_\ell,\ldots,i_k\}$. In the inductive step, assume that

$$\phi\left[R'_J, R_{-J}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in B^1.$$

We will show that $\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i) = \psi^f[R](i)$ for all $i \in B^1$.

Consider preference profile $(R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}})$. By Claim 2,

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in A^1. \tag{9}$$

By *strategy-proofness* of ϕ,

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i_\ell)\; R_{i_\ell} \underbrace{\phi\left[R'_J, R_{-J}\right](i_\ell)}_{=d_{i_{\ell+1}}} \text{ and}$$

$$\underbrace{\phi\left[R'_J, R_{-J}\right](i_\ell)}_{=d_{i_{\ell+1}}} R'_{i_\ell} \phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i_\ell)$$

and this together with construction of R_{i_ℓ} (for which Case 1 applies) imply

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i_\ell) = \phi\left[R'_J, R_{-J}\right](i_\ell) = \psi^f[R](i_\ell) = d_{i_{\ell+1}}, \tag{10}$$

where the second equality follows from the inductive assumption.

By *individual rationality* of ϕ, Equation 9, and construction of $R'_{J\setminus\{i_\ell\}}$ (for which Case 1 applies) we have

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i_m) \in \left\{d_{i_m}, d_{i_{m+1}}\right\} \qquad \text{for all } m \in \{\ell+1,\ldots,k\}. \tag{11}$$

Then,

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i_{\ell+1}) = d_{i_{\ell+2}} \qquad \text{by Eqn 10 and Eqn 11,}$$
$$\vdots$$
$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i_k) = d_{i_{k+1}} \qquad \text{by above relation and Eqn 11.}$$

Hence, we showed that

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i) = \phi\left[R'_J, R_{-J}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in J. \tag{12}$$

We are ready to complete the induction by invoking consistency: Upon removing patients in $J = \{i_\ell, \ldots, i_k\}$ and their assignments

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](J) = \phi\left[R'_J, R_{-J}\right](J) = \left\{d_{i_{\ell+1}}, \ldots, d_{i_{k+1}}\right\} \tag{13}$$

from problems $\left(R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right)$ and (R'_J, R_{-J}), the reduced problems are not only well-defined (recall that $d_{i_{k+1}}$ is a GS-donor) but also identical. Therefore, for any $i \in I \setminus J$,

$$\begin{aligned}
\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i) &= \phi\left[R_{-J}^{-\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](J)}\right](i) && \text{by } \textit{consistency} \text{ of } \phi \\
&= \phi\left[R_{-J}^{-\phi\left[R'_J, R_{-J}\right](J)}\right](i) && \text{by Eqn 13} \\
&= \phi\left[R'_J, R_{-J}\right](i) && \text{by } \textit{consistency} \text{ of } \phi
\end{aligned}$$

and this together with Equation 12 imply

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right] = \phi\left[R'_J, R_{-J}\right]. \tag{14}$$

Equation 14 and inductive assumption imply that

$$\phi\left[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\right](i) = \psi^f[R](i) \qquad \text{for all } i \in B^1,$$

completing the induction and the proof of Claim 4. $\diamond$

We are ready to complete the proof of Proposition 2. By Claim 2 and Claim 4,

$$\phi\left[R\right](i) = \psi^{f}\left[R\right](i) \qquad \text{for all } i \in A^1 \cup B^1. \tag{15}$$

Since for any $i \in A^1 \cup B^1$, the donor $\psi[R](i)$ is either the GS-donor $d_{i_{k+1}}$ or the paired-donor of a patient in $A^1 \cup B^1$, upon removing the patients in $A^1 \cup B^1$ and their assigned donors $\phi\left[R\right](A^1 \cup B^1) = \psi^{f}\left[R\right](A^1 \cup B^1)$ from the problem R, the reduced problem $R_{-A^1\cup B^1}^{-\phi[R]\left(A^1\cup B^1\right)}$ is well-defined. For any $i \in A^2 \cup B^2$, we have

$$\begin{aligned}
\phi\left[R\right](i) &= \phi\left[R_{-A^1\cup B^1}^{-\phi[R]\left(A^1\cup B^1\right)}\right](i) && \text{by } \textit{consistency} \text{ of } \phi \\
&= \psi^{f}\left[R_{-A^1\cup B^1}^{-\phi[R]\left(A^1\cup B^1\right)}\right](i) && \text{by application of Claims 2 and 4 to } R_{-A^1\cup B^1}^{-\phi[R]\left(A^1\cup B^1\right)} \text{ for } A^2 \cup B^2 \\
&= \psi^{f}\left[R_{-A^1\cup B^1}^{-\psi^{f}[R]\left(A^1\cup B^1\right)}\right](i) && \text{by Eqn 15} \\
&= \psi^{f}\left[R\right](i) && \text{by } \textit{consistency} \text{ of } \psi^{f}.
\end{aligned}$$

We iteratively continue with patients in $A^3 \cup B^3$, and so on to obtain

$$\phi\left[R\right] = \psi^{f}\left[R\right]$$

completing the proof. $\diamondsuit$

6 Independence of the Axioms

The following examples establish the independence of the axioms.

Example 1: Let mechanism ϕ assign each patient $i \in I$ her paired-donor d_i for each problem $\langle I, D, R\rangle$.

Mechanism ϕ is *individually rational*, *strategy-proof*, *weakly neutral* and *consistent* but not *Pareto efficient*.

Example 2: Fix an ordering $f \in \mathcal{F}$ and let mechanism ϕ be the *serial dictatorship induced by f*: For any problem $\langle I, D, R\rangle$, the highest priority patient in I is assigned her top choice, the second highest priority patient is assigned her top choice among remaining donors, etc.

Mechanism ϕ is *Pareto efficient*, *strategy-proof*, *weakly neutral* and *consistent* but not *individually rational*.

Example 3: Fix an ordering $f \in \mathcal{F}$. Let $g \in \mathcal{F}$ be constructed from f by demoting patient $f(1)$ to the very end of the ordering (so that the highest priority patient under f is the lowest priority patient under g) but otherwise keeping the rest of the priority ordering as in f. For any problem $\langle I, D, R \rangle$, let

$$\phi[R] = \begin{cases} \psi^g[R] & \text{if } dR_i d_{f(1)} \text{ for all } i \in I \text{ and } d \in D, \\ \psi^f[R] & \text{if otherwise.} \end{cases}$$

That is, mechanism ϕ picks the outcome of the YRMD-IGYT mechanism induced by ordering g if each patient (including patient $f(1)$) ranks the paired-donor of patient $f(1)$ as her last choice, and picks the outcome of the YRMD-IGYT mechanism induced by ordering f otherwise.

Mechanism ϕ is *Pareto efficient*, *individually rational*, *weakly neutral* and *consistent* but not *strategy-proof*.

Example 4: Let $\mathcal{I}, \mathcal{D}$ be such that $|\mathcal{I}| \geq 2$ and $|\mathcal{D}| \geq |\mathcal{I}| + 2$. Let $i_1, i_2 \in \mathcal{I}$ and $d^* \in \mathcal{D} \setminus \{d_i\}_{i \in \mathcal{I}}$. Let $f, g \in \mathcal{F}$ be such that $f(1) = g(2) = i_1$, $f(2) = g(1) = i_2$ and $f(i) = g(i)$ for all $i \in \mathcal{I} \setminus \{i_1, i_2\}$. For any problem $\langle I, D, R \rangle$, let

$$\phi[R] = \begin{cases} \psi^f[R] & \text{if } i_1 \in I,\ d^* \in D \text{ and } d^* R_{i_1} d \text{ for all } d \in D \setminus \{d_i\}_{i \in I}, \\ \psi^g[R] & \text{if otherwise.} \end{cases}$$

That is, mechanism ϕ picks the outcome of the YRMD-IGYT mechanism induced by ordering f if both patient i_1 and GS-donor d^* are present and patient i_1 prefers GS-donor d^* to any other GS-donor, and mechanism ϕ picks the outcome of the YRMD-IGYT mechanism induced by ordering g otherwise.

Mechanism ϕ is *Pareto efficient*, *individually rational*, *strategy-proof*, and *consistent* but not *weakly neutral.*

Example 5: Let $f, g \in \mathcal{F}$ be such that $f \neq g$. For any problem $\langle I, D, R \rangle$, let

$$\phi[R] = \begin{cases} \psi^f[R] & \text{if there are odd number of GS-donors,} \\ \psi^g[R] & \text{if there are even number of GS-donors.} \end{cases}$$

Mechanism ϕ is *Pareto efficient*, *individually rational*, *strategy-proof*, and *weakly neutral* but not *consistent*.

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