

Calibration Results for Betweenness Functionals*

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Abstract

A reasonable level of risk aversion with respect to small gambles leads to a high, and absurd, level of risk aversion with respect to large gambles. This was demonstrated by Rabin [15] for expected utility theory. Later, Safra and Segal [18] extended this result by showing that similar arguments apply to almost all non-expected utility theories, provided they are Gâteaux differentiable. In this paper we drop the differentiability assumption and by restricting attention to betweenness theories we show that much weaker conditions are sufficient for the derivation of similar calibration results.

1 Introduction

The hypothesis that in risky environments decision makers evaluate actions by considering possible final wealth levels is widely used in economics. But this hypothesis leads to the following conclusion: A reasonable level of risk aversion with respect to small gambles leads to a high, even absurd, level of risk aversion with respect to large gambles. This was demonstrated by

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Rabin [15] for the case of expected utility theory (see also Epstein [6]). Later, Safra and Segal [18] extended this result by showing that similar arguments apply to almost all non-expected utility theories, provided they are Gâteaux differentiable. To understand these arguments, consider the following two properties (D1 and D2) and the two subsequent assumptions (B3 and B4):

D1 Actions are evaluated by considering possible *final* wealth levels.

D2 Risk aversion: If lottery Y is a mean preserving spread of lottery X , then X is preferred to Y .

B3 Rejection of a small actuarially favorable lottery in a certain range. For example, rejection of the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all wealth levels below 300,000.

B4 Acceptance of large, very favorable lotteries. For example, acceptance of the lottery $(-5,000, \frac{1}{2}; 10,000,000, \frac{1}{2})$.

Rabin [15] showed that, within expected utility theory, properties D1 and D2 imply a tension between B3 and B4. If for given small and relatively close $\ell < g$, the decision maker rejects $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all wealth levels $x \in [a, b]$, then he also rejects $(-L, \frac{1}{2}; G, \frac{1}{2})$ at x^* for some L, G , and $x^* \in [a, b]$ where G is huge while L is not. For example, if a risk averse decision maker is rejecting the lottery $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all positive wealth levels below 300,000, then at wealth level 290,000 he will also reject the lottery $(-10,000, \frac{1}{2}; 5,503,790, \frac{1}{2})$.

In [18] we extended this surprising result to general non-expected utility theories. We used the technical tool of local utilities (Machina [12]) and, assuming Gâteaux differentiability, we showed that D1 and D2 imply a tension between B4 and a modified version of B3.¹ This analysis applies to all Gâteaux differentiable models, including (the differentiable versions of) betweenness (Dekel [5], Chew [1]), quadratic utility (Machina [12], Chew, Epstein, and Segal [4]), rank-dependent utility (Quiggin [14]) and constant (absolute and relative) risk aversion models (see Safra and Segal [17]).

Gâteaux differentiability is weaker than Fréchet differentiability (used by Machina [12]), but is still rather strong. It requires two properties.

¹This version of B3 requires that for every lottery X , the decision maker prefers X to an even chance lottery of playing $X - \ell$ and $X + g$.

1. The representation function V is differentiable with respect to α along the segment $(1 - \alpha)F + \alpha F'$ for all distributions F and F' .
2. For a fixed F , the derivative with respect to α of $V((1 - \alpha)F + \alpha F')$ at $\alpha = 0$ is linear in F' .

These requirements are too strong for the analysis of most economic applications, as in many cases it is sufficient to assume that only indifference sets are well behaved. And as has been demonstrated by Dekel [5], preferences with well behaved indifference sets need not be differentiable.

In this paper we show that when preferences satisfy betweenness, the Gâteaux differentiability requirement can be dropped. Moreover, the modified version of B3 can be significantly weakened.² The set of betweenness preferences is of interest since it is the weakest extension of expected utility: all indifference sets are still hyperplanes (though not necessarily parallel to each other). In section 2 we provide definitions and state two facts that are taken from Safra and Segal [18]. In section 3 we prove two theorems for betweenness preferences satisfying Machina's behavioral H1 and H2 assumptions (or their opposites, see section 2), without assuming Gâteaux differentiability. In section 4 we use a modified version of assumption B3 that is much weaker than that of [18] and prove a general theorem that does not require any differentiability assumptions. Examples are provided in section 5 and all proofs are relegated to the Appendix.

2 Preliminaries

We assume throughout that preferences over distributions are representable by a functional V which is risk averse with respect to mean-preserving spreads, monotonically increasing with respect to first order stochastic dominance, and continuous with respect to the topology of weak convergence. Denote the set of all such functionals by $\mathcal{V}$.

According to the context, utility functionals are defined over lotteries (of the form $X = (x_1, p_1; \dots; x_n, p_n)$) or over cumulative distribution functions (denoted F, H). Degenerate cumulative distribution functions are denoted δ_x . For x, ℓ , and g , $H_{x, \ell, g}$ denotes the distribution of the lottery $(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$. When ℓ and g are fixed, we write H_x instead.

²The condition of footnote 1 must only be satisfied with respect to lotteries X with two outcomes at most.

Betweenness functionals in $\mathcal{V}$ (Chew [1, 2], Fishburn [7], and Dekel [5]) are characterized by linear indifference sets. That is, if F and H are in the indifference set $\mathcal{I}$, then for all $\alpha \in (0, 1)$, so is $\alpha F + (1 - \alpha)H$. Formally:

Definition 1 V satisfies betweenness if for all F and H satisfying $V(F) \geq V(H)$ and for all $\alpha \in (0, 1)$, $V(F) \geq V(\alpha F + (1 - \alpha)H) \geq V(H)$.

The fact that indifference sets of the betweenness functional are hyperplanes implies that for all F , the indifference set through F can also be viewed as an indifference set of an expected utility functional with vNM utility $u(\cdot; F)$. Following Machina [12], this function is called the local utility of V at F . Unlike Machina [12] and Safra and Segal [18], we do not require the functional to be Fréchet, or even Gâteaux, differentiable.

Machina [12] introduced the following assumptions.

Definition 2 The functional $V \in \mathcal{V}$ satisfies Hypothesis 1 (H1) if for any distribution F , $-\frac{u''(x; F)}{u'(x; F)}$ is a nonincreasing function of x .

The functional $V \in \mathcal{V}$ satisfies Hypothesis 2 (H2) if for all F and H such that F dominates H by first order stochastic dominance and for all x

$$-\frac{u''(x; F)}{u'(x; F)} \geq -\frac{u''(x; H)}{u'(x; H)}$$

In the sequel, we will also use the opposite of these assumptions, denoted $\neg$ H1 and $\neg$ H2. $\neg$ H1 says that for any distribution F , $-\frac{u''(x; F)}{u'(x; F)}$ is a nondecreasing function of x while $\neg$ H2 says that if F dominates H by first order stochastic dominance, then for all x

$$-\frac{u''(x; F)}{u'(x; F)} \leq -\frac{u''(x; H)}{u'(x; H)}$$

In Safra and Segal we strengthen Rabin's [15] results and proved the following claims. [18]³

Fact 1 Let $V \in \mathcal{V}$ be expected utility. Let $g > \ell > 0$ and $G > b - a$, and let

$$L > \left[(\ell + g) \frac{1 - \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}}}{1 - \frac{\ell}{g}} + (G + a - b) \left(\frac{\ell}{g}\right)^{\frac{b-a}{\ell+g}} \right] \frac{(1-p)}{p}. \quad (1)$$

³See proofs of Theorems 1 and 2 in [18].

If for all $x \in [a, b]$,

$$V(\delta_x) \geq V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$$

then

$$V(a, 1) \geq V(a - L, p; a + G, 1 - p).$$

That is, rejection of the small lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all $x \in [a, b]$ implies a rejection of the large lottery $(-L, p; G, 1 - p)$ at a . For example, a rejection of $(-100, \frac{1}{2}; 105, \frac{1}{2})$ at all wealth levels between 100,000 and 140,000 implies a rejection at wealth level 100,000 of the lottery $(-5,035, \frac{1}{2}; 10,000,000, \frac{1}{2})$.

Fact 2 Let $V \in \mathcal{V}$ be expected utility. Let $0 < \ell < g < L$ and let $b - a = L + g$. If for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then for all p and G satisfying

$$G < \frac{p}{1 - p}(\ell + g) \frac{\left(\frac{g}{\ell}\right)^{\frac{b-a}{\ell+g}} - 1}{\frac{g}{\ell} - 1}, \quad (2)$$

it follows that

$$V(b, 1) \geq V(b - g - L, p; b - g + G, 1 - p).$$

For example, a rejection of $(-100, \frac{1}{2}; 110, \frac{1}{2})$ at all wealth levels between 150,000 and 200,000 implies a rejection, at wealth level 200,000, of the lottery $(-50,000, 6.6 \cdot 10^{-6}; 100,000,000, 1 - 6.6 \cdot 10^{-6})$.

In the next definition we consider a stochastic version of B3 where the decision maker rejects the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at both deterministic and stochastic wealth levels. This modified version of assumption B3 is much weaker than the one used in [18], as the lottery X here can have only two outcomes at most.

Definition 3 (*Bi-Stochastic B3*): The functional V satisfies (ℓ, g) bi-stochastic B3 on $[a, b]$ if for all $X = (x, p; y, 1 - p)$ with support in $[a, b]$, $V(X) > V(X - \ell, \frac{1}{2}; X + g, \frac{1}{2})$ where $X + \alpha = (x + \alpha, p; y + \alpha, 1 - p)$.

3 Hypotheses 1 and 2

The results of this section imply that betweenness functionals are susceptible to Rabin-type criticism whenever they satisfy Hypotheses 1 or 2, or their opposites, $\neg H1$ or $\neg H2$.

Theorem 1 *Let $V \in \mathcal{V}$ be a betweenness functional satisfying $\neg H1$ or $\neg H2$. Let $g > \ell > 0$ and $G > b - a$, and let L satisfy inequality (1). If for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then*

$$V(a, 1) > V(a - L, p; a + G, 1 - p).$$

Theorem 2 *Let $V \in \mathcal{V}$ be a betweenness functional satisfying $H1$ or $H2$. Let $0 < \ell < g < L$, let $b - a = L + g$, and let p and G satisfy inequality (2). If for all $x \in [a, b]$, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$, then*

$$V(b, 1) \geq V(b - g - L, p; b - g + G, 1 - p).$$

In other words, calibration results like those of Facts 1 and 2 apply not only to expected utility theory, but also to betweenness functions satisfying Hypotheses 1, 2, or their opposites.

4 Betweenness and Bi-Stochastic B3

The lottery X in Definition 3 serves as background risk—risk to which the binary lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is added. Guiso, Jappelli and Terlizzese [9] find that when investors face greater background risk or uninsurable income risk, they reduce their holdings of risky assets. Hochguertel [11] finds that “higher subjective uncertainty about incomes obtaining in five years from now is associated with safer and more liquid portfolios.” Paiella and Guiso [13] too provide some data showing that decision makers are more likely to reject a given lottery in the presence of background risk.⁴ Accordingly, if rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is likely when added to non-stochastic wealth, it is even more likely when added to a lottery. In other words, (ℓ, g) bi-stochastic B3 is as acceptable as the behavioral assumption that decision makers reject the lottery $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ used by Rabin. Note that for expected utility functionals the stochastic version of B3 is equivalent to the deterministic one: A rejection of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ at all deterministic wealth levels implies its rejection at all stochastic wealth levels.

The next theorem provides calibration results when bi-stochastic B3 replaces Hypotheses 1 and 2 and their opposites.

⁴Rabin and Thaler [16] on the other hand seem to claim that a rejection of a small lottery is likely only when the decision maker is unaware of the fact that he is exposed to many other risks.

Theorem 3 *Let $V \in \mathcal{V}$ be a betweenness functional. Let $0 < \ell < g$, $G > c$, and let L satisfy eq. (1) for $b - a = c$. Also, let $\bar{p}$ and $\bar{G}$ satisfy eq. (2) for $b - a = \bar{L} + g$. If V satisfies (ℓ, g) bi-stochastic B3 on $[w - g - \bar{L}, w + c]$ then either*

1. $V(w, 1) > V(w - L, p; w + G, 1 - p)$; or
2. $V(w, 1) > V(w - g - \bar{L}, \bar{p}; w - g + \bar{G}, 1 - \bar{p})$.

In other words, a betweenness functional satisfying bi-stochastic B3 is bound to reject lotteries from at least one of the two Facts of section 2. That these results are weaker than those of expected utility (where lotteries are rejected according to both Facts 1 and 2) is besides the point. Theorems 1-3 mimic Rabin's claim against expected utility for betweenness functionals: Seemingly reasonable behavior in the small leads to absurd behavior in the large.

5 Some Betweenness Functionals

In this section we analyze two specific betweenness functionals: Chew's [1] weighted utility and Gul's [10] disappointment aversion theories. Both are supported by simple sets of axioms.

5.1 Weighted Utility

We show first that the expected utility analysis of Facts 1 and 2 can be extended to the case of weighted utility without the need to assume either H1 — $\neg$ H2 or bi-stochastic B3. This functional, suggested by Chew [1], is given by

$$V(F) = \frac{\int v dF}{\int h dF} \quad (3)$$

for some functions $v, h : \mathfrak{R} \rightarrow \mathfrak{R}$. The local utility of V at δ_w is a function for which the indifference set through δ_w coincides with the corresponding indifference set of V , that is

$$\begin{aligned} \int u(x; \delta_w) dF(x) = u(w; \delta_w) &\iff \frac{\int v(x) dF(x)}{\int h(x) dF(x)} = \frac{v(w)}{h(w)} \\ &\iff \int \left[v(x) - \frac{v(w)}{h(w)} h(x) \right] dF(x) = 0 \end{aligned}$$

As vNM functions are unique up to positive linear transformations, we can assume, without loss of generality, that

$$u(x; \delta_w) = v(x) - \frac{v(w)}{h(w)}h(x) \quad (4)$$

Suppose that for every $x \in [a, b]$, $V(\delta_x) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$ and that $h(x) \neq \frac{1}{2}h(x - \ell) + \frac{1}{2}h(x + g)$. As $(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$ is below the indifference curve of V through δ_x , and as this is also an indifference curve of the expected utility preferences with the vNM function $u(\cdot; \delta_x)$, it follows that $u(x; \delta_x) - [\frac{1}{2}u(x - \ell; \delta_x) + \frac{1}{2}u(x + g; \delta_x)] > 0$. Replacing u by the expression on the right hand side of Eq. (4) implies that at $w = x$

$$v(x) - \frac{1}{2}[v(x - \ell) + v(x + g)] - \frac{v(w)}{h(w)} \left[h(x) - \frac{1}{2}[h(x - \ell) + h(x + g)] \right] > 0$$

Differentiating the left hand side with respect to w we obtain

$$-\frac{v'(w)h(w) - v(w)h'(w)}{h^2(w)} \left[h(x) - \frac{1}{2}[h(x - \ell) + h(x + g)] \right] \quad (5)$$

Monotonicity with respect to first order stochastic dominance implies that $v(w)h'(w) - v'(w)h(w)$ does not change sign [1, Corollary 5].⁵ The sign of the derivative of the expression in (5) with respect to w thus depends on the sign of $h(x) - \frac{1}{2}[h(x - \ell) + h(x + g)]$, which was assumed to be different from zero. In other words, for a given x , the inequality $u(x; \delta_w) > \frac{1}{2}[u(x - \ell; \delta_w) + u(x + g; \delta_w)]$ is satisfied either for all $w \in [x, b]$ (denote the set of x having this property K_1) or for all $w \in [a, x]$ (likewise, denote the set of x having this property K_2).

For every $x \in [a, b]$, either $x \in K_1$, or $x \in K_2$. At least half of the points $a, \dots, a + i(\ell + g), \dots, b$ therefore belong to the same set, K_1 or K_2 . In order to obtain calibration results, we need enough wealth levels at which $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ is rejected, and moreover, the distance between any two such points should be at least $\ell + g$. In [18] we divided the segment $[a, b]$ into $\frac{b-a}{\ell+g}$

⁵In Chew's [1] notation, $V(F) = \int \alpha \phi dF / \int \alpha dF$. To obtain the representation (3), let $h = \alpha$ and $\phi = v/h$. Monotonicity in x of $\alpha(x)(\phi(x) - \phi(s))$ for all s is equivalent to monotonicity of $v(x) - h(x)v(s)/h(s)$, that is, to $v'(x) - h'(x)v(s)/h(s) \neq 0$.

segments to obtain Facts 1 and 2. Here we can get only $\frac{b-a}{2(\ell+g)}$ such points, hence we need to observe rejections of $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ on twice the ranges needed for these Facts. If K_1 contains these points, take $w = b$ and apply Fact 2. If K_2 contains these points, take $w = a$ and apply Fact 1.⁶

5.2 Disappointment Aversion

Gul's [10] disappointment aversion is a special case of betweenness. According to this theory the value of the lottery $p = (x_1, p_1; \dots; x_n, p_n)$ is given by

$$V(p) = \gamma(\alpha) \sum_x q(x)u(x) + [1 - \gamma(\alpha)] \sum_x r(x)u(x)$$

where 1. $p = \alpha q + (1 - \alpha)r$ for a lottery q with positive probabilities on outcomes that are better than p and a lottery r with positive probabilities on outcomes that are worse than p ; and 2. $\gamma(\alpha) = \alpha/[1 + (1 - \alpha)\beta]$ for some $\beta \in (-1, \infty)$. Risk aversion behavior is obtained whenever u is concave and $\beta \geq 0$ [10, Th. 3].

We show that for this functional, (ℓ, g) bi-stochastic B3 with respect to binary lotteries in $[r, s]$ is almost equivalent to the requirement that an expected utility maximizer with the vNM utility function u will reject the lotteries $(-\ell, \frac{1}{2}; g, \frac{1}{2})$ on this segment.

Claim 1 *Let $V \in \mathcal{V}$ be a disappointment aversion functional with the associated utility function u . If for every x , $u(x) > \frac{1}{2}u(x - \ell) + \frac{1}{2}u(x + g)$, then V satisfies bi-stochastic B3. Conversely, if V satisfies bi-stochastic B3 with respect to binary random lotteries then except maybe for a segment of length $g + \ell$, $u(x) > \frac{1}{2}u(x - \ell) + \frac{1}{2}u(x + g)$.*

By claim 1, disappointment aversion theory is compatible with (ℓ, g) bi-stochastic B3. Moreover, to a certain extent, the requirements for this form of risk aversion are similar to Rabin's requirement under expected utility theory. By Theorem 3, disappointment aversion is vulnerable to both Facts 1 and 2.

⁶Since weighted utility is Gâteaux differentiable, risk aversion implies the concavity of $u(\cdot; \delta_w)$. (See Lemma 1 in [18]).

5.3 Chew & Epstein on Samuelson

Samuelson [19] proved that if a lottery X is rejected by an expected utility maximizer at all wealth levels, then the decision maker also rejects an offer to play the lottery n times, for all values of n . Like Rabin's, this result indicates a problem with expected utility theory, as positive expected value of X implies that the probability of losing money after playing the lottery many times goes to zero. Therefore, even though a rejection of a small lottery with positive expected value may seem reasonable, rejection of an option to play the same lottery many times may be a mistake (see also Foster and Hart [8]).

Chew and Epstein [3] showed, by means of an example, that this result does not necessarily hold for non-expected utility preferences. They suggested a special form of a betweenness functional V , given implicitly by

$$\int \varphi(x - V(F))dF = 0$$

and proved that under some conditions, a lottery with a positive expected payoff may be rejected when played once, but many repetitions of it may still be attractive. They similarly identified rank dependent functionals with the same property.

We don't find these results to contradict ours. We do not claim that *all* large lotteries with high expected return will be rejected. Facts 1 and 2 identify *some* very attractive large lotteries that under some conditions are rejected. In the same way in which one cannot dismiss the Allais paradox by claiming that there are many other decision problems that are consistent with expected utility theory, one cannot claim here that the fact that non-expected utility theories are not vulnerable to Samuelson's criticism implies that they are also immune to Rabin's. Moreover, if the number of times one needs to play a lottery before it becomes attractive is absurdly high (say, one million), Samuelson's identification of unreasonable behavior is still valid, as is Rabin's.

Appendix

Proof of Theorems 1 and 2 Let $\mathcal{I}$ be the indifference set of V through δ_w and let $u(\cdot; \delta_w)$ be one of the increasing vNM utilities obtained from $\mathcal{I}$

(note that all these utilities are related by positive affine transformations). Obviously, $\mathcal{I}$ is also an indifference set of the expected utility functional defined by $U(F) = \int u(x; \delta_w) dF(x)$. By monotonicity, V and U increase in the same direction relative to the indifference set $\mathcal{I}$.

We show first that $u(\cdot; \delta_w)$ is concave. Suppose not. Then there exist z and H such that $z = E[H]$ and $U(H) > u(z; \delta_w)$. By monotonicity and continuity, there exist p and z' such that $(z - w)(z' - w) < 0$ and $(z, p; z', 1 - p) \in \mathcal{I}$. As $U(H) > u(z; \delta_w)$, it follows that $(H, p; z', 1 - p)$ is above $\mathcal{I}$ with respect to the expected utility functional U . Therefore, $(H, p; z', 1 - p)$ is also above $\mathcal{I}$ with respect to V , which is a violation of risk aversion, since $(H, p; z', 1 - p)$ is a mean preserving spread of $(z, p; z', 1 - p)$.

By betweenness, $V(x, 1) > V(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$ implies that, for all $\varepsilon > 0$, $V(x, 1) > V(x, 1 - \varepsilon; x - \ell, \frac{\varepsilon}{2}; x + g, \frac{\varepsilon}{2})$. Using the local utility at δ_x we obtain

$$u(x; \delta_x) > \frac{1}{2}u(x - \ell; \delta_x) + \frac{1}{2}u(x + g; \delta_x) \quad (6)$$

THEOREM 1: Assume $\neg H1$ or $\neg H2$. We show that for every $x \in [a, b]$, $u(x; \delta_a) > \frac{1}{2}u(x - \ell; \delta_a) + \frac{1}{2}u(x + g; \delta_a)$. Eq. (6) holds for $x = a$, and $\neg H1$ implies

$$u(x; \delta_a) > \frac{1}{2}u(x - \ell; \delta_a) + \frac{1}{2}u(x + g; \delta_a) \quad (7)$$

Also, since $x \geq a$, $\neg H2$ implies that the local utility $u(\cdot; \delta_a)$ is more concave than the local utility $u(\cdot; \delta_x)$ and hence eq. (7).

By Fact 1, it now follows that the expected utility decision maker with the vNM function $u(\cdot; \delta_a)$ satisfies $u(a; \delta_a) > pu(a - L; \delta_a) + (1 - p)u(a + G; \delta_a)$ and the lottery $(a - L, p; a + G, 1 - p)$ lies below the indifference set $\mathcal{I}$. Therefore,

$$V(a, 1) > V(a - L, p; a + G, 1 - p)$$

THEOREM 2: Assume H1 or H2. Here we show that for every $x \in [a, b]$, $u(x; \delta_b) > \frac{1}{2}u(x - \ell; \delta_b) + \frac{1}{2}u(x + g; \delta_b)$. Eq. (6) holds in particular for $x = b$, and therefore H1 implies that for $x \leq b$,

$$u(x; \delta_b) > \frac{1}{2}u(x - \ell; \delta_b) + \frac{1}{2}u(x + g; \delta_b) \quad (8)$$

Also, since $x \leq b$, H2 implies that the local utility $u(\cdot; \delta_b)$ is more concave than the local utility $u(\cdot; \delta_x)$. Hence starting at eq. (6) and moving from δ_x to δ_b , H2 implies eq. (8).

By Fact 2 it now follows that the expected utility decision maker with the vNM function $u(\cdot; \delta_b)$ satisfies $u(b; \delta_b) > pu(b-L; \delta_b) + (1-p)u(b+G; \delta_b)$ and the lottery $(b-L, p; b+G, 1-p)$ lies below the indifference set $\mathcal{I}$. Therefore,

$$V(b, 1) > V(b-g-L, p; b-g+G, 1-p) \quad \blacksquare$$

Proof of Theorem 3 For every x and z such that $w+c \geq x > w > z \geq w-\bar{L}$ there is a probability q such that $X = (x, q; z, 1-q)$ satisfies $V(X) = V(w, 1)$. By (ℓ, g) bi-stochastic B3,

$$V(x, q; z, 1-q) > V(x-\ell, \frac{q}{2}; x+g, \frac{q}{2}; z-\ell, \frac{1-q}{2}; z+g, \frac{1-q}{2}) \quad (9)$$

Denote the distribution functions of the two lotteries in eq. (9) by F and F' , respectively. Betweenness implies that the local utilities at δ_w and at F are the same (up to a positive linear transformation). Hence, by using the local utility $u(\cdot; \delta_w)$, we obtain,

$$\begin{aligned} & q[u(x; \delta_w) - \frac{1}{2}(u(x-\ell; \delta_w) + u(x+g; \delta_w))] + \\ & (1-q)[u(z; \delta_w) - \frac{1}{2}(u(z-\ell; \delta_w) + u(z+g; \delta_w))] > 0 \end{aligned} \quad (10)$$

Suppose there are $w+c > x^* > w > z^* > w-\bar{L}$ such that $u(x^*; \delta_w) \leq \frac{1}{2}[u(x^*-\ell; \delta_w) + u(x^*+g; \delta_w)]$ and $u(z^*; \delta_w) \leq \frac{1}{2}[u(z^*-\ell; \delta_w) + u(z^*+g; \delta_w)]$. Then inequality (10) is reversed, and by betweenness, the inequality at (9) is reversed; a contradiction to bi-stochastic B3. Therefore, at least one of the following holds.

1. For all $x \in [w, w+c]$,

$$u(x; \delta_w) > \frac{1}{2}[u(x-\ell; \delta_w) + u(x+g; \delta_w)]$$

and, similarly to the proof of Theorem 1 (replace a with w in eq. (7)), betweenness implies that $V(w, 1) > V(w-L, p; w+G, 1-p)$ and the first claim of the theorem is satisfied.

2. For all $x \in [w-\bar{L}, w]$,

$$u(x; \delta_w) > \frac{1}{2}[u(x-\ell; \delta_w) + u(x+g; \delta_w)]$$

and, using an argument similar to that of the proof of Theorem 2 (replace b with w in eq. (8)), betweenness implies $V(w, 1) > V(w-g-\bar{L}, \bar{p}; w-g+\bar{G}, 1-\bar{p})$ and the second claim of the theorem is satisfied. $\blacksquare$

Proof of Claim 1 Suppose V satisfies bi-stochastic B3. The local utility of V at δ_w is

$$u(x; \delta_w) = \begin{cases} u(x) & x \leq w \\ \frac{u(x) + \beta u(w)}{1 + \beta} & x > w \end{cases}$$

(see Gul [10]). By the proof of Theorem 3, for every w , either for all $x \leq w$ or for all $x \geq w$,

$$u(x; \delta_w) > \frac{1}{2}u(x - \ell; \delta_w) + \frac{1}{2}u(x + g; \delta_w) \quad (11)$$

Let $w^* = \sup\{w : \forall x \leq w, \text{ eq. (11) is satisfied}\}$. If this set is empty, define $w^* = -\infty$. By the definition of the local utility, for all $x \leq w^* - g$, $u(x) > \frac{1}{2}u(x - \ell) + \frac{1}{2}u(x + g)$. Also, by the definition of w^* and by the proof of Theorem 3, for all $x > w^* + \ell$ let $w = x - \ell > w^*$ and obtain

$$u(x; \delta_w) > \frac{1}{2}u(x - \ell; \delta_w) + \frac{1}{2}u(x + g; \delta_w)$$

hence $v(x) > \frac{1}{2}v(x - \ell) + \frac{1}{2}v(x + g)$, where $v(x) = \frac{u(x) + \beta u(w)}{1 + \beta}$. As v is an affine transformation of u , u too satisfies this property.

Suppose now that for all x , $u(x) > \frac{1}{2}u(x - \ell) + \frac{1}{2}u(x + g)$. Then for every w and x , eq. (11) is satisfied. The local utility $u(\cdot; \delta_w)$ therefore rejects $(x - \ell, \frac{1}{2}; x + g, \frac{1}{2})$ at all x . Starting from $X = (x_1, p; x_2, 1 - p) \sim \delta_w$, we observe that by betweenness the local utility at X is $u(\cdot; \delta_w)$, and that this local utility rejects the (ℓ, g) randomness that is added to w . By betweenness, this randomness is also rejected by the functional V . ■

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