Transitive Regret

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Abstract

Preferences may arise from regret, i.e., from comparisons with alternatives forgone by the decision maker. We ask whether regret-based behavior is consistent with non-expected utility theories of transitive choice. We show that the answer is no. If choices are governed by ex ante regret and elation then non-expected utility preferences must be intransitive.

Key words: Transitivity, regret, expected utility

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1   Introduction

Standard models of choice assume that decision makers act as if they maximize a preference relation over sets of options and these preferences are assumed to be independent of the environment. There are however good reasons to challenge this assumption. Preferences may depend on the decision maker's holding (reference point), on other people's holdings (envy), or on the choice set itself.

One such model is "regret theory" (Bell [4] and Loomes and Sugden [10]). According to this theory the decision maker anticipates his future feelings about the choice he is about to make and acts according to these feelings. This approach is natural when the decision maker has to choose between two (or more) random variables. Once the uncertainty is resolved he will know what outcome he received, but also what outcome he could have received had he chosen an alternative option. This comparison may cause him elation — if his actual outcome is better than the alternative — or regret.

Formally, let X and Y be two random variables with money outcomes. Let $\psi(x, y)$ measure the regret or elation a person feels when observing that he won x while the alternative choice would have landed him y. Choosing X over Y thus leads, ex ante, to a lottery $\Psi(X, Y)$ where the outcomes are $\psi(x, y)$. Choice is based on regret and elation if there is a functional V over regret/elation lotteries such that X is chosen over Y iff $V(\Psi(X, Y)) > 0$.

The question we ask is simple: What functionals V and regret/elation functions ψ are consistent with transitive choice? That is, when is it true that if $V(\Psi(X, Y)) > 0$ and $V(\Psi(Y, Z)) > 0$, then $V(\Psi(X, Z)) > 0$ as well? The main result of the paper is that transitive choice implies expected utility. This conclusion does not depend on V being linear in probabilities or separable across states.

One can read this result in two different ways. It offers an axiomatization of expected utility theory without making any references to mixture spaces (see Kreps [9] for summary of terms and basic results). But the real contribution is the impossibility result that shows that regret is inherently intransitive. If so, then one must either conclude that (i) Regret, despite its clear psychological appeal, cannot be used in standard economic models; or that (ii) Regret must be analyzed in a more involved way than in Bell [4] and Loomes and Sugden [10] — for example, as it is done in Hayashi [7]

and Sarver [14], or by defining regret with respect to foregone distributions rather than foregone outcomes (see Machina [13] and Starmer [15] for some steps in this direction); or that (*iii*) Models of intransitive preferences *must* be incorporated into economics as in Fishburn and LaValle [6] or Loomes and Sugden [11].[1]

The paper is organized as follows. The model and the main result are presented in the next section. Section 3 offers an outline of the proof while the details of the proof appear in the appendix. Section 4 concludes with a discussion of some possible extensions.

2   The model and main result

Let $\mathcal{L}$ be the set of real finite-valued random variables over (S, Σ, P) with $S = [0, 1]$, Σ being the standard Borel σ algebra on S, $\mathrm{P} = \mu$, the Lebesgue measure, and the set of outcomes being the bounded interval $[\underline{x}, \bar{x}]$. The decision maker has a complete, transitive, continuous, and monotonic preference relation $\succeq$ over $\mathcal{L}$.

Definition 1 *The continuous function* $\psi : [\underline{x}, \bar{x}] \times [\underline{x}, \bar{x}] \to \Re$ *is a* regret *function if for all* x, $\psi(x, x) = 0$, $\psi(x, y)$ *is strictly increasing in* x, *and strictly decreasing in* y.

If in some event X yields x and Y yields y then $\psi(x, y)$ is a measure of the decision maker's *ex post* feelings (of regret if $x < y$ or elation if $x > y$) about the choice of X over Y. This leads to the next definition:

Definition 2 *Let* $X, Y \in \mathcal{L}$ *where* $X = (x_1, s_1; \dots; x_n, s_n)$ *and* $Y = (y_1, s_1; \dots; y_n, s_n)$. *The* regret lottery *evaluating the choice of* X *over* Y *is*

$$\Psi(X, Y) = (\psi(x_1, y_1), p_1; \dots; \psi(x_n, y_n), p_n)$$

where $p_i = \mathrm{P}(s_i)$, $i = 1, \dots, n$. *Denote the set of regret lotteries by* $\mathcal{R} = \{\Psi(X, Y) : X, Y \in \mathcal{L}\}$.

For brevity we refer to ψ and Ψ as regret function and regret lottery respectively, even though they encompass both regret and elation.

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[1] See also Starmer [15] for further references.

Definition 3 *The preference relation $\succeq$ is* regret based *if there is a regret function ψ and a continuous functional V which is defined over regret lotteries such that for any $X, Y \in \mathcal{L}$*

$$X \succeq Y \quad \textit{if and only if} \quad V(\Psi(X,Y)) \geqslant 0$$

The main result of this paper is the following.

Theorem 1 *Let $\succeq$ be a complete, transitive, continuous, and monotonic preference relation over the set $\mathcal{L}$ of random variables. The relation $\succeq$ is regret based if and only if it is expected utility.*

This theorem implies, in particular, that unless they reduce to expected utility, the regret models of Bell [4], Loomes and Sugden [10], and Sugden [16] are intransitive.[2] Recently, Hayashi [7] and Sarver [14] presented non-expected utility models of regret that are transitive but these papers depart from the standard regret model of [4] and [10]. In Hayashi [7], the decision maker has multiple priors and selects the option that minimizes the maximum possible ex ante expected regret under this set of priors. If the prior is unique then Hayashi's model reduces to expected utility.

According to Sarver [14], the decision maker chooses between menus of lotteries and a lottery from the selected menu. At the time these two choices are made, the decision maker does not know his preferences over lotteries. Later, when he learns his preferences, the selected lottery may turn out to be inferior to another lottery that is in the menu he selected, causing ex post regret. Each of the decision maker's possible preference relations is transitive but this is not inconsistent with Theorem 1 because in our model the decision maker knows his preferences at the time of choice while in [14] the decision maker is using a set of preferences. Indeed, if the set of preferences is a singleton then Sarver's model too is reduced to expected utility.

Theorem 1 is proved as follows. It is straightforward to show that expected utility is regret based (with $\psi(x,y) = u(x) - u(y)$ and $V(\Psi(X,Y)) =$

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[2] An important exception is the case where the choice set consists of statistically independent random variables and for the two lotteries $(x_1, p_1; \ldots; x_n, p_n)$ and $(y_1, q_1; \ldots; y_m, q_m)$ the probability of the regret $\psi(x_i, y_j)$ is $p_i q_j$ (see Starmer [15, pp. 355–6] and Machina [13, pp. 138–140]). For example, Chew's [5] weighted utility theory is consistent with this form of regret.

$\sum_i p_i \psi(x_i, y_i)$). That any transitive regret-based preferences must be expected utility is proved in a sequence of steps summarized below:

1. Preferences are probabilistically sophisticated. That is, if X and Y have the same distribution, then $X \sim Y$ (Section 3.1, Proposition 1).

2. The indifference curve of V through zero, $\{R : V(R) = 0\}$, is linear in probabilities (Section 3.3, Lemmas 3–5).

3. There exists V as in Def. 3 which is linear in probabilities for all regret lotteries R (Section 3.4, Lemma 6).

4. The preference relation $\succeq$ is expected utility (Section 3.4, Lemma 7).

3 Proof of the Theorem

3.1 Probabilistic Sophistication

When preferences are regret based, the decision maker cares about what events will happen as this will tell him what are the alternative outcomes he could have received had he chosen differently. When the decision maker learns that #4 on a die yields \$100 under X and \$150 under Y, the fact that these two outcomes are linked to the same state of the world is important, but the state itself is not. Consequently, only the probabilities of the underlying states are relevant for regret between X and Y. As long as the probability of #1 is the same as that of #4, it makes no difference whether the regret $\psi(100, 150)$ is obtained when the number is 1 or 4. This is why regret lotteries are evaluated with respect to their probabilities and not with respect to the generating events. Proposition 1 shows that this observation has a significant implication to the evaluation of random variables. For $X \in \mathcal{L}$ let F_X be the distribution of X, that is, $F_X(x) = \mathrm{P}(X \leqslant x)$. Also, for a permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, let $\pi(X) = (x_{\pi(1)}, s_1; \ldots; x_{\pi(n)}, s_n)$.

Proposition 1 (Probabilistic sophistication): *Let $\succeq$ be a continuous and transitive regret based preference relation over $\mathcal{L}$. For any two random variables $X, Y \in \mathcal{L}$, if $F_X = F_Y$ then $X \sim Y$.*

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3.2 Preliminary results

We assumed that outcomes are in a finite interval $[\underline{x}, \bar{x}]$. Let $\underline{r} = \psi(\underline{x}, \bar{x})$ and $\bar{r} = \psi(\bar{x}, \underline{x})$. Clearly $-\infty < \underline{r} < 0 < \bar{r} < \infty$. As $\psi(x, y)$ is continuous, increasing in x and decreasing in y, it follows that the set of regret lotteries $\mathcal{R}$ defined in Def. 2 is the set of finite-valued lotteries with outcomes in the interval $[\underline{r}, \bar{r}]$. The following monotonicity properties of V are inherited from the monotonicity of $\succeq$.

Lemma 1 *Let R, R' be two distinct regret lotteries such that R dominates R' by first-order stochastic dominance (FOSD).*

1. *If $V(R) = 0$ then $V(R') < 0$.*

2. *If $V(R') = 0$ then $V(R) > 0$.*

The next lemma permits a selection of regret lotteries that are skew symmetric in regret and elation.

Lemma 2

1. *If $\psi(x, y) = \psi(x', y')$ then $\psi(y, x) = \psi(y', x')$.*

2. *$\psi(x, y) = -\psi(y, x)$ is without loss of generality.*

We will assume throughout that $\psi(x, y) = -\psi(y, x)$ and that $\Psi(X, Y) = -\Psi(Y, X) \equiv (-\psi(y_1, x_1), p_1; \ldots; -\psi(y_n, x_n), p_n)$. Moreover, $\underline{r} = -\bar{r}$.

3.3 The indifference curve through zero is linear

A regret lottery R is generated by a permutation if there exists a random variable $X = (x_1, s_1; \ldots; x_n, s_n)$, $\mathrm{P}(s_i) = \frac{1}{n}$, and a permutation π of X such that $\Psi(X, \pi(X)) = R$. By Proposition 1, if R is generated by a permutation then $V(R) = 0$. The next lemma shows that the subset of $\{R : V(R) = 0\}$ that is generated by permutations is convex.

Lemma 3 *If R and R' are generated by permutations then so is $\frac{1}{2}R + \frac{1}{2}R'$.*

As R, R' are generated by permutations we have $V(R) = V(R') = 0$, and by Lemma 3, $V(\frac{1}{2}R + \frac{1}{2}R') = 0$. As is shown by the next example, one cannot guarantee that every regret lottery $R = (r_1, \frac{1}{n}; \ldots; r_n, \frac{1}{n})$ such that $V(R) = 0$ is generated by a permutation.

Example 1 Consider an expected value maximizer whose choice set consists of random variables with prizes in the interval $[-3, 3]$. This individual's regret function is $\psi(x, y) = x - y$ and he is indifferent between X and Y defined below, where $P(s_i) = 0.2$:

$$\begin{aligned} X &= (3, s_1; 3, s_2; -1, s_3; -1, s_4; -1, s_5) \\ Y &= (-3, s_1; -3, s_2; 3, s_3; 3, s_4; 3, s_5) \end{aligned}$$

As $X \sim Y$, $V(\Psi(X, Y)) = V(6, 0.2; 6, 0.2; -4, 0.2; -4, 0.2; -4, 0.2) = 0$. But there does not exist a random variable $\hat{Z}$ with outcomes in the interval $[-3, 3]$ and a permutation π such that $\Psi(X, Y) = \Psi(\hat{Z}, \pi(\hat{Z}))$. To see why, observe that the elation 6 must be generated by the outcomes -3 and 3. From outcome 3 only regret is possible, and as the only regret level is -4, the outcome 3 must be paired with -1. From outcome -1 one cannot generate elation 6, nor can one have regret -4.[3] $\qquad\square$

The problem is that the outcomes in X and Y are far. However, as is shown by the next example, one can find in Example 1 a random variable Z whose outcomes are sufficiently close to both X and Y such that $X \sim Z \sim Y$ and the regret lotteries $\Psi(X, Z)$ and $\Psi(Z, Y)$ are generated by permutations.

Example 2 Using the notation of Example 1, let $Z = (0, s_1; 0, s_2; 1, s_3; 1, s_4; 1, s_5)$. Thus

$$\Psi(X, Z) = \Psi(Z, Y) = (3, 0.2; 3, 0.2; -2, 0.2; -2, 0.2; -2, 0.2)$$

Define

$$\begin{aligned} \hat{Z} &= (3, s_1; 0, s_2; -3, s_3; -1, s_4; 1, s_5) \\ \pi(\hat{Z}) &= (0, s_1; -3, s_2; -1, s_3; 1, s_4; 3, s_5) \end{aligned}$$

Then $\Psi(\hat{Z}, \pi(\hat{Z})) = \Psi(X, Z) = \Psi(Z, Y)$. $\qquad\square$

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[3]If, instead, we had assumed that the set of outcomes was $(-\infty, \infty)$, then any $R = (r_1, \frac{1}{n}; \ldots; r_n, \frac{1}{n})$ such that $V(R) = 0$ would be generated by a permutation leading to a simpler proof of Theorem 1.

This idea is formalized below.

Lemma 4 *Let $X \sim Y$ where $X = (x_1, s_1; \ldots; x_n, s_n)$, $Y = (y_1, s_1; \ldots; y_n, s_n)$, and $\mathrm{P}(s_i) = \frac{1}{n}$. Then there is a sequence $X = Z_1 \sim Z_2 \sim \ldots \sim Z_k = Y$ such that for every $\ell = 1, \ldots, k-1$ there is a regret lottery $\hat{Z}_\ell$ and a permutation π_ℓ so that $\Psi(Z_\ell, Z_{\ell+1}) = \Psi(\hat{Z}_\ell, \pi_\ell(\hat{Z}_\ell))$.*

Thus, even if a regret lottery $R = (r_1, \frac{1}{n}; \ldots; r_n, \frac{1}{n})$ with $V(R) = 0$ is not generated by a permutation, one can find a sequence of random variables $Z_1 \sim \ldots \sim Z_k$ such that each $\Psi(Z_\ell, Z_{\ell+1})$ is generated by a permutation and $R = \Psi(Z_1, Z_k)$. This is used to prove that the set $\{R : V(R) = 0\}$ is convex.

Lemma 5 *If $V(R) = V(R') = 0$, then $V\left(\frac{1}{2}R + \frac{1}{2}R'\right) = 0$.*

3.4 V is linear in probabilities and $\succeq$ is expected utility

The following lemma establishes that all indifference curves of V are linear.

Lemma 6

1. *There is a function $v : [-\bar{r}, \bar{r}] \to \Re$ such that $V(R) \gtreqqless 0$ iff $\mathrm{E}[v(R)] \gtreqqless 0$.*

2. *Moreover, v is strictly increasing with $v(0) = 0$ and $v(\psi(x, y)) = -v(\psi(y, x))$ for all x, y.*

We now use the function v to create a function u on outcomes which will turn out to be the vNM utility claimed by Theorem 1.

Lemma 7 *There exists an increasing function $u : [\underline{x}, \bar{x}] \to \Re$ such that*

$$v(\psi(x, y)) = u(x) - u(y)$$

From the last two lemmas we have for $X = (x_1, s_1; \ldots; x_n, s_n)$ and $Y = (y_1, s_1; \ldots; y_n, s_n)$ where $\mathrm{P}(s_i) = p_i$,

$$X \succeq Y \Longleftrightarrow$$
$$V(\Psi(X,Y)) \geqslant 0 \Longleftrightarrow$$
$$\sum_i p_i v(\psi(x_i, y_i)) \geqslant 0 \Longleftrightarrow$$
$$\sum_i p_i [u(x_i) - u(y_i)] \geqslant 0 \Longleftrightarrow$$
$$\mathrm{E}[u(X)] \geqslant \mathrm{E}[u(Y)]$$

which is the claim of the theorem. ∎

4   A concluding remark

The intuition of regret with respect to an alternative that was not taken is the basis of the above analysis. A somewhat similar situation is often discussed in the literature, where preferences depend on a certain reference point. This reference point may be the decision maker's current holding (eg. Kahneman, Knetsch, and Thaler [8] or Tversky and Kahneman [17]), or the holdings of other people (eg. Bagwell and Bernheim [2], Ball et. al. [3], or Maccheroni, Marinacci, and Rustichini [12]). Formally, such preferences are indexed by the reference point. That is, $X \succeq_Z Y$ means that given the reference point Z, X is preferred to Y. As long as Z is fixed, these preferences behave like standard preferences. The interesting question is what is the connection between preferences that are conditioned on different reference points. Our analysis may contribute to this issue in the following way.

Suppose that a decision maker has to choose between two random variables X and Y. He knows that another person he envies has (or will have) either X or Y, both equally likely. One approach is to say that the other person holds the lottery $\frac{1}{2}X + \frac{1}{2}Y$ and that this mixture should serve as the reference point for the decision maker. This approach misses the main feature of envy, namely the feelings the decision maker will experience once he knows his position relative to the other person. We suggest that this decision maker's behavior should be modeled as a choice between the envy lotteries $(X|X, \frac{1}{2}; X|Y, \frac{1}{2})$ and $(Y|X, \frac{1}{2}; Y|Y, \frac{1}{2})$, where $A|B$ means "A when the other person holds B." Obviously $X|X$ and $Y|Y$ lead to no envy, so the choice

between the above two envy lotteries depends on the comparison of $X|Y$ with $Y|X$. Our formal analysis extends naturally to such situations.

5 Appendix

Proof of Proposition 1: Let $X = (x_1, s_1; \ldots; x_n, s_n)$ and $Y = (y_1, s'_1; \ldots; y_n, s'_n)$ be such that $F_X = F_Y$.

Case 1: $s_i = s'_i$ and $\mathrm{P}(s_i) = \frac{1}{n}$, $i = 1, \ldots, n$. Then there is a permutation $\hat{\pi}$ such that $Y = \hat{\pi}(X)$. Obviously, $\Psi(X, \hat{\pi}(X)) = \Psi(\hat{\pi}^i(X), \hat{\pi}^{i+1}(X))$, hence, as $\hat{\pi}^n(X) = X$, it follows by transitivity that for all i, $X \sim \hat{\pi}^i(X)$. In particular, $X \sim Y$.

Case 2: For all i, j, $\mathrm{P}(s_i \cap s'_j)$ is a rational number. Let N be a common denominator of all these fractions. X and Y can now be written as in Case 1 with equiprobable events $t_1, \ldots, t_N$.

Case 3: There exist i, j, s.t. $\mathrm{P}(s_i \cap s'_j)$ is irrational. Any random variable $Z = (z_1, t_1; \ldots; z_n, t_n)$ is the limit of $Z^k = (z_1^k, t_1^k; \ldots; z_{2^k}^k, t_{2^k}^k)$ where for all k and ℓ, $\mathrm{P}(z_\ell^k) = 2^{-k}$. This case follows by continuity from case 2. ∎

Proof of Lemma 1: Let R, R' be two regret lotteries. As usual, R dominates R' by FOSD iff R and R' can be written as $R = (r_1, p_1; \ldots; r_n, p_n)$ and $R' = (r'_1, p_1; \ldots; r'_n, p_n)$ where for all i, $r_i \geqslant r'_i$.

From the continuity of ψ we know that for every $r \in [\underline{r}, \bar{r}]$ there exist $x, y \in [\underline{x}, \bar{x}]$ such that $r = \psi(x, y)$. Hence there are $X, Y \in \mathcal{L}$ such that $\Psi(X, Y) = R$. By the continuity and monotonicity of ψ we can find X' and Y' such that $x'_i \leqslant x_i$, $y'_i \geqslant y_i$, $\psi(x'_i, y'_i) = r'_i$ for each i and $\Psi(X', Y') = R'$. Either X strictly dominates X' by FOSD or Y' strictly dominates Y by FOSD (or both). Monotonicity of $\succeq$ implies that $X \succeq X'$ and $Y' \succeq Y$ with at least one of these preferences being strict.

1. If $V(R) = 0$, then $X \sim Y$. By transitivity $X' \prec Y'$ and hence $V(R') = V(\Psi(X', Y')) < 0$.

2. If $V(R') = 0$, then $X' \sim Y'$. By transitivity $X \succ Y$ and therefore $V(R) = V(\Psi(X, Y)) > 0$. ∎

Proof of Lemma 2:

1. Let $S = \{s_1, s_2\}$ with $P(s_1) = P(s_2) = 0.5$. Define the lotteries $X = (x, s_1; y, s_2)$, $Y = (y, s_1; x, s_2)$, $X' = (x', s_1; y', s_2)$ and $Y' = (y', s_1; x', s_2)$. Let $r = \psi(x, y) = \psi(x', y')$. Then

$$\begin{aligned}
\Psi(X, Y) &= (r, 0.5; \psi(y, x), 0.5) \\
\Psi(X', Y') &= (r, 0.5; \psi(y', x'), 0.5)
\end{aligned}$$

By Proposition 1, $X \sim Y$ and $X' \sim Y'$, thus, we have $V(\Psi(X, Y)) = V(\Psi(X', Y')) = 0$. But if $\psi(y, x) \neq \psi(y', x')$ then $\Psi(X, Y)$ either dominates or is dominated by $\Psi(X', Y')$, contradicting Lemma 1.

2. Recall that $\psi(x, x) = 0$. Let $f : [\underline{r}, \bar{r}] \to [-\bar{r}, \bar{r}]$ be defined as follows:

$$f(r) = \begin{cases} -\psi(y, x) & \text{if } r < 0 \text{ and } x < y \text{ is such that } \psi(x, y) = r \\ r & \text{if } r \geqslant 0 \end{cases}$$

By the first part of this lemma, the value of $f(r)$ for $r < 0$ does not depend on the choice of x, y in the above definition, hence f is well defined. Monotonicity of ψ implies that f is strictly increasing. We can therefore define

$$V^*(r_1, p_1; \ldots; r_n, p_n) = V(f^{-1}(r_1), p_1; \ldots; f^{-1}(r_n), p_n)$$

Let

$$\psi^*(x, y) = \begin{cases} \psi(x, y) & \text{if } x \geqslant y \\ f(\psi(x, y)) & \text{if } x < y \end{cases}$$

Now

$$\begin{aligned}
X \succeq Y &\Longleftrightarrow \\
V(\Psi(X, Y)) \geqslant V(\Psi(Y, X)) &\Longleftrightarrow \\
V^*(\Psi^*(X, Y)) \geqslant V^*(\Psi^*(Y, X))
\end{aligned}$$

where $\Psi^*(X, Y)$ is obtained from $\Psi(X, Y)$ by replacing $\psi(x, y)$ with $\psi^*(x, y)$. ∎

Proof of Lemma 3: In the sequel, random variables Q with m (not necessarily distinct) outcomes are of the form $(q_1, s_1^m; \ldots; q_m, s_m^m)$ for some canonical partition where $P(s_i^m) = \frac{1}{m}$, $i = 1, \ldots, m$. For Q and Q' with m outcomes each, let

$$\langle Q, Q' \rangle = \left(q_1, s_1^{2m}; \ldots; q_m, s_m^{2m}; q_1', s_{m+1}^{2m}; \ldots; q_m', s_{2m}^{2m} \right)$$

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where $\mathrm{P}(s_i^{2m}) = \frac{1}{2m}$.

Let R and R' be generated by permutations π of $X = (x_1, s_1; \ldots; x_n, s_n)$ and π' of $Y = (y_1, s_1'; \ldots; y_n, s_n')$ respectively, where $\mathrm{P}(s_i) = \mathrm{P}(s_i') = \frac{1}{n}$, $i = 1, \ldots, n$. That is, $R = \Psi(X, \pi(X))$, $R' = \Psi(Y, \pi'(Y))$. (The assumption that X and Y are of the same length is without loss of generality). Define $Z = \langle X, Y \rangle$ and $\pi^* : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ by

$$\pi^*(i) = \begin{cases} \pi(i) & i \leqslant n \\ \pi'(i-n) + n & i > n \end{cases}$$

to obtain $\Psi(Z, \pi^*(Z)) = \Psi(\langle X, Y \rangle, \pi^* \langle X, Y \rangle) = \frac{1}{2}R + \frac{1}{2}R'$. ■

Proof of Lemma 4: All random variables in this proof have n outcomes on the equiprobable events $s_1, \ldots, s_n$. For $Z = (z_1, s_1; \ldots; z_n, s_n)$ and $Z' = (z_1', s_1; \ldots; z_n', s_n)$, define $\|Z - Z'\| = \max_i |z_i - z_i'|$.

The proof follows from Claims 1–2.

Claim 1 *Let $X \sim Y$. For any $\delta > 0$ there exist $Z_1, \ldots, Z_k$ such that $X = Z_1 \sim \ldots \sim Z_k = Y$ and $\|Z_{\ell-1} - Z_\ell\| \leqslant \delta$, $\ell = 2, \ldots, k$.*

Proof: We construct the sequence $Z_1, \ldots$ inductively. Suppose that $X \neq Y$ and that we have already defined $X = Z_1 \sim \ldots \sim Z_\ell$ such that $\|Z_{i-1} - Z_i\| \leqslant \delta$, $i = 2, \ldots, \ell$. If $Z_\ell \sim Y$ we are through. Otherwise, define $L_+^\ell = \{i : z_i^\ell > y_i\}$ and $L_-^\ell = \{i : z_i^\ell < y_i\}$. As $Z_\ell \sim Y$ and $Z_\ell \neq Y$, both L_+^ℓ and L_-^ℓ are non-empty. Let

$$\delta_+^\ell = \min_{i \in L_+^\ell}\{z_i^\ell - y_i\}$$
$$\delta_-^\ell = \min_{i \in L_-^\ell}\{y_i - z_i^\ell\}$$

Define $f_\ell(\theta)$ such that $Z_\ell \sim Z_{\ell+1}(\theta) \equiv (z_1^{\ell+1}(\theta), s_1; \ldots; z_n^{\ell+1}(\theta), s_n)$ where

$$z_i^{\ell+1}(\theta) = \begin{cases} z_i^\ell - \theta & \text{if } i \in L_+^\ell \\ z_i^\ell + f_\ell(\theta) & \text{if } i \in L_-^1 \\ z_i^\ell & \text{otherwise} \end{cases}$$

By continuity and monotonicity of $\succeq$, $f_\ell(\theta)$ is well defined (for small θ), continuous and increasing. Its inverse exists and is continuous. Define $\theta^\ell =$

$\min\left\{\delta, \delta_+^\ell, f_\ell^{-1}(\delta_-^\ell)\right\}$ and let $Z_{\ell+1} = Z_{\ell+1}(\theta^\ell)$. Note that $Z_1, \ldots, Z_{\ell+1}$ satisfy the hypothesis of the claim.

If $\theta^\ell = \delta$, then $\|Z_{\ell+1} - Y\| \leqslant \|Z_\ell - Y\| - \delta$. If $\theta^\ell = \delta_+^\ell$, then $|L_+^{\ell+1}| \leqslant |L_+^\ell| - 1$. If $\theta^\ell = f_\ell^{-1}(\delta_-^\ell)$, then $|L_-^{\ell+1}| \leqslant |L_-^\ell| - 1$. Thus, this process terminates in a finite number of steps with $Z_k = Y$. ∎

Claim 2 *There exists $\varepsilon_n > 0$ such that if for all i, $|r_i| < \varepsilon_n$, then there exist a random variable $\hat{Z}$ and a permutation π such that $R = (r_1, \frac{1}{n}; \ldots; r_n, \frac{1}{n})$ satisfies $R = \Psi(\hat{Z}, \pi(\hat{Z}))$.*

Proof: The domain of outcomes is $[\underline{x}, \bar{x}]$. Let

$$z_1 = \frac{\bar{x} + \underline{x}}{2}, \qquad \delta_n = \frac{\bar{x} - \underline{x}}{2n} = \frac{z_1 - \underline{x}}{n} = \frac{\bar{x} - z_1}{n} > 0$$

Thus, $z_1 + n\delta_n = \bar{x}$ and $z_1 - n\delta_n = \underline{x}$.

The function ψ is continuous on the compact segment $[\underline{x}, \bar{x}]$, therefore for any $\delta_n > 0$ there exists $\varepsilon_n > 0$ such that $|\psi(x, y)| < \varepsilon_n$ implies $|x - y| < \delta_n$. Thus, with $|r_i| < \varepsilon_n$ we can construct $\hat{Z}$ such that:

Event	s_1	s_2	s_3	s_4	$\ldots$	s_{n-1}	s_n
$\hat{Z}$	z_1	z_2	z_3	z_4	$\ldots$	z_{n-1}	z_n
$\pi(\hat{Z})$	z_2	z_3	z_4	z_5	$\ldots$	z_n	z_1
$\Psi(\hat{Z}, \pi(\hat{Z}))$	r_1	r_2	r_3	r_4	$\ldots$	r_{n-1}	$\psi(z_n, z_1)$

z_1 is chosen to be the midpoint between $\underline{x}$ and $\bar{x}$, and each $z_{\ell+1}$ is chosen so that $\psi(z_\ell, z_{\ell+1}) = r_\ell$, $\ell = 1, 2, \ldots, n-1$. As $|r_\ell| < \varepsilon_n$, we have $|z_\ell - z_{\ell+1}| < \delta_n$ and each $z_\ell \in [\underline{x}, \bar{x}]$. As $V(R) = V(\Psi(\hat{Z}, \pi(\hat{Z}))) = 0$, it must be that $\psi(z_n, z_1) = r_n$. Otherwise, R either dominates or is dominated by $\Psi(\hat{Z}, \pi(\hat{Z}))$, contradicting Lemma 1. Thus, $R = \Psi(\hat{Z}, \pi(\hat{Z}))$. ∎

Proof of Lemma 5: For $R = (r_1, \frac{1}{n}; \ldots; r_n, \frac{1}{n})$ and $R' = (r_1', \frac{1}{n}; \ldots; r_n', \frac{1}{n})$ such that $\psi(R) = \psi(R') = 0$, let X, Y, X', Y' be such that $\Psi(X, Y) = R$ and $\Psi(X', Y') = R'$. By Lemma 4, there exist sequences $X = Z_1 \sim \ldots \sim Z_k = Y$ and $X' = Z_1' \sim \ldots \sim Z_k' = Y'$ such that for all $\ell = 1, \ldots, k-1$ there

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exist $\hat{Z}_\ell, \pi_\ell, \hat{Z}'_\ell, \pi'_\ell$ satisfying $\Psi(\hat{Z}_\ell, \pi_\ell(\hat{Z}_\ell)) = \Psi(Z_\ell, Z_{\ell+1})$ and $\Psi(\hat{Z}'_\ell, \pi'_\ell(\hat{Z}'_\ell)) = \Psi(Z'_\ell, Z'_{\ell+1})$.[4] Thus, for each $\ell = 1, 2, \ldots, k-1$, the pair of regret lotteries $\Psi(Z_\ell, Z_{\ell+1})$ and $\Psi(Z'_\ell, Z'_{\ell+1})$ satisfies the hypothesis of Lemma 3. Therefore,

$$V\left(\tfrac{1}{2}\Psi(Z_\ell, Z_{\ell+1}) + \tfrac{1}{2}\Psi(Z'_\ell, Z'_{\ell+1})\right) = 0$$

Note that $\tfrac{1}{2}\Psi(Z_\ell, Z_{\ell+1}) + \tfrac{1}{2}\Psi(Z'_\ell, Z'_{\ell+1}) = \Psi(\langle Z_\ell, Z'_\ell\rangle, \langle Z_{\ell+1}, Z'_{\ell+1}\rangle)$ where $\langle\,\cdot\,,\,\cdot\,\rangle$ is defined in the proof of Lemma 3. Consequently,

$$\langle X, X'\rangle = \langle Z_1, Z'_1\rangle \sim \ldots \sim \langle Z_k, Z'_k\rangle = \langle Y, Y'\rangle$$

Hence

$$V(\Psi(\langle X, X'\rangle, \langle Y, Y'\rangle)) = 0$$

But

$$\Psi(\langle X, X'\rangle, \langle Y, Y'\rangle) = \tfrac{1}{2}R + \tfrac{1}{2}R'$$

and we obtain $V(\tfrac{1}{2}R + \tfrac{1}{2}R') = 0$.

As each $X \in \mathcal{L}$ is the limit of a sequence $\{X^k\}$ where for each k, $X_k = (x_1^k, \tfrac{1}{n_k}; \ldots; x_{n_k}^k, \tfrac{1}{n_k})$, the lemma now follows by continuity for all R and R' such that $V(R) = V(R') = 0$. ∎

Proof of Lemma 6: Recall that $V(\delta_{\bar{r}}) > 0 > V(\delta_{-\bar{r}})$ where δ_t is the constant lottery yielding t.

1. For a regret lottery R such that $V(R) > 0$, let $\alpha(R)$ be defined by $V(\alpha(R)R + (1 - \alpha(R))\delta_{-\bar{r}}) = 0$ and for R such that $V(R) < 0$, let $\alpha(R)$ be defined by $V(\alpha(R)R + (1 - \alpha(R))\delta_{\bar{r}}) = 0$. By Lemma 1 and the continuity of V, $\alpha(R)$ is well defined and $\alpha(R) < 1$. Let α^* satisfy $V(\alpha^*\delta_{\bar{r}} + (1 - \alpha^*)\delta_{-\bar{r}}) = 0$.

We show first that α is a continuous function. Let $R_k \to R_0$ and suppose that $\alpha(R_k) \to \alpha'$.[5] Suppose without loss of generality that for all k, $V(R_k) \geqslant 0$. By the continuity of V,

$$V(\alpha'R_0 + (1 - \alpha')\delta_{-\bar{r}}) = \lim_k V(\alpha(R_k)R_k + (1 - \alpha(R_k))\delta_{-\bar{r}}) = 0$$

---

[4]We use the same k in both sequences without loss of generality as the sequences may become stationary from a certain point on.

[5]If $\alpha(R_k)$ does not have a limit, then we take a subsequence that has a limit.

hence $\alpha' = \alpha(R_0)$.

Define now $U(R)$ by

$$
U(R) = \begin{cases}
\frac{\alpha^*}{\alpha(R)} - \alpha^* & V(R) > 0 \\
0 & V(R) = 0 \\
1 - \alpha^* - \frac{1-\alpha^*}{\alpha(R)} & V(R) < 0
\end{cases}
$$

For R such that $V(R) \neq 0$, $\alpha(R) < 1$, hence $U(R) \gtreqless 0$ iff $V(R) \gtreqless 0$. The continuity of $\alpha(\cdot)$ implies that $U(R)$ is continuous. We show next that U is linear. That is, for all R and R', $U\left(\frac{1}{2}R + \frac{1}{2}R'\right) = \frac{1}{2}U(R) + \frac{1}{2}U(R')$.

By Lemma 5 and the continuity of V we have:

Conclusion 1 *If $V(R) = V(R') = 0$, then for all $\alpha \in [0,1]$, $V(\alpha R + (1-\alpha)R') = 0$.*

For arbitrary R and R', the four regret lotteries $R, R', \delta_{\bar{r}}, \delta_{-\bar{r}}$ determine (at most) a three-dimensional simplex. By taking an appropriate linear transformation we can assume without loss of generality that $\delta_{-\bar{r}} = (0,0,-1)$, $\delta_{\bar{r}} = (0,0,\frac{1-\alpha^*}{\alpha^*})$, $R = (x^*, y^*, z^*)$, $R' = (x', y', z')$, and, by Conclusion 1, $V(x,y,z) = 0$ iff $z = 0$. It follows that for $z > 0$, $\alpha(x,y,z)$ solves

$$
\alpha z - (1 - \alpha) = 0 \implies \alpha(x, y, z) = \frac{1}{z+1}
$$

and for $z < 0$, $\alpha(x,y,z)$ solves

$$
\alpha z + (1 - \alpha)\frac{1 - \alpha^*}{\alpha^*} = 0 \implies \alpha(x, y, z) = \frac{1 - \alpha^*}{1 - \alpha^* - \alpha^* z}
$$

In both cases, $U(x, y, z) = \alpha^* z$.

Define now a preference relation $\succeq^*$ on regret lotteries by $R \succeq^* R'$ iff $U(R) \geqslant U(R')$. Since U is continuous, so is $\succeq^*$ and since U is linear, $\succeq^*$ satisfies the independence axiom. Therefore, there is a function v such that $U(R) \gtreqless 0$ iff $\mathrm{E}[v(R)] \gtreqless 0$. The lemma follows since $U(R) \gtreqless 0$ iff $V(R) \gtreqless 0$.

2. Suppose that $v(\cdot)$ is not strictly increasing. Then, there exist $r_1 < r_2$ such that $v(r_1) \geqslant v(r_2)$. Take $R = (r_1, p_1; r_2, p_2; \ldots; r_n, p_n)$ such that $V(R) =$

0. The continuity of V implies that such an R exists. Construct $R' = (r_1, p_1 - \varepsilon; r_2, p_2 + \varepsilon; \ldots; r_n, p_n)$. If $v(r_1) = v(r_2)$ then $0 = V(R) = V(R')$ but R' dominates R by FOSD, contradicting Lemma 1. If, instead, $v(r_1) > v(r_2)$ then $0 = V(R) > V(R')$ but R' dominates R by FOSD, once again contradicting Lemma 1. The fact that $v(0) = 0$ follows from $V(0,1) = 0$.

Finally, let $S = \{s_1, s_2\}$ with $\mathrm{P}(s_1) = \mathrm{P}(s_2) = 0.5$. Define $X = (x, s_1; y, s_2)$ and $Y \equiv (y, s_1; x, s_2)$. By Proposition 1, $X \sim Y$. Thus $v(\psi(x,y)) = -v(\psi(y,x))$. ∎

Proof of Lemma 7: The following claim follows from a theorem in Aczél [1].

Claim 3 *If $G(x,y) + G(y,z) = G(x,z)$ for all $x < y < z$, then there exists a function $g : \Re \to \Re$ such that $G(x,y) = g(x) - g(y)$.*

Proof: Define

$$
H(x,y) = \begin{cases} G(x,y) & x < y \\ 0 & x = y \\ -G(y,x) & x > y \end{cases}
$$

It may be verified that for all x, y, z,

$$
H(x,y) + H(y,z) = H(x,z)
$$

Therefore, Aczél [1] (Theorem 1, p. 223) implies that there exists $g : \Re \to \Re$ such that $H(x,y) = g(x) - g(y)$. ∎

Select $x_1 < x_2 < x_3$ and $p, q > 0$, $p \neq q$, $p + q < \frac{1}{3}$. Define lotteries X, Y as below.

Event	s_1	s_4	s_7	s_2	s_5	s_8	s_3	s_6	s_9
$\mathrm{P}(s_i)$	p	p	p	q	q	q	$\frac{1}{3} - p - q$	$\frac{1}{3} - p - q$	$\frac{1}{3} - p - q$
X	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
Y	x_3	x_1	x_2	x_2	x_3	x_1	x_1	x_2	x_3

Proposition 1 implies $X \sim Y$ as each of these lotteries gives x_1, x_2, and x_3 with probability $\frac{1}{3}$ each. Thus, $V(\Psi(X,Y)) = 0$ and by Lemma 6,

$E[v(\Psi(X,Y))] = 0$. As $v(\psi(x,y)) = -v(\psi(y,x))$ and $v(\psi(x,x)) = 0$ (see Lemma 6), it follows that

$$[q-p]v(\psi(x_1,x_2)) + [q-p]v(\psi(x_2,x_3)) + [p-q]v(\psi(x_1,x_3)) = 0$$

Since $p \neq q$ we obtain for all $x_1 < x_2 < x_3$, $v(\psi(x_1,x_2)) + v(\psi(x_2,x_3)) = v(\psi(x_1,x_3))$. By Claim 3, there exists a function $u : \Re \to \Re$ such that $v(\psi(x_1,x_2)) = u(x_1) - u(x_2)$. Monotonicity of u follows from the monotonicity of $\succeq$. ∎

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