Profit-Maximizing Matchmaker*

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This paper considers a resource allocation mechanism that utilizes a profit-maximizing auctioneer/matchmaker in the Kelso-Crawford (1982) (many-to-one) assignment problem. We consider general and simple (individualized price) message spaces for firms' reports following Milgrom (2010). We show that in the simple message space, (i) the matchmaker's profit is always zero and an acceptable assignment is achieved in every Nash equilibrium, and (ii) the sets of stable assignments and coalition-proof Nash equilibria are equivalent. By contrast, in the general message space, the matchmaker may make a positive profit in a Nash equilibrium. This shows that restricting message space not only reduces the information requirement but also improves resource allocation.

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1 Introduction

In their influential paper, Shapley and Shubik (1971) introduce an assignment problem that is a transferrable utility (cooperative) game in a two-sided one-to-one matching problem. Kelso and Crawford (1982) generalize the assignment model to a many-to-one setting: they allow firms to choose how many workers to hire, and they analyze the resulting market equilibrium and the core. They consider a central planning authority that matches up firms and workers and propose a price adjustment mechanism by generalizing the Gale-Shapley deferred acceptance algorithm (Gale and Shapley 1962; Roth and Sotomayor 1990). Their algorithm finds the firm-optimal stable assignment that is a market equilibrium and a core allocation. As in many centralized market clearing mechanisms successfully used in the real world, such as entry-level medical markets and school choice problems, Kelso and Crawford (1982) assume that the matchmaker is a benevolent central planner who tries to achieve a desirable allocation—a market equilibrium.

By contrast, in this paper, we consider another matching mechanism that utilizes an auctioneer (matchmaker) who chooses a matching of firms and workers that maximizes profit in an environment of heterogeneous firms and workers. Specifically, we consider a two-stage noncooperative game in a many-to-one assignment problem with a matchmaker. In the first stage, each firm proposes how much it is willing to pay workers if they are matched, and each worker proposes what salary she is willing to accept from each firm if they are matched. These proposals are made simultaneously. Then, in the second stage, the matchmaker matches up firms and workers in order to maximize profits (the sum of the differences between the offering and asking salaries from each matched firm-worker(s)). This matchmaker game can be regarded as a resource allocation mechanism with an auctioneer in a two-sided matching problem.

Recently, Milgrom (2010) proposes a framework that analyzes the effect on equilibria of restricting the message space of a game. He defines a certain "outcome closure property" on a simplification of message space, and shows that if the condition is satisfied, then every (ϵ)-Nash equilibrium in the simplified mechanism is an (ϵ)-Nash equilibrium of the original mechanism. Moreover, he illustrates the benefits of working with the simplified mechanism by noting that the set of Nash equilibria is intact by simplifying message space through adopting simple (individualized price) strategies in a combinatorial auction game, and also that the Gale-Shapley algorithm selects the same outcome even with individualized prices in the Kelso-Crawford assignment game under (gross)-substitute assumption. Thus, it is interesting to investigate the performance of using simple (individualized price) strategies in our matchmaker game, which combines a two-sided matching problem and a combinatorial auction game.

Our matchmaker game can be considered a two-sided version of a combinatorial auction game. It satisfies the outcome closure property, so a Nash equilibrium in simple (individualized price) strategies is a Nash equilibrium in general (package price) strategies. However, in contrast to Milgrom' observation on a combinatorial auction game and the Gale-Shapley algorithm, restricting the message space significantly reduces the set of Nash equilibria in our matchmaker game. In particular, all Nash equilibria in simple strategies generate zero profit for the matchmaker (Theorem 1), but Nash equilibria in general strategies may generate positive profits (Example 3). This result shows that while the simple strategy restriction excludes some of Nash equilibria in our matchmaker game, the performance of the mechanism improves with the restriction since profit for the matchmaker is a waste of resource. We also use a stronger equilibrium concept and investigate the equilibrium outcomes. A strong Nash equilibrium is a strategy profile that is immune to every coordinated change in strategies for any coalition (Aumann 1959). In our matchmaker game, a strong Nash equilibrium in simple strategies is a strong Nash equilibrium in general strategies as well (Proposition 1). We show that every strong Nash equilibrium outcome in simple strategies is a stable assignment (a core allocation) (Theorem 3).

Applying the above theorems, we obtain results on the implementation of popular social choice correspondences in the Kelso-Crawford many-to-one assignment problem with monetary transfers. Alcalde et al. (1998) show that the stable correspondence (competitive equilibrium correspondence) is subgame-perfect-Nash-implementable by a simple two-stage game. Hayashi and Sakai

(2009) characterize the stable correspondence by Nash implementation. Note that their results cannot treat the one-to-one problem or a many-to-one problem with quotas. By noting that the set of Nash equilibrium outcomes is equivalent to the set of acceptable assignments, we can show that the acceptable correspondence is Nash-implementable by our simple matchmaker game by applying Theorem 1 (Corollary 1). Theorem 2 directly shows that a stable correspondence is strong-Nash-implementable in a simple matchmaker game (Corollary 3). These results are not dependent on the presence of monetary transfers (Theorem 4) or quotas.[1]

Our matchmaker game is related to the *menu auction game* introduced by Bernheim and Whinston (1986), although the results in the literature of the menu auction game do not have much to do with ours except for the one-to-one problem. In a menu auction game, there are multiple principals (players) and an agent, and a set of actions. All players and the agent have preferences over actions, and each player offers a contribution schedule to the agent, which is a function from the action set to a monetary contribution. The agent sees the players' contribution schedules and chooses the action with the highest total payoff. We show that the class of our matchmaker games in general (package price) strategies can be embedded into that of the menu auction games (Proposition 2). In this sense, our game is related to the menu auction game. However, many important results in the literature of menu auction games have something to do with Nash equilibrium in restricted strategies: *truthful strategies* as defined in Bernheim and Whinston (1986). Unfortunately, in general, truthful strategies and simple (individualized price) strategies are incompatible with each other except for a special domain of one-to-one assignment problems, and we cannot apply the results to our many-to-one assignment case. Still, our Theorem 1 implies that the one-to-one assignment problem is a new domain that satisfies the *no-rent property* introduced by Laussel and Le Breton (2001), under which many nice results hold.

The rest of the paper is organized as follows. In Section 2, the (many-to-one) Kelso-Crawford assignment problem and our matchmaker game are introduced with a few examples. Section 3 presents our main results. Section

[1] The effect of quota can be muted by setting quota equal to the size of the labor force.

4 provides applications of our main results to the implementation of acceptable and stable matchings and discusses the relationship of our results with menu auction games. Section 5 contains the proof of the main theorem.

2 The Model

2.1 A Many-to-One Matching Problem

We consider the Kelso-Crawford many-to-one assignment problem without imposing complementarity or substitutability of workers (Kelso and Crawford 1982). There are two disjoint finite sets of players: the set of firms F and the set of workers W. Let $N = F \cup W$. Each firm $f \in F$ has a finite **quota** q_f and each of q_f positions can hold one worker. Production technology is described by a function $Y : F \times 2^W \to \mathbb{R}_+$ such that $Y(f, W_f) \geq 0$ is the output that firm f can produce by hiring $W_f \subseteq W$ workers. We assume that $Y(f, W_f) = 0$ when $W_f = \varnothing$ or $|W_f| > q_f$ for all $f \in F$. Let $\mathcal{Y}$ be the set of all possible production technologies. Each worker $w \in W$ hired by firm f has some disutility from working d_{wf} independent of his or her position. If unemployed, then w receives zero disutility ($d_{w\varnothing} = 0$). We assume that $d_{wf} \geq 0$ for all $f \in F$ and all $w \in W$. Let $D = (d_{wf})_{w\in W, f\in F\cup\{\varnothing\}}$ be a disutility matrix, and let $\mathcal{D}$ be the set of all possible disutility matrices. A **many-to-one matching** $\mu : W \cup F \twoheadrightarrow W \cup F$ is a mapping such that (i) $\mu(f) \subseteq W$ and $\mu(w) \in F \cup \{\varnothing\}$ for all $f \in F$ and all $w \in W$; (ii) $|\mu(f)| \leq q_f$ for all $f \in F$; (iii) $w \in \mu(f)$ if $f = \mu(w)$; (iv) $\mu(w) = f$ for all $w \in \mu(f)$. Let $\mathcal{M}$ be the set of all matchings μ. An **efficient matching** is $\mu^* \in \arg\max_{\mu\in\mathcal{M}} \sum_{f\in F} \left[Y(f, \mu(f)) - \sum_{w\in\mu(f)} d_{wf}\right]$. We denote payoffs of firm f and worker w by v_f and u_w, respectively. Let $v = (v_f)_{f\in F}$ and $u = (u_w)_{w\in W}$ be firms' and workers' payoff vectors. A (nonwasteful) **allocation** is a list $(v, u, \mu) \in \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$ such that (i) $v_f = 0$ for all $f \in F$ with $\mu(f) = \varnothing$, (ii) $u_w = 0$ for all $w \in W$ with $\mu(w) = \varnothing$ and (iii) $v_f + \sum_{w\in\mu(f)} u_w = Y(f, \mu(f)) - \sum_{w\in\mu(f)} d_{wf}$ for all $f \in F$. An allocation (v, u, μ) is **efficient** if μ is an efficient matching. An allocation is **individually rational** if for all $f \in F$ and all $w \in W$, $v_f \geq 0$ and $u_w \geq 0$. An allocation is an **acceptable assignment** if (i) it is individually rational and (ii) $Y(f, W_f) - \sum_{w \in W_f} d_{wf} \leq v_f + \sum_{w \in W_f} u_w$ for all $f \in F$ and all $W_f \subset \mu(f)$. Condition (ii) of acceptability requires that firm f cannot be better off by firing some of its workers. Note that individual rationality is equivalent to acceptability in the one-to-one assignment problem, but not in the many-to-one problem. An allocation is a **stable assignment** if (i) it is individually rational and (ii) there is no pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $Y(f, W_f) - \sum_{w \in W_f} d_{wf} > v_f + \sum_{w \in W_f} u_w$. Clearly, stability requires acceptability.

2.2 The Matchmaker Game

Consider a mechanism by which a matchmaker matches up firms and workers under complete information. This matchmaker can be regarded as an auctioneer, or as a central planning authority who chooses a matching based on information submitted by firms and workers. In the first stage, a matchmaker asks each worker what salary she demands from each firm, and asks each firm how much it is willing to offer workers if they are matched. Thus, each worker w submits $s_w : F \to \mathbb{R}$ (or $s_w = (s_w(f))_{f \in F}$). However, the strategy for the firm has two possible formulations. One is a **simple strategy** (or an individualized price strategy) such that each firm $f \in F$ submits $\sigma_f : W \to \mathbb{R}$ (or $\sigma_f = (\sigma_f(w))_{w \in W}$). That is, irrespective of other workers assigned to firm f, f always pays $\sigma_f(w)$ to the matchmaker for getting worker w. The other is a **general strategy** (or a package price strategy) such that each firm $f \in F$ submits $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}$ where $\mathcal{S}_f = \{W_f \subseteq W : |W_f| \leq q_f\}$. Clearly, simple strategies are special cases of general strategies. The matchmaker is allowed to take the difference between $\sigma_f(w)$ and $s_w(f)$ if she matches f and w in the case of simple strategies, and the matchmaker is allowed to take the difference between $\tilde{\sigma}_f(W_f)$ and $\sum_{w \in W_f} s_w(f)$ from matching up f and W_f in the case of general strategies. Needless to say, the matchmaker would not match a pair (f, w) if $\sigma_f(w) < s_w(f)$ in the case of a simple strategy, and would not match (f, W_f) if $\tilde{\sigma}_f(W_f) < \sum_{w \in W_f} s_w(f)$ in the case of a general strategy: the matchmaker would rather leave them unmatched.

In the second stage, using these submitted strategies, the matchmaker chooses a matching $\mu \in \mathcal{M}$ that maximizes profit. This game is called a

matchmaker game, and the matching games with firms' simple and general strategies are called **simple** and **general matchmaker games**, respectively.

In a *simple matchmaker game*, the matchmaker has a payoff function $U : \mathbb{R}^{F\times W} \times \mathbb{R}^{W\times F} \times \mathcal{M} \to \mathbb{R}$ with $U(\sigma, s, \mu) = \sum_{f\in F}\sum_{w\in\mu(f)} (\sigma_f(w) - s_w(f))$. Let the set $M(\sigma, s) \subset \mathcal{M}$ be

$$M(\sigma, s) \equiv \operatorname*{argmax}_{\mu\in\mathcal{M}} U(\sigma, s, \mu).$$

Each firm f, worker w, and the matchmaker obtain the following payoffs under $\mu \in M(\sigma, s)$:

$$v_f(\sigma, s, \mu) = Y(f, \mu(f)) - \sum_{w\in\mu(f)} \sigma_f(w),$$

$$u_w(\sigma, s, \mu) = s_w(\mu(w)) - d_{w\mu(w)},$$

and

$$U(\sigma, s, \mu) = \sum_{f\in F}\sum_{w\in\mu(f)} (\sigma_f(w) - s_w(f)),$$

respectively.

In a *general matchmaker game*, the matchmaker has a payoff function $\tilde{U} : \mathbb{R}^{\times_{f\in F}\mathcal{S}_f} \times \mathbb{R}^{W\times F} \times \mathcal{M} \to \mathbb{R}$ with $\tilde{U}(\tilde{\sigma}, s, \mu) = \sum_{f\in F}\left(\tilde{\sigma}_f(\mu(f)) - \sum_{w\in\mu(f)} s_w(f)\right)$. Let the set $\tilde{M}(\tilde{\sigma}, s) \subset \mathcal{M}$ be

$$\tilde{M}(\tilde{\sigma}, s) \equiv \operatorname*{argmax}_{\mu\in\mathcal{M}} \tilde{U}(\tilde{\sigma}, s, \mu).$$

Each firm f, worker w, and the matchmaker obtain the following payoffs under $\mu \in \tilde{M}(\tilde{\sigma}, s)$:

$$\tilde{v}_f(\tilde{\sigma}, s, \mu) = Y(f, \mu(f)) - \tilde{\sigma}_f(\mu(f)),$$

$$u_w(\tilde{\sigma}, s, \mu) = s_w(\mu(w)) - d_{w\mu(w)},$$

and

$$\tilde{U}(\tilde{\sigma}, s, \mu) = \sum_{f\in F}\left(\tilde{\sigma}_f(\mu(f)) - \sum_{w\in\mu(f)} s_w(f)\right),$$

respectively.

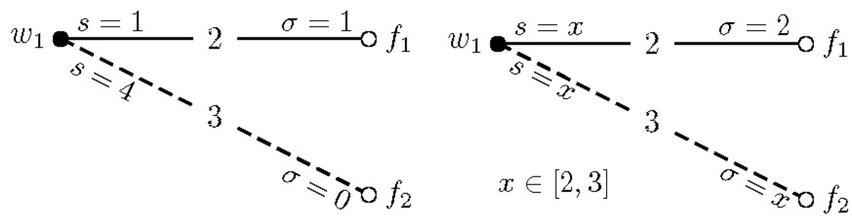
Note that each firm f cares only about $\mu(f)$. The rest of the matching is irrelevant. Similarly, each worker w cares only about $\mu(w)$. A list $((\sigma^*, s^*), \mu^*)$ is a **Nash equilibrium** in a *simple matchmaker game* if (i) $\mu^* \in M(\sigma^*, s^*)$, (ii) there is no $f \in F$ such that $\sigma_f : W \to \mathbb{R}$ and $\mu \in M(\sigma_f, \sigma^*_{-f}, s^*)$ such that $v_f(\sigma_f, \sigma^*_{-f}, s^*, \mu) > v_f(\sigma^*, s^*, \mu^*)$, and (iii) there is no $w \in W$ such that $s_w : F \to \mathbb{R}$ and $\mu \in M(\sigma^*, s_w, s^*_{-w})$ such that $u_w(\sigma^*, s_w, s^*_{-w}, \mu) > u_w(\sigma^*, s^*, \mu^*)$.[2] An **outcome** of a Nash equilibrium $((\sigma^*, s^*), \mu^*)$ in a *simple matchmaker game* is a list $(v, u, \mu) \in \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$ such that $v_f = Y(f, \mu^*(f)) - \sum_{w \in \mu^*(f)} \sigma^*_f(w)$ for all $f \in F$, $u_w = s^*_w(\mu^*(w)) - d_{w\mu^*(w)}$ for all $w \in W$ and $\mu = \mu^*$. A list $((\sigma^*, s^*), \mu^*)$ is a **(strictly) strong Nash equilibrium (SNE)** in a *simple matchmaker game* if (i) $\mu^* \in M(\sigma^*, s^*)$, and (ii) there is no coalition $C \subseteq N$ with their strategies $(\sigma_{C\cap F}, s_{C\cap W}) = ((\sigma_f)_{f \in C\cap F}, (s_w)_{w \in C\cap W})$, and a matching $\mu \in M(\sigma_{C\cap F}, s_{C\cap W}, \sigma^*_{-C\cap F}, s^*_{-C\cap W})$ such that $v_f(\sigma_{C\cap F}, s_{C\cap W}, \sigma^*_{-C\cap F}, s^*_{-C\cap W}, \mu) \geq v_f(\sigma^*, s^*, \mu^*)$ for all $f \in C \cap F$, and $u_w(\sigma_{C\cap F}, s_{C\cap W}, \sigma^*_{-C\cap F}, s^*_{-C\cap W}, \mu) \geq u_w(\sigma^*, s^*, \mu^*)$ for all $w \in C \cap W$, with at least one being strict. An outcome of a strong Nash equilibrium in a *simple matchmaker game* is defined similarly. Corresponding definitions in a *general matchmaker game* are given in the same manner.

2.3 Examples

In this subsection, we illustrate what Nash and strong Nash equilibria look like. We start with a very simple one-to-one matching example.

Example 1. There are two firms $\{f_1, f_2\}$ and one worker $\{w_1\}$. Each firm has one position $q_{f_1} = q_{f_2} = 1$. Let $Y(f_1, \{w_1\}) = 2$, $Y(f_2, \{w_1\}) = 3$ and $d_{w_1 f_1} = d_{w_1 f_2} = 0$. Even in this simple example, there are multiple Nash equilibria with different matchings. Let $\sigma_{f_1}(w_1) = 1$ and $\sigma_{f_2}(w_1) = 0$, and $s_{w_1}(f_1) = 1$ and $s_{w_1}(f_2) = 4$. Under this strategy profile, the matchmaker chooses $\mu(f_1) = w_1$ and $\mu(f_2) = \varnothing$, and makes no profit. This is a Nash equilibrium, but the resulting matching is inefficient. This inefficiency is due to a coordination failure. Firm f_2 has no incentive to hire w_1 by changing its strategy unilaterally since w_1 is asking an unreasonable salary, while worker

[2] Although strictly speaking the game is a two-stage game, because the second stage is a mere maximization problem by the matchmaker, we can regard the game as static (see Bernheim and Whinston 1986; Laussel and Le Breton 2001).



(a) NE can be inefficient. (b) SNE achieves efficiency.

Figure 1: Illustration for Example 1.

w_1 has no incentive to try to be hired by changing her strategy unilaterally since f_2 is offering zero salary. However, if both firm f_2 and worker w_1 jointly change their strategies, then both can be better off by being matched up, thus achieving efficiency.

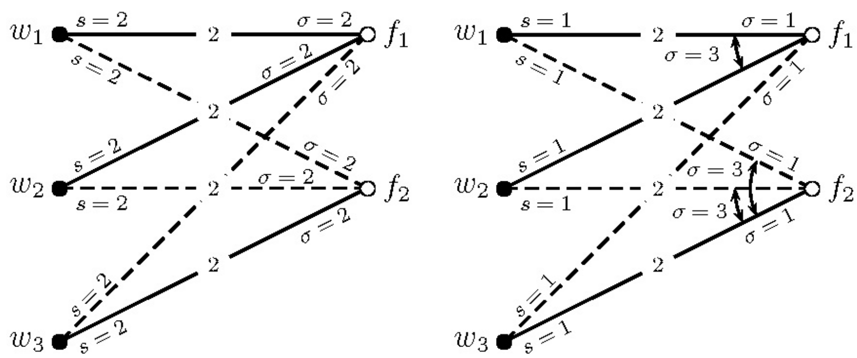
In contrast, let $\sigma_{f_2}(w_1) = x$ and $\sigma_{f_1}(w_1) = 2$, and $s_{w_1}(f_1) = x$ and $s_{w_1}(f_2) = x$, where $x \in [2, 3]$. If the matchmaker chooses $\mu'(f_2) = w_1$ and $\mu'(f_1) = \varnothing$ (indeed, unless $x = 2$, it must choose μ'), this is a strong Nash equilibrium, since there is no profitable deviation. Thus, any salary $x \in [2, 3]$ can be supported by a strong Nash equilibrium, and efficiency is achieved. Note that each of these allocations is a stable assignment.

Example 1 shows that Nash equilibria in matchmaker games can generate inefficient matchings. The matchmaker's profit is zero in all Nash equilibria. In the next example, we consider more general situations and show that the matchmaker's profit is still zero.

Example 2. There are two firms $\{f_1, f_2\}$ and two workers $\{w_1, w_2\}$. Each firm has one position $q_{f_1} = q_{f_2} = 1$. Let $Y(f_1, \{w_1\}) = Y(f_2, \{w_2\}) = 3$ and $Y(f_1, \{w_2\}) = Y(f_2, \{w_1\}) = 0$, and let $d_{w_j f_i} = 0$ for all $i, j = 1, 2$. Clearly, the efficient matching is $\mu(f_1) = w_1$, and $\mu(f_2) = w_2$. Suppose that the matchmaker is earning a positive profit in a Nash equilibrium at least from the pair $\{f_1, w_1\}$ by choosing μ, that is, $\sigma_{f_1}(w_1) > s_{w_1}(f_1)$. If $\sigma_{f_2}(w_2) = s_{w_2}(f_2)$, we have $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_2}(w_1) - s_{w_1}(f_2)$ and $\sigma_{f_1}(w_1) - s_{w_1}(f_1) = \sigma_{f_1}(w_2) - s_{w_2}(f_1)$ to prevent firm f_1 from offering less salary and worker w_1 from asking more salary. Then, a matching μ' with $\mu'(f_1) = w_2$, and $\mu'(f_2) = w_1$ generates a higher profit than μ. Hence, $\sigma_{f_2}(w_2) > s_{w_2}(f_2)$. Note that unless $\sigma_{f_1}(w_2) > s_{w_2}(f_1)$ or $\sigma_{f_2}(w_1) > s_{w_1}(f_2)$, f_1 can gain by reducing $\sigma_{f_1}(w_1)$ because the matchmaker would still choose μ. Without loss of generality, assume $\sigma_{f_1}(w_2) > s_{w_2}(f_1)$. Then f_1 can earn more by reducing $\sigma_{f_1}(w_1)$ and $\sigma_{f_1}(w_2)$ by the same amount without affecting the resulting matching. As a result, $\sigma_{f_1}(w_1) = s_{w_1}(f_1)$ must hold in every Nash equilibrium. Thus, in this example again, the matchmaker's profit must be zero in every Nash equilibrium. It is easy to see that the set of strong Nash equilibrium outcomes is equivalent to the set of stable assignments.

Now we consider a many-to-one problem. The following simple example illustrates a very important point: in general matchmaker games, a Nash equilibrium may yield a positive profit to the matchmaker.

Example 3. There are two firms $\{f_1, f_2\}$ and three workers $\{w_1, w_2, w_3\}$. All firms and workers are symmetric. Each firm has two positions $q_{f_1} = q_{f_2} = 2$. For all $i = 1, 2$ and all $j, k = 1, 2, 3$ $(j \neq k)$, $d_{w_j f_i} = 0$ and $Y(f_i, \{w_j\}) = 2$ and $Y(f_i, \{w_j, w_k\}) = 4$. In a simple matchmaker game, the wage offered to each worker is individualized, and similar arguments as above follow, since the matchmaker cares only about how much it can earn from each match of a firm with a worker. Thus, we can show that all Nash equilibria in this simple matchmaker game generate zero profit to the matchmaker. The unique strong Nash equilibrium (up to permutations) in the simple matchmaker game is $((\sigma, s), \mu)$ such that $\sigma_{f_i}(w_j) = 2$ and $s_{w_j}(f_i) = 2$ for all i and j, and $\mu(f_1) = \{w_1, w_2\}$ and $\mu(f_2) = \{w_3\}$. The salaries are pinned down owing to excess demand for workers. Note that this strong Nash equilibrium generates a stable assignment. Clearly, there is no profit for the matchmaker in the strong Nash equilibrium of the simple matchmaker game. From the above strong Nash equilibrium in a simple matchmaker game, let $\tilde{\sigma}_{f_i}(w_j) = 2$ and $\tilde{\sigma}_{f_i}(\{w_j, w_k\}) = 4$, and $s_{w_j}(f_i) = 2$ for all i, j, and k. This is indeed a strong Nash equilibrium in this general matchmaker game. However, in the general matchmaker game, there are Nash equilibria with positive profits. Consider the following strategy profile $((\tilde{\sigma}, s), \mu)$: $\tilde{\sigma}_{f_i}(\{w_j\}) = 1$ for all i and j, and $\tilde{\sigma}_{f_i}(\{w_j, w_k\}) = 3$ (if firm f_i is willing to pay 3 in total if it is matched with subset $\{w_j, w_k\}$) for all i,



(a) Unique SNE with zero profit in simple matchmaker game.

(b) A NE with positive profit in general matchmaker game.

Figure 2: Illustration for Example 3.

j, and k, and $s_{w_j}(f_i) = 1$ for all i and j. This results in $\mu(f_1) = \{w_1, w_2\}$ and $\mu(f_2) = \{w_3\}$ (up to permutations). This is a Nash equilibrium, and firms are indifferent between hiring one or two workers. However, the matchmaker receives a profit of 1 from f_1. Note that firms are better off in this Nash equilibrium in the general matchmaker game: they obtain positive profits.

This example shows that unlike the one-to-one matching problem, restrictions on firms' strategy sets may affect the outcomes of a matchmaker game. A "simple" matchmaker game selects zero-profit Nash equilibria from the larger set of equilibria in the general matchmaker game.

In the next section, we will investigate whether the above observations hold in general.

3 The Results

3.1 Preliminaries

We first review Milgrom's recent contribution. Let (N, X, ω) be a normal-form mechanism where N is the set of players, $X = (X_i)_{i \in N}$ is the set of strategy profiles, Ω is the set of possible outcomes where Ω is endowed with a topology, and $\omega : X \to \Omega$ is an outcome function. A normal form game can be constructed given utility functions $u = (u_i)_{i \in N}$ where $u_i : \Omega \to \mathbb{R}$. A normal form mechanism $(N, \hat{X}, \omega|_{\hat{X}})$ is a **simplification** of (N, X, ω) if $\hat{X} \subseteq X$. A simplification $(N, \hat{X}, \omega|_{\hat{X}})$ of (N, X, ω) has the **outcome closure property** if, for every i, every $\hat{x}_{-i} \in \hat{X}_{-i}$, every $x_i \in X_i$, and every open neighborhood O of $\omega(x_i, \hat{x}_{-i})$, there exists $\hat{x}_i \in \hat{X}_i$ such that $\omega(\hat{x}) \in O$. The simplification $(N, \hat{X}, \omega|_{\hat{X}})$ of (N, X, ω) is **tight** if, for every continuous function u and every $\varepsilon \geq 0$, every pure strategy profile x that is an ε-Nash equilibrium of $(N, \hat{X}, \omega|_{\hat{X}})$ is also an ε-Nash equilibrium of (N, X, ω). Milgrom (2010) shows the following simplification theorem.

Theorem 0. (Milgrom 2010) Any simplification $(N, \hat{X}, \omega|_{\hat{X}})$ of (N, X, ω) that has the outcome closure property is tight.

In a matchmaker game, the set of players is the set of firms and workers, $N = F \cup W$. For player $w \in W$, a strategy is $s_w : F \to \mathbb{R}$ and X_w is a collection of all strategies for w. For player $f \in F$, a (general) strategy is $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}$, and X_f is the collection of all possible general strategies for f. The restriction $\hat{X}_f$ is the set of all general strategies that can be created from simple strategies.[3] The set of possible outcomes is denoted by $\Omega = \mathbb{R}^F \times \mathbb{R}^W \times \mathcal{M}$, where $\omega = (v, u, \mu) \in \Omega$, and an outcome function is $\omega : X \to \Omega$ such that $v_f = Y(f, \mu(f)) - \tilde{\sigma}_f(\mu(f))$ for all $f \in F$, $u_w = s_w(\mu(w)) - d_{w\mu(w)}$ for all $w \in W$, and $\mu \in \tilde{M}(\tilde{\sigma}, s)$. A simple matchmaker game is a simplification of a general matchmaker game, and the simplification satisfies the outcome-closure property. Then the following observation immediately emerges by selecting the appropriate outcome function to support each Nash equilibrium:[4]

Observation. Every Nash equilibrium in a simple matchmaker game is a Nash equilibrium in the general matchmaker game.

[3]Setting $\tilde{\sigma}_f(S) = \sum_{w \in S} \sigma_f(w)$ for all $S \subseteq W$ with $S \neq \emptyset$, we can create a general strategy $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}$ from a simple strategy $\sigma_f : W \to \mathbb{R}$.

[4]In a Nash equilibrium of a (simple and general) matchmaker game, the matchmaker is indifferent among at least two actions.

3.2 Main Result

Given the above Observation, it makes sense to analyze the Nash equilibrium in the simple matchmaker game. The first and most important result of this paper is as follows.

Theorem 1. In every simple matchmaker game, the matchmaker's profit is zero in every Nash equilibrium.

The proof of this theorem is complicated, and we defer it to the last section of the paper. If there is only one firm, it is not surprising that the firm can reduce wages without changing the matching if the matchmaker is getting a positive profit as in Example 1. However, if multiple firms are competing for workers, a firm's reducing its wage offers may not improve the firm's payoff, since the matchmaker may match other firms with workers whom the firm could have had if it had not reduced wages. Thus the result of Theorem 1 is more subtle than the argument that leaving the profit margin to the matchmaker is never a best response. To provide some intuition behind this result, we briefly describe the proof for a special case of a one-to-one assignment problem: $q_f = 1$ for all $f \in F$ (the formal proof is postponed to Section 5). Suppose that there is a Nash equilibrium with a positive profit, and let $((\sigma, s), \mu)$ be a Nash equilibrium with the highest profit. Pick a firm-worker pair f and w such that $\mu(f) = w$ and $\sigma_f(w) > s_w(f)$. Since μ is the outcome of a Nash equilibrium, firm f and worker w do not deviate for the fear of μ not being chosen. Since the matchmaker is profit-maximizing, if f deviates, the matchmaker chooses a matching $\mu' \neq \mu$ with $\mu'(w) \neq f$ that generates exactly the same profit as μ does (see Corollary 4 in Section 5 for the formal statement). Similarly, if w deviates, the matchmaker chooses matching $\mu'' \neq \mu$ with $\mu''(f) \neq w$ that generates exactly the same profit as μ does. By combining μ' and μ'' with some adjustments we can create a new matching without a match between f and w, which generates an even higher profit than μ. Then the matchmaker can improve its profit by choosing the new matching, which contradicts that $((\sigma, s), \mu)$ is a Nash equilibrium. Thus, even with interactions among firm-worker pairs, leaving the profit margin to the matchmaker cannot be supported by a Nash equilibrium of a simple matchmaker game.

The result of Theorem 1 provides a stark contrast with Nash equilibria in the general matchmaker game. Example 3 in the previous section showed that there might be Nash equilibria that give a positive profit to the matchmaker. Thus, unlike the Nash equilibrium in a (one-sided) combinatorial auction game and the Gale-Shapley algorithm in the two-sided matching problem, restricting the message space to simple strategies has a real impact on the set of Nash equilibria. Is this result bad news for simple strategies? We think that it is actually good news. In a resource allocation problem, a positive profit for the matchmaker (or the auctioneer) is a waste of resources. If a restriction in message space eliminates profit made by the matchmaker, thus achieving a nonwasteful allocation, then it should be considered a desirable property.

Although this result is somewhat surprising by itself, it also turns out to be quite useful when we consider a refinement of Nash equilibrium. With the zero profit result for Nash equilibrium, we will have a strong Nash version of Observation.

Proposition 1. Every strong Nash equilibrium in a simple matchmaker game is a strong Nash equilibrium in the general matchmaker game.

Proof. Suppose that a strong Nash equilibrium in a simple matchmaker game is not immune to a coalitional deviation with general strategies. Then, at least one player improves by the deviation. Suppose that firm f is such a player. Then, after the deviation, f is matched with a subset of workers W_f. Clearly, all $w \in W_f$ cannot be made worse off by the deviation. That is, $Y(f, W_f) - \sum_{w \in W_f} d_{wf}$ must achieve a higher value than the sum of their strong Nash equilibrium payoffs. However, by Theorem 1, every Nash (thus strong Nash) equilibrium leaves zero profit to the matchmaker. Thus, all output is divided up by firms and workers, and the strong Nash equilibrium outcome is a nonwasteful allocation. Since $Y(f, W_f)$ would improve over the allocation, the original matching is not a stable assignment. This is a contradiction. The same logic applies to the case where no firm is strictly better off (but there is a worker who is better off). $\square$

That is, "simple" strategies refine the Nash equilibrium and the strong Nash equilibrium in a general matchmaker game. From previous examples, it is easy to observe that every Nash equilibrium outcome is an acceptable assignment.

Theorem 2. In every many-to-one assignment problem, the set of Nash equilibrium outcomes in the simple matchmaker game is equivalent to the set of acceptable assignments.

Proof. Let (v, u, μ) be the outcome of a Nash equilibrium $((\sigma, s), \mu)$. It is clearly individually rational, as negative payoffs can be avoided. Suppose for firm f there exists some $C \subset \mu(f)$ such that $Y(f, C) - \sum_{w \in C} d_{wf} > v_f + \sum_{w \in C} u_w$. From Theorem 1, that the matchmaker earns zero profit implies $v_f + \sum_{w \in \mu(f)} u_w = Y(f, \mu(f)) - \sum_{w \in \mu(f)} d_{wf}$ and $\sigma_f(w) = s_w(f) = u_w + d_{wf}$ for all $w \in \mu(f)$. Consider $\sigma'_f(w) = \sigma_f(w) + \varepsilon$ for all $w \in C$ and $\sigma'_f(w) = 0$ for all $w \notin C$, where $\varepsilon > 0$ satisfies $\varepsilon < \frac{1}{|C|}\Big[Y(f, C) - Y(f, \mu(f)) + \sum_{w \in \mu(f) \setminus C}(u_w + d_{wf})\Big]$. The matchmaker can make a positive profit by matching f and C. Hence, $((\sigma, s), \mu)$ cannot be a Nash equilibrium. Thus, a Nash equilibrium outcome is an acceptable assignment.

Consider an acceptable assignment (v, u, μ). For every matched firm f, consider for all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = u_w + d_{wf}$, and for all $w' \notin \mu(f)$, $\sigma_f(w') = 0$ and $s_{w'}(f)$ is prohibitively high. For each single firm, let its salary offer be zero for all workers, and for each single worker, let her salary demand be at a prohibitively high level. It is easy to see $((\sigma, s), \mu)$ is a Nash equilibrium.$\square$

We notice in Example 3 that if a Nash equilibrium is refined by a strong Nash equilibrium, then a stable assignment is achieved. The next theorem shows that this is not a coincidence. Using Theorem 1, we obtain the following.

Theorem 3. In every many-to-one assignment problem, the set of strong Nash equilibrium outcomes in the simple matchmaker game is equivalent to the set of stable assignments.

Proof. From Theorem 1, the matchmaker earns zero profit in every Nash equilibria, hence earns zero profit in every strong Nash equilibrium. Let (v, u, μ)

be a strong Nash equilibrium outcome, and suppose that it is not a stable assignment. Then, there is a pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $Y(f, W_f) - \sum_{w \in W_f} d_{wf} > v_f + \sum_{w \in W_f} u_w$. Consider $\sigma'_f(w) = u_w + d_{wf} + \epsilon$ for all $w \in W_f$ and $\sigma'_f(w') = 0$ for all $w' \notin W_f$, and $s'_w(f) = u_w + d_{wf} + \frac{\epsilon}{2}$ for all $w \in W_f$ and $s'_{w'}(f)$ is prohibitively high for all $w' \notin W_f$, where $\epsilon > 0$ satisfies $\epsilon < \frac{1}{|W_f|}\left[Y(f, W_f) - \sum_{w \in W_f} d_{wf} - \left(v_f + \sum_{w \in W_f} u_w\right)\right]$. Since the matchmaker gets no profit, she is happy to match up f and W_f to make a positive profit. This cannot be a strong Nash equilibrium. Thus, a strong Nash equilibrium outcome is a stable assignment.

Now, let (v, u, μ) be a stable assignment. Consider the following strategy. For all matched firms $f \in F$ and all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = u_w + d_{wf}$ and $\sigma_f(w') = 0$ and $s_{w'}(f)$ is prohibitively high for $w' \notin \mu(f)$. For each single firm, let its salary offer be zero for all workers, and for each single worker, let her salary demand be at a prohibitively high level. The matchmaker chooses μ and gets zero profit. Given the strategy (σ, s), the matchmaker would create a new match only when a pair $(f', W_{f'}) \in F \times 2^W$ with $|W_{f'}| \leq q_{f'}$ provides a positive profit. However, by the definition of a stable assignment, there is no pair $(f'', W_{f''}) \in F \times 2^W$ with $|W_{f''}| \leq q_{f''}$ such that $Y(f'', W_{f''}) - \sum_{w \in W_{f''}} d_{wf''} > v_{f''} + \sum_{w \in W_{f''}} u_w$. Thus, there is no subset of players who agree to offer a positive profit to the matchmaker to create a new matching. Therefore, a stable assignment is supportable by a strong Nash equilibrium. $\square$

From Example 3 in the previous section, we know that some Nash equilibria in a general matchmaker game leave positive profits to the matchmaker, which implies that some Nash outcomes are not nonwasteful allocations. We conclude that in matchmaker games, restricting the strategy space to simple ones is socially beneficial.

4 Discussion

In this section, we discuss the issues of implementation in matching problems. We then discuss the relationship between our matchmaker games and the menu auction games in Bernheim and Whinston (1986).

4.1 Implementation

Here we discuss the implementation of popular social choice correspondences by using our matchmaker games. We then show how our results can be connected with the literature on matching problems without money. Let us first introduce some notation. A mapping $\varphi : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a **social choice correspondence** if $\varphi(Y, D) \neq \varnothing$ for all $(Y, D) \in \mathcal{Y} \times \mathcal{D}$. An **individually rational correspondence** $\varphi^{IR} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a social choice correspondence such that $\varphi^{IR}(Y, D) \subset \mathbb{R}^{F \cup W} \times \mathcal{M}$ is the set of all individually rational allocations (v, u, μ) for (Y, D). An **acceptable correspondence** $\varphi^{A} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a social choice correspondence such that $\varphi^{A}(Y, D) \subset \mathbb{R}^{F \cup W} \times \mathcal{M}$ is the set of all acceptable allocations (v, u, μ) for (Y, D). A **stable correspondence** $\varphi^{S} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is a social choice correspondence such that $\varphi^{S}(Y, D) \subset \mathbb{R}^{F \cup W} \times \mathcal{M}$ is the set of all stable assignments (v, u, μ) for (Y, D).

By Theorem 2, we know that the set of Nash equilibrium outcomes and the set of acceptable assignments are equivalent. Thus, we have the following implementation result.

Corollary 1. In every many-to-one assignment problem, the acceptable correspondence $\varphi^{A} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is implemented by the simple matchmaker game in Nash equilibria.

In the one-to-one matching problem, the acceptable allocations and individual rational allocations are the same, and there is no difference between simple and general strategies. Thus, the above corollary implies the following.

Corollary 2. In every one-to-one assignment problem, the individually rational correspondence $\varphi^{IR} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is implemented by the matchmaker game in Nash equilibria.

Theorem 3 directly implies the following.

Corollary 3. In every many-to-one assignment problem, if workers are gross substitutes for each firm, then the stable correspondence $\varphi^{S} : \mathcal{Y} \times \mathcal{D} \twoheadrightarrow \mathbb{R}^{F \cup W} \times \mathcal{M}$ is implemented by the simple matchmaker game in strong Nash equilibria.

Without the gross substitutability assumption, φ^S may be empty valued. This is why we require the assumption. Note that Corollaries 1 and 2 are not affected by the presence of quotas. Hayashi and Sakai (2009) characterize the stable correspondence by Nash implementation. Note that their results cannot treat the one-to-one problem or a many-to-one problem with quotas.

Finally, we connect our results with the implementation literature in a matching problem without money: a many-to-one assignment problem when salaries between each firm and worker are fixed exogenously.[5] Roth (1985) and Shin and Suh (1996) show that under any stable mechanism, the individually rational (acceptable matching, in our definition) correspondence and the stable correspondence are implemented in Nash and strong Nash equilibria, respectively.[6]

Our simple matchmaker game can generate similar results. Suppose for each firm f and each worker w, the salary has been fixed at x_{fw}. Then if firm f hires $W_f \subseteq W$ workers, the payoff for f is $Y(f, W_f) - \sum_{w \in W_f} x_{fw}$. Similarly, if worker w works for firm f, the payoff for w would be $x_{fw} - d_{wf}$. Firms without any workers pay no salary, and unemployed workers receive no salary, so that being unmatched would still result in a payoff of 0. Under this setting, it is easy to see that the definitions in Section 2.1 can be expressed in similar fashion in models of matching without money. Since salaries are fixed here, a firm's offer and a worker's demand are considered as an additional monetary transfer. The matchmaker takes the difference between these two bids. For simplicity, we assume preference orderings are *strict*. Firm f's preference $\succ_f$ is a linear ordering over subsets of workers $\mathcal{S}_f$, while worker w's preference $\succ_w$ is a linear ordering over firms F. An **NTU matching problem** is a list $\{F, W, (\succ_f)_{f \in F}, (\succ_w)_{w \in W}\}$. A **many-to-one matching** $\mu : W \cup F \twoheadrightarrow W \cup F$ is a mapping such that (i) $\mu(f) \subseteq W$ and $\mu(w) \in F \cup \{\varnothing\}$ for all $f \in F$ and for all $w \in W$; (ii) $|\mu(f)| \leq q_f$ for all $f \in F$; (iii) $w \in \mu(f)$ if $f = \mu(w)$; (iv) $\mu(w) = f$ for all $w \in \mu(f)$. A matching μ is **individually rational** if $\mu(f) \succeq_f \varnothing$ for all $f \in F$ and $\mu(w) \succeq_w \varnothing$ for all $w \in W$. A matching μ is

[5] See, say, Chapters 5 and 6.1 in Roth and Sotomayor (1990).

[6] Sonmez (1997) generalizes these results to the class of all efficient and individually rational mechanisms. The results by Suh and Shin (1996) and Sonmez (1997) are on one-to-one matching problems.

acceptable if it is individually rational and $\mu(f) \succ_f C$ for all $C \subsetneqq \mu(f)$. A matching μ is **stable** if there is no pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $W_f \succ_f \mu(f)$ and $f \succeq_w \mu(w)$ for all $w \in W_f$.[7] Let $Ch_f : 2^W \to \mathcal{S}_f$ be firm f's choice function such that $Ch_f(C) = \{S \subseteq C : S \in \mathcal{S}_f$ and $S \succeq_f S'$ for all $S' \subseteq C$ with $S' \in \mathcal{S}_f\}$. Firms' preferences are **substitutable** if for all $f \in F$, all $C \in 2^W$, and all $w \in Ch_f(C)$, $Ch_f(C)\backslash\{w\} \subseteq Ch_f(C\backslash\{w\})$ holds.

We restrict available monetary transfers by firms and workers to the set $\{-L, 0, K\}$, where $L > \max_{w \in W} \max_{f \in F} (x_{fw} - d_{wf})$ and $K > \max_{f \in F} \max_{W_f \subseteq W} (Y(f, W_f) - \sum_{w \in W_f} x_{fw})$. Each firm's offer will be chosen from the set $\{-L, 0\}$, since for any firm K is an amount of money that is not worthwhile to pay to any worker. Similarly, each worker's request will be chosen from $\{0, K\}$, since for any worker $-L$ is an amount of money that is not worthwhile to request from any firm. We assume the following **tie-breaking rule**: the matchmaker matches up a pair of a firm and a worker if she is indifferent between matching them up or not.[8] What remains is exactly the same as a simple matchmaker game. Call this game a **simple NTU matchmaker game**. We can show the following result.

Theorem 4. In every many-to-one matching problem without transfer, if firms' preferences are substitutable, then the set of Nash equilibrium matchings in the simple NTU matchmaker game is equivalent to the set of acceptable matchings, and the set of strong Nash equilibrium matchings in the simple NTU matchmaker game is equivalent to the set of stable matchings.

Proof. First, we show that a Nash equilibrium matching is individually rational and acceptable. A Nash equilibrium matching is individually rational because, by construction, every worker will not be matched up with a firm if she requests $K > 0$ from it, and every firm will not be matched with a worker if it offers $-L < 0$ to her. This implies that in every Nash equilibrium the matchmaker earns zero profit. We can show that for every Nash equilibrium

[7] With strict preferences, this definition is the same as requiring no (f, W_f) such that firm f and all workers in W_f are weakly better off and at least one of them is strictly better off.

[8] This tie-breaking rule is sufficient to pin down the Nash equilibrium under strict preferences. However, if indifference in preferences is allowed, more careful treatment is needed in the NTU setting. See Ko (2010) for further discussion.

$((\sigma, s), \mu)$, the matching μ is an acceptable matching. Suppose not. Then, there exist a firm f and a subset of workers $C \subsetneqq \mu(f)$ such that $C \succ_f \mu(f)$. However, firm f can improve its payoff by switching its strategy to σ'_f such that $\sigma'_f(w) = 0$ if $w \in C$ and $\sigma'_f(w) = -L$ if $w \notin C$, which is a contradiction.

Second, an acceptable matching can be implemented by a Nash equilibrium. Consider an acceptable matching μ. For each matched firm f, consider for all $w \in \mu(f)$, $\sigma_f(w) = s_w(f) = 0$, and for all $w' \notin \mu(f)$, $\sigma_f(w') = -L$ and $s_{w'}(f) = K$. Given the tie-breaking rule by the matchmaker, the matching μ is chosen given the strategy profile (σ, s). This is a Nash equilibrium because all unmatched pairs would never be matched up by choosing other strategies.

Third, we show that a strong Nash equilibrium matching is stable. Let $((\sigma, s), \mu)$ be a strong Nash equilibrium. Suppose it is not a stable matching. Then there is a pair $(f, W_f) \in F \times 2^W$ with $|W_f| \leq q_f$ such that $W_f \succ_f \mu(f)$ and $f \succ_w \mu(w)$ for all $w \in W_f$ (strict preference). Consider a deviation by (f, W_f) such that (i) $\sigma'_f(w) = 0$ for all $w \in W_f$ and $\sigma'_f(w) = -L$, otherwise, and (ii) for all $w \in W_f$, $s'_w(f) = 0$ and $s'_w(f') = K$ for $f' \neq f$. Since the matchmaker would still make zero profit by matching (f, W_f) and no player in $\{f\} \cup W_f$ can be matched with outsiders, the matchmaker matches them up by the tie-breaking rule. Thus, $((\sigma, s), \mu)$ cannot be a strong Nash equilibrium.

Finally, we show that a stable matching can be implemented by a strong Nash equilibrium. Let μ be a stable matching. Consider the following strategy. For each matched firm f, consider a strategy profile (σ, s) such that $\sigma_f(w) = s_w(f) = 0$ if and only if $w \in \mu(f)$. Given the tie-breaking rule, matching μ is chosen by the matchmaker. Since this is a stable matching, it is immune to coalitional deviations, which implies that $((\sigma, s), \mu)$ is a strong Nash equilibrium.□

4.2 Relationship with Menu Auction Games

A menu auction game is a complete information multi-principal-one-agent game, introduced by Bernheim and Whinston (1986). The agent is going to choose an action, which will affect her own payoff as well as the payoffs to principals. Principals can affect the agent's decision by offering a menu of side payments: a side payment schedule for each possible action. The agent maximizes the sum of her own utility and side payments from the principals when choosing an action. We can consider our matchmaker's problem as a menu auction game by interpreting a matching μ as an action, and letting the matchmaker be intrinsically indifferent over μ (except for side payments).

A **menu auction problem** Γ is described by $(N+2)$ tuples:

$$\Gamma \equiv \left\{A, (V_k)_{k \in N \cup \{0\}}\right\},$$

where A is the set of actions, $V_k : A \to \mathbb{R}$ is k's (quasi-linear) payoff function, 0 denotes the agent, and N is the set of principals. In the extensive form of the game, the principals simultaneously offer contingent payment schedules to the agent, who subsequently chooses an action that maximizes her total payoff. A strategy for each principal $k \in N$ is a function $T_k : A \to [b_k, \infty)$, which is a monetary reward (or punishment) of $T_k(a)$ to the agent for selecting a, where b_k is the lower bound for payment from principal k. For each action a, principal k receives a net payoff:

$$U_k(a, T) = V_k(a) - T_k(a),$$

where $T = (T_{k'})_{k' \in N}$ is a strategy profile. The set of all possible strategies for principal k is denoted by $\mathcal{T}_k$. The agent chooses an action that maximizes her total payoff: the agent selects an action in the set $M(T)$, where

$$M(T) \equiv \underset{a \in A}{\operatorname{argmax}} \left[V_0(a) + \sum_{k \in N} T_k(a) \right].$$

A **menu auction game** $(\Gamma, \mathcal{T})$ is a pair consisting of a menu auction problem Γ and a set of strategies for all principals $\mathcal{T} = (\mathcal{T}_k)_{k \in N}$. This menu auction game is merely a game among principals, although, strictly speaking, a tie-breaking rule among $M(T)$ needs to be specified for the agent.

Let $\mathcal{T}_k^I \equiv \{T_k \in \mathcal{T}_k : T_k(a) = T_k(a') \text{ for all } a, a' \in A \text{ with } V_k(a) = V_k(a')\}$ be the restricted domain of strategies that requires principal k must bid the same amount for all actions among which principal k is indifferent. If all principals' strategy spaces belong to this domain, then we say that the principals' strategy spaces belong to the set of strategy spaces $\mathcal{T}^I = \left(\mathcal{T}_k^I\right)_{k \in N}$.[9]

An outcome of a menu auction game $(\Gamma, \mathcal{T})$ is (a, T). An outcome (a^*, T^*) is a **Nash equilibrium** if $a^* \in M(T^*)$ and there is no $k \in N$ such that $T_k : A \to [b_k, \infty)$ and $a \in M\left(T_k, T_{-k}^*\right)$ such that $U_k\left(a, T_k, T_{-k}^*\right) > U_k(a^*, T^*)$. However, the set of Nash equilibria in a menu auction game is quite large owing to coordination problems. So, Bernheim and Whinston (1986) propose a refinement of Nash equilibrium by using what they call "truthful strategies." A strategy T_k is **truthful relative to** $\bar{a}$ if and only if for all $a \in A$ either (i) $U_k(a, T) = U_k(\bar{a}, T)$ or (ii) $U_k(a, T) < U_k(\bar{a}, T)$ and $T_k(a) = b_k$. Clearly, truthful strategies belong to the domain $\mathcal{T}^I$. An outcome (a^*, T^*) is a **truthful Nash equilibrium (TNE)** if and only if it is a Nash equilibrium, and T_k^* is truthful relative to a^* for all $k \in N$.

It is clear that if workers are objects (with no preferences), and if firms are bidding on workers, then we can easily formulate a combinatorial auction game by this menu auction.[10] In the following, we show that our *general* matchmaker game can also be embedded in the class of menu auction games by reinterpreting players' strategies. In a general matchmaker game, firm f's strategy $\tilde{\sigma}_f : \mathcal{S}_f \to \mathbb{R}_+$ is **truthful relative to** W_f if and only if for all $S \in \mathcal{S}_f$ either (i) $Y(f, S) - \tilde{\sigma}_f(S) = Y(f, W_f) - \tilde{\sigma}_f(W_f)$ or (ii) $Y(f, S) - \tilde{\sigma}_f(S) < Y(f, W_f) - \tilde{\sigma}_f(W_f)$ and $\tilde{\sigma}_f(S) = 0$.

Proposition 2. A general matchmaker game can be embedded in the class of menu auction games with strategy space $\mathcal{T}^I$. A strategy in a general matchmaker game is truthful if and only if the corresponding strategy is a truthful strategy in the corresponding menu auction game.

Proof. Let the matchmaker be the agent, and firms and workers be principals. Let $\mathcal{M}$ be the set of actions A. Firm f receives monetary payoff $V_f : \mathcal{M} \to \mathbb{R}$ where $V_f(\mu) \equiv Y(f, \mu(f))$, worker w receives monetary payoff $V_w : \mathcal{M} \to \mathbb{R}$ with $V_w(\mu) \equiv -d_{w\mu(w)}$, and the matchmaker's (denoted by 0) monetary payoff

[9] Although this restriction is needed for the formal statement of Proposition 1, the set of Nash equilibrium payoffs with the restriction is the same as the set of Nash equilibrium payoffs without the restriction.

[10] Milgrom (2004) discusses menu auction games in the context of a combinatorial auction problem.

is $V_0(\mu) = 0$ for all $\mu \in \mathcal{M}$.[11]

Under $\mathcal{T}^I$, principals are able to choose any contribution menu over potential partners but not over the entire matching. A strategy for firm f that is generated from σ_f is a function $T_f : \mathcal{M} \to \mathbb{R}_+$, where $T_f(\mu) \equiv \sigma_f(\mu(f))$. A strategy for worker w that is generated from s_w is a function $T_w : \mathcal{M} \to \mathbb{R}_-$, where $T_w(\mu) \equiv -s_w(\mu(w))$. We can set a lower bound for the value for $T_w(\mu)$ without losing anything, since worker w would not be matched anyway, if $T_w(\mu) < -Y(f, \mu(f))$ holds. Thus, we assume that for each $k \in N = W \cup F$, there is a lower bound b_k: $T_k(\mu) \geq b_k$ that must be satisfied for all $k \in N$. Thus, a matchmaker game can be represented as a menu auction game. Clearly, a truthful strategy $\tilde{\sigma}_f$ or s_w trivially can be extended to a truthful strategy T_k, and vice versa. This completes the proof.$\square$

Remark. Note that in a one-to-one assignment problem, the general strategy and the simple strategy are equivalent. Thus, Proposition 2 together with Theorem 1 implies that the agent earns zero rent in every Nash equilibrium in a menu auction game that is generated from a matchmaker game in a one-to-one assignment problem.

Laussel and Le Breton (2001) define a menu auction game as possessing the **no-rent property** if and only if all truthful Nash equilibrium (TNE) outcomes leave no profit to the agent. They prove that if a cooperative game from a menu auction game Γ is convex,[12] then Γ possesses the no-rent property. However, although convexity is satisfied in interesting classes of menu auction games such as the public good provision game, in our assignment problem convexity is clearly not satisfied.[13] Moreover, the following example shows that even if convexity holds, there exists a Nash equilibrium such that the agent earns a positive profit.

Example 4 (discrete public good provision). Consider a public good pro-

[11] We normalize $V_0(\mu) = 0$ because the matchmaker has no preferences over the matchings themselves.

[12] A system $(v(S))_{S \subseteq N}$ is **convex** if and only if for all $S, T \subseteq N$, $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ holds.

[13] For example, imagine $N = \{f_1, w_1, w_2\}$ with $y_{11} = y_{12} = 1$. Letting $S = \{f_1, w_1\}$ and $T = \{f_1, w_2\}$, we can see a violation of convexity.

vision problem with two principals (consumers) $N = \{1, 2\}$ and an agent (the government) with two actions $A = \{a_1, a_2\}$. Actions a_2 and a_1 are regarded as provision and no provision of a discrete public good. Consumers prefer a_2 to a_1 but a_2 is more costly for the government: $V_i(a_1) = 0$ and $V_i(a_2) = 5$ for $i = 1, 2$ and $V_0(a_1) = 0$ and $V_0(a_2) = -1$ (public good provision cost is 1). This creates a transferrable utility cooperative game (N, v) such that $v(\{1, 2\}) = 9$, $v(\{1\}) = v(\{2\}) = 4$, and $v(\varnothing) = 0$, where $v(S)$ is the value of coalition $S \subseteq N$. This is a convex game, and Le Breton-Laussel's no-rent property holds. Consider $T_1(a_1) = 2$, $T_1(a_2) = T_2(a_1) = 0$, and $T_2(a_2) = 3$. Then (a_2, T) is a Nash equilibrium where the agent earns a positive profit. However, the set of truthful Nash equilibria is $\{(a_2, \tilde{T}) : \tilde{T}_1(a_1) = \tilde{T}_2(a_1) = 0$ and $\tilde{T}_1(a_2) + \tilde{T}_2(a_2) = 1\}$ since the game satisfies the no-rent property.

In contrast, in our one-to-one matchmaker game, the matchmaker always earns zero profit not only in all truthful Nash equilibria but also in all Nash equilibria. Since the simple strategy and the general strategy are the same in the one-to-one matchmaker game, Theorem 1 provides another interesting class of menu auction games that possess the no-rent property. However, in many-to-one matching, we cannot obtain the same result by Proposition 2 (and Example 3).

Readers who are familiar with the menu auction game literature may find it odd that we have not mentioned **coalition-proof Nash equilibrium** (**CPNE**: Bernheim and Whinston 1986; Bernheim, Peleg, and Whinston 1987), which is the central solution concept in menu auction games.[14] The standard definition of coalition-proof Nash equilibrium requires that all reduced games (where the outsiders of a coalition keep their strategies fixed, and the members of the coalition play the game) belong to the same class of games. Unfortunately, however, in our game, this is not true. If outsiders make their salary offers and demands to coalition-members, then the matchmaker will have preferences over the matchings it chooses. However, we have assumed that the matchmaker cares only about the profit made from matching

[14] Bernheim and Whinston (1986) show that TNE and CPNE are equivalent in a utility space. Under the no-rent property, Laussel and Le Breton (2001) and Konishi, Le Breton, and Weber (1999) show the equivalences of CPNE and the core of underlying TU game, and CPNE and SNE, respectively.

unlike in Bernheim and Whinston (1986). This is why we have not mentioned coalition-proof Nash equilibrium in this paper. In contrast, if we allow the matchmaker to have preferences over matchings, we can extend Theorem 3 in the domain of the one-to-one matching problem.[15]

Theorem 3'. Suppose that the matchmaker is allowed to have preferences over matchings. Then, in every one-to-one assignment problem, the sets of truthful Nash equilibrium outcomes, strong Nash equilibrium outcomes, and coalition-proof Nash equilibrium outcomes in the matchmaker game, and the set of stable assignments (the core) are all equivalent.

5 Proof of Theorem 1

In this section, we prove Theorem 1. First, we introduce some notation. For all $S \subseteq N$, let $C(S,\mu) \equiv \{k \in S : \mu(k) \in S \text{ and } \mu(k) \neq \varnothing\}$. That is, $C(S,\mu)$ is the set of members of S who have partners in S under matching μ (coupled). Given a strategy profile $(\sigma, s) \in \mathbb{R}^{F\times W} \times \mathbb{R}^{W\times F}$, let $R(S,(\sigma,s),\mu)$ be the profit (rent) generated in S under μ such that $R(S,(\sigma,s),\mu) = \sum_{f\in C(S,\mu)\cap F}(\sigma_f(\mu(f)) - s_{\mu(f)}(f))$. Let $R^*(S,(\sigma,s)) \equiv \max_{\mu\in\mathcal{M}} R(S,(\sigma,s),\mu)$ and let $A^*(S,(\sigma,s)) \equiv \operatorname{argmax}_{\mu\in\mathcal{M}} R(S,(\sigma,s),\mu)$ be the maximum profit generated in coalition S given firms' strategies σ and workers' strategies s, and its associated matching μ, respectively. We can characterize Nash equilibrium payoffs in an interesting way.

Proposition 3. In every simple matchmaker game, in every Nash equilibrium $((\sigma,s),\mu)$, (1A) for all $f \in F$ with $\mu(f) = \varnothing$, $R^*(N,(\sigma,s)) = R(N,(\sigma,s),\mu) = R^*(N\backslash\{f\},(\sigma,s))$; (1B) for all $f \in F$ with $\mu(f) \neq \varnothing$, and all $w \in \mu(f)$, there exists μ' such that (i) $\mu'(f) \subsetneqq \mu(f)\backslash\{w\}$, and (ii) $R^*(N,(\sigma,s)) = R(N,(\sigma,s),\mu')$; and (2) for all $w \in W$, there exists μ'' such that (i) $\mu''(w) = \varnothing$, and (ii) $R^*(N,(\sigma,s)) = R(N,(\sigma,s),\mu'') = R^*(N\backslash\{w\},(\sigma,s))$.

[15] The proof of equivalence is very simple. By definition, an SNE is a CPNE. And the outcome of a CPNE is a stable assignment, since otherwise, there is a pair that deviates from a CPNE. But in this domain, such a two-person deviation is credible anyway.

Proof. Since (2) is a special case of (1), we focus on case (1).

Case (1A) is trivial, since we can use the same matching μ to achieve the same profit. Thus, we will work on case (1B). Clearly, if $\sigma_f(w) = s_w(f)$ for all $w \in \mu(f)$, then we can find a μ' that satisfies all three conditions: the matchmaker makes no money by matching f with workers, so she might as well cancel the matching (let $\mu'(f) = \varnothing$). Thus, let us focus on $\mu(f) \in W$ and $\sigma_f(w) > s_w(f)$ for some $w \in \mu(f)$ for the rest of the proof.

Consider $\sigma'_f(w) = \sigma_f(w) - \epsilon$, $\sigma'_f(w') = \max\{\sigma_f(w') - \epsilon, 0\}$ for all $w' \notin \mu(f)$ and $\sigma'_f(w'') = \sigma_f(w'')$ for all $w'' \in \mu(f) \setminus \{w\}$. Let $\mu' \in A^*(N, (\sigma'_f, \sigma_{-f}, s))$. By construction, $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(N, (\sigma, s), \mu') - \epsilon |\mu'(f) \setminus \mu(f)|$ and $R(N, (\sigma'_f, \sigma_{-f}, s), \mu) = R(N, (\sigma, s), \mu) - \epsilon$. By optimalities of μ and μ', we have $R(N, (\sigma, s), \mu) \geq R(N, (\sigma, s), \mu')$ and $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') \geq R(N, (\sigma'_f, \sigma_{-f}, s), \mu)$. Since $|\mu'(f) \setminus \mu(f)| > 1$ leads to a contradiction, either $|\mu'(f) \setminus \mu(f)| = 1$ or $|\mu'(f) \setminus \mu(f)| = 0$. Suppose $|\mu'(f) \setminus \mu(f)| = 1$. This implies $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(N, (\sigma'_f, \sigma_{-f}, s), \mu)$. However, if this is the case, then firm f can improve its payoff by $\epsilon > 0$ by choosing σ''_f such that $\sigma''_f(w) = \sigma_f(w) - \epsilon$, $\sigma''_f(w') = 0$ for all $w' \notin \mu(f)$ and $\sigma''_f(w'') = \sigma_f(w'')$ for all $w'' \notin \mu(f) \setminus \{w\}$ as the matchmaker is forced to choose μ. This is a contradiction. Hence, we have $|\mu'(f) \setminus \mu(f)| = 0$ or $\mu'(f) \subseteqq \mu(f)$. Hence, $R(N, (\sigma, s), \mu') = R(N, (\sigma'_f, \sigma_{-f}, s), \mu')$.

(i) Suppose $w \in \mu'(f)$. By construction, $R(\{f, \mu'(f)\}, (\sigma, s), \mu') > R(\{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}, s), \mu')$ and $R(N \setminus \{f, \mu'(f)\}, (\sigma, s), \mu') = R(N \setminus \{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}, s), \mu')$. Since $R(N, (\sigma, s), \mu') = R(\{f, \mu'(f)\}, (\sigma, s), \mu') + R(N \setminus \{f, \mu'(f)\}, (\sigma, s), \mu')$ and $R(N, (\sigma'_f, \sigma_{-f}, s), \mu') = R(\{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}, s), \mu') + R(N \setminus \{f, \mu'(f)\}, (\sigma'_f, \sigma_{-f}, s), \mu')$, we have $R(N, (\sigma, s), \mu') > R(N, (\sigma'_f, \sigma_{-f}, s), \mu')$. This is a contradiction. Thus, $\mu'(f) \subseteqq \mu(f) \setminus \{w\}$.

(ii) Suppose not. Then $R^*(N, (\sigma, s)) > R(N, (\sigma, s), \mu')$. Consider $\delta \equiv R^*(N, (\sigma, s)) - R(N, (\sigma, s), \mu') > 0$. Since $R(N, (\sigma, s), \mu') = R(N, (\sigma'_f, \sigma_{-f}, s), \mu')$, firm f can improve its payoff by $\epsilon < \delta$ by choosing σ'''_f such that $\sigma'''_f(w) = \sigma_f(w) - \epsilon$, $\sigma'''_f(w') = 0$ for all $w' \notin \mu(f)$ and $\sigma'''_f(w'') = \sigma_f(w'')$ for all $w'' \notin \mu(f) \setminus \{w\}$. This is a contradiction.□

Although Theorem 1 deals with a simple matchmaker game in a many-to-one matching problem, it is more convenient to start with a one-to-one matching problem, since the result of a one-to-one matching problem can be extended to the case of a many-to-one matching problem. Let $q_f = 1$ for all $f \in F$. In the one-to-one matching problem, Proposition 3 becomes the following simple statement.

Corollary 4. In every one-to-one matchmaker game, in every Nash equilibrium $((\sigma, s), \mu)$, $R^*(N, (\sigma, s)) = R^*(N \backslash \{k\}, (\sigma, s))$ for all $k \in N$.

Let $S_k = \{k' \in N \backslash \{k\} : \mu(k') \neq \varnothing\}$. This implies that $R^*(S_k, (\sigma, s)) = R(S_k, (\sigma, s), \mu) = R^*(N \backslash \{k\}, (\sigma, s))$. Then, Corollary 4 says that in every Nash equilibrium $((\sigma, s), \mu)$, for all $k \in N$, there exists $S_k \subseteq N \backslash \{k\}$ such that the following equation holds:

$$R^*(N, (\sigma, s)) = R^*(S_k, (\sigma, s)).$$

This **system of Nash equations** characterizes a Nash equilibrium $((\sigma, s), \mu)$ of the one-to-one matchmaker game.[16] The following is the first main result of this section.

Proposition 4. In every one-to-one matchmaker game, the matchmaker's profit is zero in every Nash equilibrium.

Proof. We will prove the theorem by contradiction. Assume that there is a Nash equilibrium allocation $((\sigma, s), \mu)$ with a positive profit ($R(N, (\sigma, s), \mu) = R^*(N, (\sigma, s)) > 0$), and we will reach a contradiction.

First, note that $R(N, (\sigma, s), \mu) = \sum_{f \in C(N,\mu) \cap F} R(\{f, \mu(f)\}, (\sigma, s), \mu)$. Pick up a pair $(f_1, w_1) \subset N$ that generates the *highest* positive profit under (σ, s) and μ:

$$R(\{f_1, w_1\}, (\sigma, s), \mu) > 0. \tag{$*$}$$

[16] Our system of Nash equations is inspired by the system of fundamental equations given by Laussel and Le Breton (2001). However, these two systems of equations are very different from each other. Laussel and Le Breton's (2001) system of fundamental equations is constructed from each coalition's value (the maximal value of the sum of the payoffs of the agent and the principals in the coalition), and all truthful equilibrium payoff vectors satisfy the same system of equations. In contrast, our system of Nash equations is constructed from the matchmaker's (the agent's) total profit for each coalition when a Nash strategy profile is picked.

The relevant Nash equations for f_1 and w_1 can be written as

$$\sum_{f\in C(S_{f_1},\mu')\cap F} R(\{f,\mu'(f)\},(\sigma,s),\mu') = \sum_{f\in C(N,\mu)\cap F} R(\{f,\mu(f)\},(\sigma,s),\mu),$$

$$\sum_{f\in C(S_{w_1},\mu'')\cap F} R(\{f,\mu''(f)\},(\sigma,s),\mu'') = \sum_{f\in C(N,\mu)\cap F} R(\{f,\mu(f)\},(\sigma,s),\mu)$$

where $\mu' \in A^*\left(S_{f_1},(\sigma,s)\right)$ and $\mu'' \in A^*\left(S_{w_1},(\sigma,s)\right)$.

Our first lemma is the following.

Lemma 1. We have $w_1 \in S_{f_1}$ and $R(\{\mu'(w_1),w_1\},(\sigma,s),\mu') > 0$. Similarly, $f_1 \in S_{w_1}$ and $R(\{f_1,\mu''(f_1)\},(\sigma,s),\mu'') > 0$.

Proof of Lemma 1. We will prove the first half (the second half follows by a symmetric argument). Suppose $w_1 \notin S_{f_1}$ or $\mu'(w_1) = \varnothing$. Then, we can construct a new matching μ^* such that $\mu^*(k) = \mu'(k)$ for all $k \in S_{f_1}$, $\mu^*(f_1) = w_1$, and $\mu^*(k) = \varnothing$ for all $k \in N\setminus\left(S_{f_1}\cup\{w_1,f_1\}\right)$. Then, we have

$$\begin{aligned} R\left(N,(\sigma,s),\mu^*\right) &= R\left(S_{f_1},(\sigma,s),\mu'\right) + R\left(\{w_1,f_1\},(\sigma,s),\mu\right) \\ &> R\left(S_{f_1},(\sigma,s),\mu'\right) = R\left(N,(\sigma,s),\mu\right). \end{aligned}$$

Note that the last equality comes from the Nash equation. This is in contradiction with $\mu \in A^*(N,(\sigma,s))$.

Now, suppose $R(\{\mu'(w_1),w_1\},(\sigma,s),\mu') = 0$ (if profit is negative, the matchmaker would rather leave them unmatched). Then, we have

$$\begin{aligned} R\left(S_{f_1},(\sigma,s),\mu'\right) &= R\left(S_{f_1}\setminus\{w_1,\mu'(w_1)\},(\sigma,s),\mu'\right) + R(\{\mu'(w_1),w_1\},(\sigma,s),\mu') \\ &= R\left(S_{f_1}\setminus\{w_1,\mu'(w_1)\},\mu'\right). \end{aligned}$$

Then we could construct μ^* such that $\mu^*(k) = \mu'(k)$ for all $k \in S_{f_1}$, $\mu^*(f_1) = w_1$, and $\mu^*(k) = \varnothing$ for all $k \in N\setminus\left(S_{f_1}\cup\{w_1,\mu'(w_1)\}\right)$. Then we have

$$\begin{aligned} R\left(N,(\sigma,s),\mu^*\right) &= R\left(S_{f_1}\setminus\{w_1,\mu'(w_1)\},(\sigma,s),\mu'\right) + R\left(\{w_1,f_1\},(\sigma,s),\mu'\right) \\ &= R\left(S_{f_1},(\sigma,s),\mu'\right) + R(\{w_1,f_1\},(\sigma,s),\mu) > R\left(N,(\sigma,s),\mu\right). \end{aligned}$$

This violates $\mu \in A^*(N, (\sigma, s))$.$\square$

Recall μ' and μ'' are matchings that achieve values $R^*(S_{f_1}, (\sigma, s))$ and $R^*(S_{w_1}, (\sigma, s))$, respectively. By using Lemma 1, we will construct chains of pairs from matchings μ, μ', and μ''. Let $f_{\ell+1} \equiv \mu'(w_\ell)$ and $w_{\ell+1} = \mu(f_{\ell+1})$ for $\ell = 1, 2, ...L$, where L is such that $\mu(f_\ell) \in C(N, \mu) \cap W$ and $\mu'(w_\ell) \in C(N, \mu) \cap F$ for all $\ell < L$ and $\mu(f_L) \notin C(N, \mu) \cap W$. Similarly, let $\tilde{w}_{\ell+1} \equiv \mu''(\tilde{f}_\ell)$ and $\tilde{f}_{\ell+1} = \mu(\tilde{w}_{\ell+1})$ for $\ell = 1, 2, ..., \tilde{L}$, where $\tilde{L}$ is such that $\mu(\tilde{w}_\ell) \in C(N, \mu) \cap F$ and $\mu'(\tilde{f}_\ell) \in C(N, \mu) \cap W$ for all $\ell < \tilde{L}$ and $\mu(\tilde{w}_{\tilde{L}}) \notin C(N, \mu) \cap F$. The following is our key lemma.

Lemma 2. Either $\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ or $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$ holds.

Proof of Lemma 2. Optimality of μ implies: $\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) \geq \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) \geq \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$. Thus, suppose to the contrary that

$$\begin{aligned} \sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) &= \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu'), \\ \sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) &= \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu''). \end{aligned} \qquad (**)$$

There are two cases: (Case 1) $\left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) = \varnothing$, and (Case 2) $\left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) \neq \varnothing$. We will analyze the two cases by noting $\{w_1, f_1\} = \{\tilde{w}_1, \tilde{f}_1\}$. Let us start with the simpler case.

(Case 1) Suppose $\left(\cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1} \{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) = \varnothing$. See Figure 3.

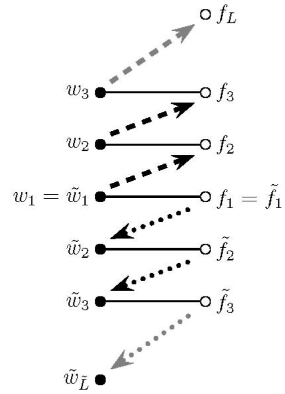


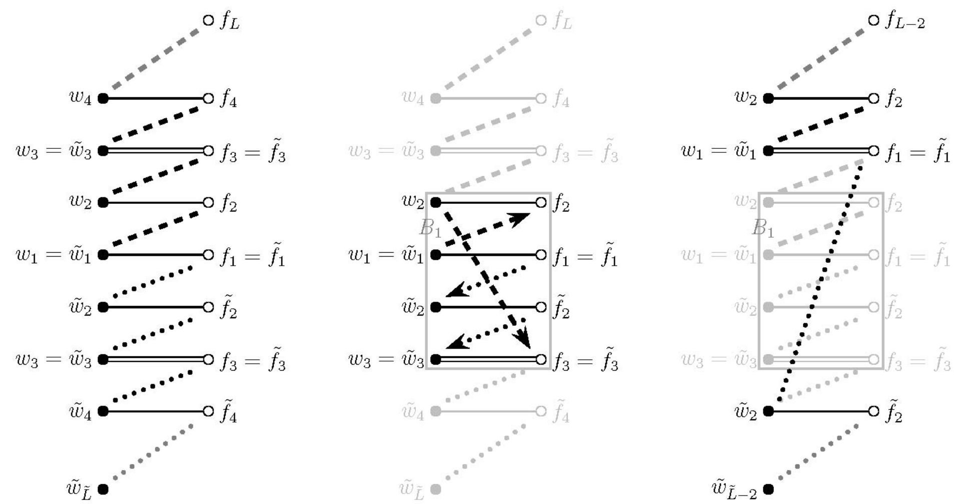
Figure 3: Illustration of (Case 1). Solid, dashed, and dotted lines represent matchings μ, μ', and μ'', respectively. Arrows represent μ^*.

Summing the two equations in $(**)$, we have

$$\begin{aligned}
& \sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu'\right) + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \\
= & \sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, (\sigma, s), \mu\right) + \sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) \\
= & \left(\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\}, (\sigma, s), \mu\right) + \sum_{\ell=2}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu)\right) \\
& + R(\{w_1, f_1\}, (\sigma, s), \mu)
\end{aligned}$$

where the last equality comes from $\{\tilde{w}_1, \tilde{f}_1\} = \{w_1, f_1\}$. Let $A \equiv \left(\cup_{\ell=1}^{L-1}\{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\})$. There is no double counting of players in A. Let $\mu^* \in \mathcal{M}$ be such that $\mu^*(w_\ell) = f_{\ell+1}$ for $\ell = 1, ..., L-1$ and $\mu^*(\tilde{f}_\ell) = \tilde{w}_{\ell+1}$ for $\ell = 1, ..., \tilde{L}-1$. Replacing μ by μ^*, the total value in A increases by $R(\{w_1, f_1\}, (\sigma, s), \mu)$. By the prevailing assumption $(*)$, $R(\{w_1, f_1\}, (\sigma, s), \mu) > 0$. This contradicts the optimality of μ.

(Case 2) $\left(\cup_{\ell=1}^{L-1}\{w_\ell, f_{\ell+1}\}\right) \cap (\cup_{\ell=1}^{\tilde{L}-1}\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}) \neq \varnothing$. Let $\bar{\ell}$ be such that for all $1 \leq \ell < \bar{\ell}$, $\tilde{w}_\ell, \tilde{f}_\ell \notin \cup_{\ell=1}^{L}\{w_\ell, f_\ell\}$, and $\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}} \in \cup_{\ell=1}^{L}\{w_\ell, f_\ell\}$. Hence, $\{\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}}\} = \{w_{\ell'}, f_{\ell'}\}$ for some $\ell' \in \{2, ..., L\}$. See Figure 4a for the case when



(a) $w_3 = \tilde{w}_3$, $f_3 = \tilde{f}_3$. **(b)** Construct B_1. **(c)** Remove B_1.

Figure 4: Illustration of (Case 2) when $\ell' = \bar{\ell} = 3$. Solid, dashed, and dotted lines represent matchings μ, μ', and μ'', respectively. Arrows represent μ^*.

$\ell' = \bar{\ell} = 3$. Denote the set of players $B_1 \equiv (\cup_{\ell=1}^{\ell'} \{w_\ell, f_\ell\}) \cup (\cup_{\ell=2}^{\bar{\ell}-1}\{\tilde{w}_\ell, \tilde{f}_\ell\})$ as in Figure 4b. There is no double counting in B_1. Now, consider two matchings in B_1: μ and μ^* such that $\mu^*(w_\ell) = \mu'(w_\ell)$ for $\ell = 1, ..., \ell'-1$, and $\mu^*(\tilde{f}_\ell) = \mu''(\tilde{f}_\ell)$ for $\ell = 1, ..., \bar{\ell}-1$ (note $\tilde{f}_1 = f_1$ and $\tilde{w}_{\bar{\ell}} = w_{\ell'}$). We now compare the values of these two. First,

$$
\begin{aligned}
& R(B_1, (\sigma, s), \mu^*) \\
= \;& \sum_{\ell=1}^{\ell'-1} R\left(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu'\right) + \sum_{\ell=1}^{\bar{\ell}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'') \\
= \;& \left[\sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu'\right) + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')\right] \\
& - \left[\sum_{\ell=\ell'}^{L-1} R\left(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu'\right) + \sum_{\ell=\bar{\ell}}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')\right]
\end{aligned}
$$

and

$$
\begin{aligned}
& R(B_1,(\sigma,s),\mu) \\
= \;& \sum_{\ell=1}^{\ell'} R\left(\{w_\ell, f_\ell\},(\sigma,s),\mu\right) + \sum_{\ell=2}^{\bar{\ell}-1} R(\{\tilde{w}_\ell,\tilde{f}_\ell\},(\sigma,s),\mu) \\
= \;& \left[\sum_{\ell=1}^{L} R\left(\{w_\ell, f_\ell\},(\sigma,s),\mu\right) + \sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell,\tilde{f}_\ell\},(\sigma,s),\mu)\right] \\
& - \left[\sum_{\ell=\ell'}^{L} R\left(\{w_\ell, f_\ell\},(\sigma,s),\mu\right) + \sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell,\tilde{f}_\ell\},(\sigma,s),\mu)\right] \\
& - \left[R(\{w_1,f_1\},(\sigma,s),\mu) - R\left(\{w_{\ell'}, f_{\ell'}\},(\sigma,s),\mu\right)\right] \\
= \;& \left[\sum_{\ell=1}^{L-1} R\left(\{w_\ell, f_{\ell+1}\},(\sigma,s),\mu'\right) + \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1},\tilde{f}_\ell\},(\sigma,s),\mu'')\right] \\
& - \left[\sum_{\ell=\ell'}^{L} R\left(\{w_\ell, f_\ell\},(\sigma,s),\mu\right) + \sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell,\tilde{f}_\ell\},(\sigma,s),\mu)\right] \\
& - \left[R(\{w_1,f_1\},(\sigma,s),\mu) - R\left(\{w_{\ell'}, f_{\ell'}\},(\sigma,s),\mu\right)\right]
\end{aligned}
$$

where the last equality follows from $(**)$. Thus, we have

$$
\begin{aligned}
& R(B_1,(\sigma,s),\mu^*) - R(B_1,(\sigma,s),\mu) \\
= \;& R(\{w_1,f_1\},(\sigma,s),\mu) - R\left(\{w_{\ell'}, f_{\ell'}\},(\sigma,s),\mu\right) \\
& + \left[\sum_{\ell=\ell'}^{L} R\left(\{w_\ell, f_\ell\},(\sigma,s),\mu\right) - \sum_{\ell=\ell'}^{L-1} R\left(\{w_\ell, f_{\ell+1}\},(\sigma,s),\mu'\right)\right] \\
& + \left[\sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell,\tilde{f}_\ell\},(\sigma,s),\mu) - \sum_{\ell=\bar{\ell}}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1},\tilde{f}_\ell\},(\sigma,s),\mu'')\right].
\end{aligned}
$$

Note that the contents in both brackets must be nonnegative since μ maximizes the total value in N. Since $\{w_1, f_1\}$ generates the highest profit under (σ, s) and μ, $R(\{w_1,f_1\},(\sigma,s),\mu) \geq R\left(\{w_{\ell'}, f_{\ell'}\},(\sigma,s),\mu\right)$ must hold. Thus, $R(B_1,(\sigma,s),\mu^*) \geq R(B_1,(\sigma,s),\mu)$ must hold. If $R(B_1,(\sigma,s),\mu^*) > R(B_1,(\sigma,s),\mu)$, we have a contradiction, so assume that $R(B_1,(\sigma,s),\mu^*) =$

$R(B_1, (\sigma, s), \mu)$. For this to happen, the following three conditions must hold:

(i) $R(\{w_1, f_1\}, (\sigma, s), \mu) = R(\{w_{\ell'}, f_{\ell'}\}, (\sigma, s), \mu)$.

(ii) $\sum_{\ell=\ell'}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) = \sum_{\ell=\ell'}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$.

(iii) $\sum_{\ell=\bar{\ell}}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) = \sum_{\ell=\bar{\ell}}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$.

Recall that $\{w_{\ell'}, f_{\ell'}\} = \{\tilde{w}_{\bar{\ell}}, \tilde{f}_{\bar{\ell}}\}$. Rename w_ℓ, f_ℓ, $\tilde{w}_\ell$, $\tilde{f}_\ell$, L, and $\tilde{L}$ as $w_{\ell-\ell'+1}$, $f_{\ell-\ell'+1}$, $\tilde{w}_{\ell-\bar{\ell}+1}$, $\tilde{f}_{\ell-\bar{\ell}+1}$, $L-\ell'+1$, and $\tilde{L}-\bar{\ell}+1$, respectively. Then, we again have exactly the same problem as before: $\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ and $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) = \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$ as in Figure 4c.

If (Case 1) applies, then we have a contradiction. If (Case 2) applies, then we again find $\{w_{\ell'}, f_{\ell'}\} = \{\tilde{w}_\ell, \tilde{f}_\ell\}$, and we can again find a cycle set B_2. If the cycle achieves a strict improvement, we reach a contradiction. So, assuming equalities, firms and workers that remain after taking B_2 out still satisfy the above three conditions. Applying this procedure repeatedly, eventually, (Case 1) applies (by a finite number of players). Hence, we conclude that $\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ or $\sum_{\ell=1}^{\tilde{L}} R(\{\tilde{w}_\ell, \tilde{f}_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{\tilde{L}-1} R(\{\tilde{w}_{\ell+1}, \tilde{f}_\ell\}, (\sigma, s), \mu'')$ holds.■

The last part of the proof of Proposition 4. Now we will complete the proof of Proposition 4. Suppose, without loss of generality, that $\sum_{\ell=1}^{L} R(\{w_\ell, f_\ell\}, (\sigma, s), \mu) > \sum_{\ell=1}^{L-1} R(\{w_\ell, f_{\ell+1}\}, (\sigma, s), \mu')$ holds. There are two possibilities: (1) $S_{f_1} = \cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}$, or (2) $S_{f_1} \supsetneqq \cup_{\ell=1}^{L-1} \{w_\ell, f_{\ell+1}\}$. In the first case, $R(S_{f_1}, (\sigma, s), \mu') < R(N, (\sigma, s), \mu)$. This contradicts the Nash equation. In the second case, the new matching created from μ and μ' is broken in the middle. There are two subcases: (i) $\mu(f_L) = \varnothing$, and (ii) $\mu'(w_L) = \varnothing$. In either subcase, $R(N \backslash \sum_{\ell=1}^{L} \{w_\ell, f_\ell\}, (\sigma, s), \mu) = R(N \backslash \sum_{\ell=1}^{L} \{w_\ell, f_\ell\}, (\sigma, s), \mu')$.[17] This again implies $R(S_{f_1}, (\sigma, s), \mu') < R(N, (\sigma, s), \mu)$. Hence, assumption $(*)$ cannot be true. Thus, no pair can generate a positive profit.□

The proof of Proposition 4 utilizes only Corollary 4 and the matchmaker's profit-maximizing behavior given the system of profit on each pair of firms

[17] This is a slight abuse of notation: in subcase (i), w_L does not exist, since f_L is single.

and workers (generated from σ and s). As the Nash equations apply to each position instead of each firm, we can extend our Proposition 3 to the simple matchmaker game in the many-to-one assignment problem. Let us separate firm f into q_f **positions** $f' = \left\{f'_1, \ldots, f'_{q_f}\right\}$ where each position offers the same wages. Denote $F' \equiv \bigcup_{f \in F} \left\{f'_1, \ldots, f'_{q_f}\right\}$ as the set of positions (decomposed firms). Then, we can generate a one-to-one matching of positions and workers. Let μ-**decomposed matching** $\bar{\mu} : W \cup F' \to W \cup F'$ be a bijection such that (i) $\bar{\mu}(f'_i) = w$ if there exists $f' \ni f'_i$ such that $w \in \mu(f)$; (ii) $\bar{\mu}(w) = f'_i$ if $\mu(w) = f$; (iii) $\bar{\mu}(f) \in F'$ implies $\bar{\mu}(f'_i) = f'_i$ for all $f'_i \in f'$ and $\bar{\mu}(w) \in W$ implies $\bar{\mu}(w) = \varnothing$. Since Proposition 3 implies that Corollary 4 applies to μ-decomposed matching in the artificial one-to-one assignment problem, Proposition 4 directly implies that the zero-profit result for the simple matchmaker game will hold in the many-to-one assignment problem. This completes the proof of Theorem 1.

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