The "Boston" School-Choice Mechanism

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Abstract

The Boston mechanism is a popular student-placement mechanism in school-choice programs around the world. We provide two characterizations of the Boston mechanism. We introduce a new axiom, respect of preference rankings. A mechanism is the Boston mechanism for some priority if and only if it respects preference rankings and satisfies consistency, resource monotonicity, and an auxiliary invariance property. In environments where each type of object has exactly one unit, as in house allocation, a characterization is given by respect of preference rankings, individual rationality, population monotonicity, and the auxiliary invariance property.

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1 Introduction

School choice is a practice in which school children and their parents can express their preferences over schools, and the school system tries to accommodate their desires. School choice has become very popular in public school systems in the United States and around the world during the past few decades, and it has also become a hotly debated topic in public policy.

The so-called "Boston" mechanism is a very popular student-placement procedure.[1] Under this mechanism, students submit their preference lists to the central clearinghouse. Given the reported preferences, the clearinghouse follows an algorithm that tries to match as many students to their stated preferred schools as possible, subject to prespecified priorities of students at each school: seats of each school are allocated to students who rank that school first, then to those who rank it second *if there is any remaining seat*, and so forth.

Abdulkadiroğlu and Sönmez (2003) find various shortcomings of the Boston mechanism: the mechanism is not stable, that is, it can cause an unfair outcome where a student is not admitted to a school she likes while a student with a lower priority than her is admitted to that school; the Boston mechanism is not strategy-proof and, even worse, is easy to manipulate. Abdulkadiroğlu and Sönmez (2003) point out that the (student-proposing) deferred acceptance mechanism (Gale and Shapley, 1962) solves both problems because it is both stable and strategy-proof. Subsequent studies find additional pitfalls of the Boston mechanism; its (complete information) Nash equilibrium outcomes are Pareto dominated by the outcome of the deferred acceptance mechanism (Ergin and Sönmez, 2006); a higher fraction of individuals misreport their preferences to manipulate the Boston mechanism than the deferred acceptance mechanism in an experimental environment (Chen and Sönmez, 2006);[2] if some students sincerely report their true preferences while others are sophisticated in the sense of taking best responses, then sophisticated students can be better off under the Boston mechanism than in the deferred acceptance mechanism at the expense of the sincere students (Pathak and Sönmez, 2008). Consistently with recommendations by these studies, the Boston mechanism was abandoned in Boston and Chicago, and the use of Boston-like mechanisms was recently prohibited by law in the U.K. (Pathak and Sönmez, 2011).

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[1]Roth (1991) observes that the two-sided version of this mechanism is used in the field. Alcalde (1996) studies implementation under that mechanism. Abdulkadiroğlu and Sönmez (2003) find that the one-sided version of the mechanism, which is the focus of our study, is used widely in school choice and named it the "Boston mechanism."

[2]Related experiments are conducted by Pais and Pinter (2007) and Calsamiglia, Haeringer, and Klijn (2009).

Despite these alleged shortcomings, the Boston mechanism continues to be a very popular student-placement mechanism. School districts in the U.S. mentioned by Abdulkadiroğlu and Sönmez (2003), such as Minneapolis and Lee County of Florida, are just a few examples. The mechanism is used in many school-choice systems around the globe. Even outside the school-choice context, mechanisms similar to the Boston mechanism have been tried in various times and contexts although they have sometimes failed: Examples include university housing assignment (Hylland and Zeckhauser, 1979) and labor market clearinghouses for doctors (Roth, 1991). Another interesting case is the school-choice program in Seattle. The district recently started using a mechanism inspired by deferred acceptance instead of the Boston mechanism, but the system will transition back to a version of the Boston mechanism starting with the 2011 enrollment season.[3]

Even in Boston, where the deferred acceptance mechanism is adopted and still being used instead of the Boston mechanism, some parents raised concerns about the change as follows:

> I'm troubled that you're considering a system that takes away the little power that parents have to prioritize... what you call this strategizing as if strategizing is a dirty word... (Recording from Public Hearing by the School Committee, 05/11/2004).

Furthermore, there is a recent surge in research that finds advantages of the Boston mechanism over the deferred acceptance mechanism. For instance, Abdulkadiroğlu, Che, and Yasuda (2009) consider an environment in which all students have the same priority at all schools and have identical ordinal preferences, but have incomplete information regarding the cardinal utilities of other students over schools. They show that, in this environment, students are better off at any symmetric Bayesian Nash equilibrium under the Boston mechanism than under the deferred acceptance mechanism, which selects any matching with uniform distribution.[4]

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[3]Seattle Public Schools, New Student Assignment Plan Transition Plan for 2010-11, January 2010 draft, retrieved from http://www.seattleschools.org/area/board/09-10agendas/010610agenda/nsaptransitionattachment.pdf on 01/27/2010.

[4]Based on their findings, Abdulkadiroğlu, Che, and Yasuda (2008) propose a mechanism that could be regarded as a hybrid of the deferred acceptance and the Boston mechanism. In a similar model, Miralles (2008) shows that the Boston mechanism outperforms the deferred acceptance mechanism with respect to certain ex-ante efficiency criteria. Featherstone and Niederle (2008) offer a setting where truth-telling is an equilibrium under the Boston mechanism and efficiency under the mechanism is higher than under the deferred acceptance mechanism. They also conduct laboratory experiments whose outcomes confirm their predictions.

Almost all research and popular opinions on the Boston mechanism focus on an important welfare property: Under the mechanism, each school admits all qualified students who rank it higher before admitting any student who ranks it lower. This property is intuitive as a welfare criterion, and also enables students to express the strength of their preferences for a particular school by ranking that school higher. However, the same property causes many of the Boston mechanism's shortcomings, since it promotes preference manipulations in a very obvious manner: A student may put a school higher than it actually is in the hope of being admitted to that school.

In this paper we refrain from endorsing a specific mechanism. Rather, we aim to provide a basic understanding of the good and bad properties of the Boston mechanism using axiomatic tools. First, we formalize the aforementioned welfare property of the Boston mechanism: We say that a mechanism respects preference rankings if whenever a qualified student prefers a school to the school assigned by the mechanism, all the seats of the former are allocated to students who rank it at least as high as the initial student. The Boston mechanism satisfies this property. We further show that all mechanisms respecting preference rankings satisfy a certain efficiency property (Proposition 2). The Boston mechanism shares this welfare property because it respects preference rankings.

We show that this new property and standard fairness axioms as well as an auxiliary invariance property we introduce fully characterize the class of the Boston mechanisms (since each priority profile induces a Boston mechanism, we refer to the class of such mechanisms as the Boston mechanisms in the plural form). We provide two independent characterizations of the Boston mechanisms using different axioms.

Our first result, Theorem 1, states that a mechanism respects preference rankings, consistency, resource monotonicity and our auxiliary invariance property if and only if it is in the class of the Boston mechanisms. A mechanism is consistent if, whenever we fix the school assignment of a student at the mechanism's outcome and rerun the mechanism for the remaining students and school seats, all remaining students are assigned the same schools. A mechanism is resource monotonic if increasing the capacity of a school makes all students weakly better off. Consistency and resource monotonicity are standard axioms and have clear interpretations as robustness properties in the school-choice context. If the mechanism is consistent, then no student has incentives to strategically file an appeal after allocations are finalized for some students because the assignment will be unchanged. Increasing seats in desirable schools, which is an important policy goal for many school districts, is facilitated if the mechanism satisfies resource monotonicity, because this axiom implies that an increase in seats

of schools is favored by every student.

Our second result, Theorem 2, offers an alternative characterization in environments where there is a single copy of each object as in housing and office allocation. We also show that our properties are independent in each of our characterizations, confirming that each axiom is indispensable.

Both of our results use respect of preference rankings, while employing different combinations of standard axioms for the two characterizations. This suggests that the respect of preference rankings, introduced in this paper, is a crucial property of the Boston mechanisms. In fact, one may suspect that the axiom is close to a direct statement of the mechanism itself. However, our analysis demonstrates that such a belief is not correct. In each of our characterizations, the mechanism coincides with the Boston mechanism only if other axioms are satisfied as well. Our results provide the entire set of properties essential to the Boston mechanism for the first time.

In this paper, the priority structure is not primitive to the model, but is found "endogenously" together with a specific mechanism. While many papers on the Boston mechanisms assume that priorities are exogenously given, the social planner can often design priorities to some extent. In fact, a priority structure is a meaningful concept only if one also specifies how to use it. Our approach is logically consistent in this respect, as our characterizations find both the priority structure and its use (that is, using priorities to define a specific rule from the class of the Boston mechanisms) simultaneously. Another advantage of this modeling decision is that, when no additional information about the priority structure is available, our axioms represent the strongest statements one can make about the Boston mechanisms. These axioms may prove useful in studying mechanisms in more specific contexts in the future. These considerations lead us to exclude the priority structure from the primitives of the environment. Our approach follows extant studies such as Papai (2000), Ehlers and Klaus (2006), Kojima and Manea (2010a), and Pycia and Ünver (2009).

2   Related Literature

While this paper is the first to characterize the Boston mechanisms, other allocation mechanisms have been previously characterized. The closest study to ours is Kojima and Manea (2010a), who axiomatize the class of the deferred acceptance mechanisms with substitutable (and acceptant) priorities. Their paper and the current paper complement each other, as these two studies together provide characterizations of the main competing mechanisms in school choice. The analysis by Kojima and

Manea (2010a) is followed by Ehlers and Klaus (2009), who axiomatize the deferred acceptance mechanisms with responsive priorities.[5] Papai (2000) and Pycia and Ünver (2009) characterize a broad class of mechanisms. Their mechanisms include the priority-based top trading cycle mechanisms of Abdulkadiroğlu and Sönmez (2003), which attracts interest as another class of desirable student-placement mechanisms. Moulin (1991, 2004) and Thomson (2008) give comprehensive surveys on allocation mechanisms and axiomatic studies on the subject.

More broadly, this study is part of the rapidly growing literature on school-choice mechanisms.[6] Practical considerations in designing school-choice mechanisms in Boston and New York City are discussed by Abdulkadiroğlu, Pathak, and Roth (2005, 2009) and Abdulkadiroğlu, Pathak, Roth, and Sönmez (2005, 2006). Erdil and Ergin (2008) and Kesten (2009a) propose alternative mechanisms that may produce more efficient student placements than those used in New York City and Boston. When there are no priorities, as in the supplementary round of New York City's student-placement process, the probabilistic serial mechanism (Bogomolnaia and Moulin, 2001) is more efficient than the current random priority mechanism, but the former mechanism is not strategy-proof.[7] Kojima and Manea (2010b) show that the probabilistic serial mechanism is incentive compatible when the number of seats in each school is sufficiently large. Che and Kojima (2010) subsequently show that these two mechanisms are asymptotically equivalent as the market becomes infinitely large.[8] Ergin (2002) shows that the deferred acceptance mechanism is Pareto efficient if and only if the priority structure is acyclic. The acyclic priority structure has proved crucial for the deferred acceptance mechanisms to satisfy a number of desirable properties (Haeringer and Klijn, 2009; Kesten, 2006a,b; Ehlers and Erdil, 2009).

Finally, this paper is part of an extensive field of matching theory initiated by Gale and Shapley (1962). The field is too large to summarize in this paper: Roth and Sotomayor (1990) provide a comprehensive survey of the early literature, while more recent advances are discussed by Roth (2008) and Sönmez and Ünver (2011).

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[5]Ehlers and Klaus (2003, 2006) characterize the class of deferred acceptance mechanisms with acyclic priority structures (Ergin, 2002).

[6]See also an earlier contribution by Balinski and Sönmez (1999).

[7]Kesten (2009b) inspects the reasons why the random priority mechanism lacks efficiency and shows that a modification to the random priority method would make it equivalent to the probabilistic serial mechanism.

[8]Manea (2009) provides sufficient conditions for asymptotic efficiency and inefficiency of the random priority mechanism.

3   The Model

Let I and C be nonempty and finite sets of **students** and **schools.** Each student shall be assigned to a school or remain unmatched. Each school $c \in C$ has a maximum capacity of students to admit, referred to as its **quota** and denoted by q_c. Let $q = (q_c)_{c \in C}$ be the quota vector associated with the schools. We refer to being unmatched as being matched to a **null school** $\emptyset$. The null school is interpreted as an outside option, such as a school in a different district or a private school. We set the quota of the null school as $q_\emptyset = \infty$. Each student i has a **strict preference relation** (denoted by P_i) over $C \cup \{\emptyset\}$. Denote the set of strict preference relations by $\mathcal{P}$. Let $P = (P_i)_{i \in I} \in \mathcal{P}^{|I|}$ be a **preference profile**. Denote by R_i the weak preference relation associated with P_i, i.e., cR_id if and only if $c = d$ or cP_id. Let $P_i(c)$ be the ranking of school c at P_i, i.e., if school c is the ℓth choice of student i under P_i, then $P_i(c) = \ell$. Thus, for all $c, d \in C \cup \{\emptyset\}$, $P_i(c) < P_i(d)$ if and only if cP_id. A (**school-choice**) **problem** is specified by I, C, P, and q.

A **matching** is a function $\mu : I \to C \cup \{\emptyset\}$ where μ_i is the school assigned to student i for all $i \in I$. For each $c \in C \cup \{\emptyset\}$, we write $\mu_c = \{i \in I : \mu_i = c\}$ for the set of students assigned to school c. We require that any matching μ satisfy $|\mu_c| \le q_c$ for all $c \in C \cup \{\emptyset\}$, i.e., no school is assigned to more students than its quota.

A (**school-choice**) **mechanism** is a systematic procedure that assigns a matching for each problem. Throughout the paper we fix I and C and denote a problem simply by its preference profile and quota vector $[P; q]$. For any problem $[P; q]$, let $\mathcal{M}[q]$ be the set of matchings and let $\varphi[P; q] \in \mathcal{M}[q]$ be the matching generated by mechanism φ at $[P; q]$.

4   The Boston Mechanisms

The most commonly used school-choice mechanisms are the so-called Boston mechanisms (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005, 2006). This class of mechanisms was used by the Boston public schools until 2005 and is currently in use in Lee County of Florida and in Minneapolis, among other districts.

To define the mechanisms, we introduce some additional terminology. For each school $c \in C$, a **priority order** $\succ_c$ is a linear order over the set of students and a vacant position denoted by $\emptyset$. We interpret that if $i \succ_c j$ then student i has a higher priority at school c than student j, and

if $i \succ_c \emptyset$ then student i is **acceptable** at school c. Let Π be the set of priority orderings. Let $\succ = (\succ_c)_{c \in C} \in \Pi^{|C|}$ be a priority order profile. Formally, the **Boston (school-choice) mechanism** at a priority order profile $\succ$ is defined through the following iterative algorithm for each problem $[P; q]$:

Algorithm 1 The Boston mechanism:

Step 1: Only the top choices of the students at P are considered. For each school c, consider the students who have listed it as their top choice and assign seats of the school c to these students one at a time following their priority order at $\succ_c$ until either there are no seats left at c (i.e., q_c students have been assigned) or there is no student left who has listed it as her top choice and is acceptable to c.

$\vdots$

Step ℓ: Consider the remaining students. Only the ℓth choices of these students at P are considered. For each school c still with available seats, consider the students who have listed it as their ℓ^{th} choice and assign the remaining seats of c to these students one at a time following their priority order at $\succ_c$ until either there are no seats left at c (i.e., all q_c seats have been assigned in the current and previous steps) or there is no student left who has listed it as her ℓ^{th} choice and is acceptable to c.

Let $\psi^\succ [P; q]$ denote the resulting matching of the Boston mechanism induced by priority profile $\succ$ in problem $[P; q]$.

Example 1 How does a Boston mechanism assign students to schools? Consider a problem with $I = \{1, 2, ..., 5\}$ and $C = \{a, b, c, d, e\}$ where the quota of each school is one. Let P be given as

P_1	P_2	P_3	P_4	P_5
d	b	d	a	b
a	d	$\vdots$	$\vdots$	$\vdots$
e	e			
$\vdots$	c			
	$\emptyset$			
	a			

Consider the Boston mechanism induced by the following priority profile $\succ$:

$\succ_a=\succ_b$	$\succ_c$	$\succ_d$	$\succ_e$
1	5	2	1
5	4	3	2
2	$\emptyset$	1	$\vdots$
4	$\vdots$	$\vdots$	
$\vdots$			

Matching $\psi^\succ [P; q]$ is found as follows.

Step 1:

- 1 and 3 apply to d. Since $3 \succ_d 1 \succ_d \emptyset$ and $q_d = 1$, only 3 is admitted to d.

- 2 and 5 apply to b. Since $5 \succ_b 2 \succ_b \emptyset$ and $q_b = 1$, only 5 is admitted to b.

- 4 applies to a. Since $4 \succ_a \emptyset$, she is admitted.

Step 2:

- 1 applies to a. Since the quota of a is already filled, she is not admitted.

- 2 applies to d. Since the quota of d is already filled, she is not admitted.

Step 3:

- 1 and 2 apply to e. Since $1 \succ_e 2 \succ_e \emptyset$ and $q_e = 1$, only 1 is admitted to e.

Step 4:

- 2 applies to c. Since $\emptyset \succ_c 2$, she is not admitted although c has an empty seat.

Step 5:

- 2 applies to $\emptyset$ and is admitted.

The algorithm terminates with the resulting matching

$$\psi^{\succ}[P;q] = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ e & \emptyset & d & a & b \end{pmatrix}.$$

An important observation about the mechanics of the algorithm is that a student may have justified envy in the resulting matching–a school that a student prefers to her assigned school may be attended by students with lower priority for it. For example, although student 1 prefers a to $\psi_1^{\succ}[P;q] = e$ and has higher priority at school a than student $\psi_a^{\succ}[P;q] = 4$, 4 is admitted while 1 is not. This is because a Boston mechanism matches seats at each school to students who rank it high. In the current problem, for instance, 4 is matched to a while 1 is not since 4 ranks a higher than 1 does. Similarly, student 2 is not assigned to a more preferred school d than her assignment $\emptyset$ while student 3 is assigned to d, even though she has lower priority at d, because 3 ranks d higher than 2 does.

Remark 1 *It is easy to see that no two distinct priority profiles induce the same Boston mechanism (except for irrelevant parts of orders among unacceptable students). Thus the characterization results we offer in the sequel lead to unique representations.*

The Boston mechanisms have an intuitive welfare maximization property, subject to some feasibility constraints. More specifically, one interpretation of the mechanism is that it assigns artificial utility indices to the alternatives consistent with each individual's preferences and finds a matching that maximizes total welfare of the economy.[9]

In what follows, we present a particular welfare maximization problem using linear programming whose solution is equivalent to the outcome of the Boston mechanism for a given priority profile. For each problem $[P;q]$, the solution of the following linear assignment program, which maximizes the sum of the induced utilities, is equivalent to the outcome of the Boston mechanism induced by a priority profile $\succ$: First, define for each $c \in C$, $I_c = \{i \in I : i \succ_c \emptyset\}$ as the set of acceptable students for school c.

Proposition 1 *For each problem $[P;q]$, the solution of the following linear program is equal to the*

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[9]Note that no information about the utility functions of individuals are provided in the description of the problem: Rather, only ordinal preferences are primitive to the problem.

outcome of the Boston mechanism:

$$\max_{[z_{i,c}]} \sum_{i \in I, c \in C \cup \{\emptyset\}} z_{i,c} u_i(c)$$

subject to

$$i \notin I_c \Rightarrow z_{i,c} = 0 \qquad \forall i \in I, c \in C.$$

$$0 \leq z_{i,c} \qquad \forall i \in I, \ c \in C \cup \{\emptyset\}$$

$$\sum_{c \in C \cup \{\emptyset\}} z_{i,c} = 1 \qquad \forall i \in I$$

$$\sum_{i \in I} z_{i,c} \leq q_c \qquad \forall c \in C \cup \{\emptyset\},$$

where $q_\emptyset = |I|$ and $(u_i)_{i \in I} \subseteq (\mathbb{R}_+^{|C| \cup \{\emptyset\}})^{|I|}$ are utility functions consistent with the given preferences P (i.e., for any $i \in I$ and $c, d \in C \cup \{\emptyset\}$, we have $u_i(c) > u_i(d)$ if and only if $c\, P_i\, d$) satisfying

$$2u_i(d) < u_j(c) \qquad \forall i, j \in I \text{ and } c, d \in C \cup \{\emptyset\} \text{ with } P_j(c) < P_i(d)$$

$$2u_i(c) < u_j(c) \qquad \forall i, j \in I \text{ and } c \in C \cup \{\emptyset\} \text{ with } P_i(c) = P_j(c) \text{ and } \succ_c(j) \ < \ \succ_c(i)$$

In the solution $[z_{i,c}]_{i \in I, c \in C \cup \{\emptyset\}}$, which is unique and where $z_{i,c} \in \{0, 1\}$ for all $i \in I$ and $c \in C \cup \{\emptyset\}$, school i is matched with school c if and only if $z_{i,c} = 1$.

The popularity of the Boston mechanisms may be due to this seemingly appealing welfare maximization property. In this regard, note that other welfare-maximizing mechanisms such as the Walrasian mechanism are widely applauded as desirable solutions for many economic problems. However, this promising property of the mechanism comes at a cost. Like many other welfare-based solutions, the Boston mechanisms are vulnerable to manipulation of preference revelations by students. For instance, in Example 1, student 1 can successfully manipulate the mechanism by submitting

$$\frac{P_1'}{a}$$
$$e$$
$$\vdots$$

instead of P_1 and be matched with a instead of e.

5 Axioms

In this section, we introduce several axioms regarding school-choice mechanisms. First, we introduce a new axiom that plays central roles in the remainder of the paper. To do this, for any $c \in C \cup \{\emptyset\}$, let $I_c = \{i \in I : \varphi_i[P; q] = c$ for some $P, q\}$ be the set of students who are matched to c for at least one preference-quota profile under φ. The set I_c is the set of "qualified" students at c in the sense that there is an instance in which they are matched to c.

Definition 1 Given mechanism φ, and hence I_c for each $c \in C \cup \{\emptyset\}$ induced by φ, matching μ **respects preference rankings** for φ at $[P; q]$ if, for all $c \in C \cup \{\emptyset\}$ and $i \in I_c$, $c P_i \mu_i$ implies $|\mu_c| = q_c$ and $P_j(c) \le P_i(c)$ for all $j \in \mu_c$.

That is, a matching respects preference rankings if whenever a student $i \in I_c$ is assigned a worse school than a school c, then c's quota is filled with students who rank c at least as high as student i. A mechanism φ **respects preference rankings** if for every problem $[P; q]$, the matching $\varphi[P; q]$ respects preference rankings for φ at $[P; q]$. Respecting preference rankings is a representative feature of the Boston mechanisms, as illustrated in Example 1. Naturally, properties similar to respect of preference rankings have been extensively studied in the literature on the Boston mechanisms.[10] However, the property has not been formalized in the form presented here.

One motivation of respecting preference rankings is based on welfare considerations. Intuitively, the social planner respecting preference rankings tries to assign school seats to students who value them as highly as possible. Of course, Pareto efficiency is not necessarily guaranteed even if respect of preference rankings holds: This is because we allow some student i to be simply unacceptable to some school c (i.e., $i \notin I_c$). However, there is a sense in which respect of preference rankings implies a milder concept of efficiency. Given mechanism φ, and hence I_c for each $c \in C \cup \{\emptyset\}$ induced by φ, a matching $\mu \in \mathcal{M}[q]$ is **constrained Pareto efficient** for φ at a problem $[P; q]$ if there exists no matching $\nu \in \mathcal{M}[q]$ such that $\nu_i R_i \mu_i$ for all $i \in I$ and $\nu_i P_i \mu_i$ for some $i \in I$ as well as $i \in I_{\nu_i}$ for every $i \in I$. In words, a matching is constrained Pareto efficient if there is no matching that makes every student weakly better off and at least one student strictly better off while no unqualified student is admitted to a school. A mechanism is **constrained Pareto efficient** if $\varphi[P; q]$ is constrained Pareto

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efficient for φ at every problem $[P; q]$. The following proposition establishes a relationship between respecting preference rankings and constrained Pareto efficiency.

Proposition 2 *If matching μ respects preference rankings, then it is constrained Pareto efficient.*

In some school districts, the law may require that all students are qualified for all schools. Under this presumption, Proposition 2 asserts that respect of preference rankings implies Pareto efficiency.

Next we introduce axioms that are standard in the literature. A mechanism φ is **resource monotonic** if for all $P \in \mathcal{P}^{|I|}$, if $q_c \geq q'_c$ for all $c \in C$, then $\varphi_i[P; q] R_i \varphi_i[P; q']$ for all $i \in I$ (Thomson, 1978). In words, a mechanism is resource monotonic if whenever the supply of school seats increases, the mechanism's outcome makes each student weakly better off than its original outcome.

A mechanism φ is **individually rational** if $\varphi_i[P; q]R_i\emptyset$ for all P, q and $i \in I$. Note that any resource-monotonic mechanism φ satisfies

$$\varphi_i[P; q] \, R_i \, \varphi_i[P; (0, \ldots, 0)] = \emptyset,$$

for any P, q, and $i \in I$, so any mechanism that is resource monotonic is individually rational in our context. Also, since $I_\emptyset = I$ (because $\varphi_i[P; (0, \ldots, 0)] = \emptyset$ for all $i \in I$) and $q_\emptyset = \infty$, any mechanism that respects preference rankings is individually rational.[11]

Next, we define two properties of mechanisms regarding variable populations. We introduce one additional notation. For any $i \in I$, let $\mathcal{P}^\emptyset$ be the set of preference relations that rank $\emptyset$ as the first choice. A mechanism φ is **population monotonic** if, for all $P \in \mathcal{P}^{|I|}$ and all $P_i^\emptyset \in \mathcal{P}^\emptyset$, we have $\varphi_j \left[P_i^\emptyset, P_{-i}; q \right] R_j \varphi_j[P; q]$ for all $j \neq i$ and $\varphi_i[P_i^\emptyset, P_{-i}; q] = \emptyset$ (Thomson, 1983a,b).[12] That is, a mechanism is population monotonic if whenever a student is removed from the problem, the mechanism's outcome makes every remaining student weakly better off than its original outcome.

Population monotonicity is usually defined for variable populations. To avoid further notational complexity, we use a fixed-population representation of population monotonicity. We interpret a change in preferences of student i from P_i to $P_i^\emptyset$ as a situation where student i leaves the school-choice problem.

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[11]As explained in Section 6.2, respect of preference rankings does not imply individual rationality in a setting in which the quota vector is fixed.

[12]This definition implies that for all $P_i^\emptyset, \tilde{P}_i^\emptyset \in \mathcal{P}^\emptyset, \varphi \left[P_i^\emptyset, P_{-i}; q \right] = \varphi \left[\tilde{P}_i^\emptyset, P_{-i}; q \right]$.

A mechanism φ is **consistent** if $\varphi_j\left[P_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \varphi_j[P;q]$ for all $j \neq i$ and $\varphi_i\left[P_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \emptyset$ for all $P \in \mathcal{P}^{|I|}$ and all $P_i^\emptyset \in \mathcal{P}^\emptyset$ (Thomson, 1988).[13],[14] In words, a mechanism is consistent if whenever a student is removed from the problem with her assigned school seat, the assignment for each remaining student is unchanged.

Consistency is usually defined for variable populations, like population monotonicity. We use a fixed-population representation of consistency, as we do for population monotonicity. We express a situation where a student leaves the problem with her assigned school by decreasing the quota of this school by one and having this student rank the null school as her top choice.

The following Lemma offers a relationship between resource monotonicity, consistency, and population monotonicity.

Lemma 1 *If a mechanism φ satisfies resource monotonicity and consistency, then it satisfies population monotonicity.*[15]

Finally, we need a new auxiliary axiom. Before introducing the axiom, we review a standard axiom in the implementation literature. We say that P_i' is a **monotonic transformation** of P_i at $c \in C \cup \{\emptyset\}$ (P_i' *m.t.* P_i at c) if every school that is ranked above c under P_i' is ranked above c under P_i, i.e.,

$$b \, P_i' \, c \;\; \Rightarrow \; b \, P_i \, c, \; \forall b \in C \cup \{\emptyset\}.$$

P' is a **monotonic transformation** of P at a matching μ (P' *m.t.* P at μ) if P_i' *m.t.* P_i at μ_i for all $i \in I$. A mechanism φ satisfies **Maskin monotonicity** (Maskin, 1999) if, for any pair of preference profiles P and P',

$$P' \; m.t. \; P \; at \; \varphi[P;q] \Rightarrow \varphi[P', q] \;\; = \;\; \varphi[P;q].$$

Maskin monotonicity is regarded as a reasonable property and is equivalent to group strategy-proofness in the current setting (Takamiya, 2001). The property is indeed satisfied by a number of promising mechanisms. Examples include the priority-based top-trading cycles mechanisms (Abdulkadiroğlu and Sönmez, 2003) and their generalizations, such as the hierarchical exchange mechanisms (Papai, 2000) and the trading cycles with brokers and owners (Pycia and Ünver, 2009).

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[13]It follows that, for all $P_i^\emptyset, \tilde{P}_i^\emptyset \in \mathcal{P}^\emptyset, \varphi\left[P_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right] = \varphi\left[\tilde{P}_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]}\right].$

[14]We stipulate $\infty - 1 = \infty$, so consistency implies that when an unmatched student is removed from the problem, assignments for all remaining students are unchanged.

[15]Although we suspect that this lemma is well known, we were unable to find a reference.

However, the Boston mechanisms may violate Maskin monotonicity. To see this point consider, for instance, a change in student 1's reported preferences in Example 1. Although $P' = (P_1', P_{-1})$ satisfies P' $m.t.$ P at $\psi^\succ[P; q]$, we have $\psi_1^\succ[P'; q] = a \neq e = \psi_1^\succ[P; q]$.

In general, Maskin monotonicity is almost always in conflict with respect of preference rankings. To see this point, assume that the mechanism φ is **non-wasteful**, that is, there exist no student i, school c, preference profile P, and quota q such that $\varphi_c[P; q] < q_c$ and $cP_i\varphi_i[P; q]$.[16] Under this presumption, the following incompatibility holds.

Proposition 3 *Let $|I| \geq 3$ and $|C| \geq 2$. There exists no mechanism that respects preference rankings and satisfies Maskin monotonicity.*

The proof in the Appendix shows that, under non-wastefulness, there exists no mechanism which respects preference rankings and satisfies strategy-proofness. In that sense, respect of preference rankings is in a direct conflict with a desirable concept of incentive compatibility.

These observations motivate a certain weakening of Maskin monotonicity. More specifically, we seek a condition that obtains the same conclusion as Maskin monotonicity does, but for which the hypothesis is restricted in such a way that the condition is not in conflict with respecting preference rankings. To proceed, define

$$U_i(P, \mu) = \{j \in I \; : \; P_j(\mu_i) \leq P_i(\mu_i), \; P_j(\mu_i) \leq P_j(\mu_j)\},$$
$$V_i(P, \mu) = \{j \in I \; : \; P_j(\mu_i) < P_i(\mu_i), \; P_j(\mu_i) \leq P_j(\mu_j)\}.$$

In words, the set $U_i(P, \mu)$ (resp. $V_i(P, \mu)$) is the group of students who, at preference profile P, rank μ_i weakly (resp. strictly) higher than i does and weakly higher than their assignments at μ. Intuitively, they are the sets of students who are potentially in competition with i for a seat in school μ_i. We say that P' is a **rank-respecting monotonic transformation** of P at a matching μ (P' $r.r.m.t.$ P at μ) if P' $m.t.$ P at μ and, for all i with $\mu_i \in C$, $U_i(P', \mu) \subseteq U_i(P, \mu)$ and $V_i(P', \mu) \subseteq V_i(P, \mu)$. A mechanism φ satisfies **rank-respecting (r.r.) invariance** if, for any pair of preference profiles P and P',

$$P' \; r.r.m.t. \; P \; at \; \varphi[P; q] \Rightarrow \varphi[P'; q] \; = \; \varphi[P; q].$$

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[16]Non-wastefulness is imposed in order to exclude trivial cases in which Maskin monotonicity and respect of preference rankings are consistent. To see this point, note that the null mechanism, that is, a mechanism such that $\varphi_i[P; q] = \emptyset$ for any student i, preference profile P, and quota q, trivially satisfies both properties.

15

In words, r.r. invariance requires that a matching is unchanged when students promote the rankings of their original assignments, as long as doing so does not increase the competition for schools assigned to others. Thus r.r. invariance does not conflict with respecting preference rankings and is satisfied by the Boston mechanisms. For instance, preference $P' = (P'_1, P_{-1})$ in Example 1 is not a rank-respecting monotonic transformation of P at $\psi^\succ[P; q]$ because $\{1\} = U_4(P', \mu) \not\subseteq U_4(P, \mu) = \emptyset$, so this example is consistent with r.r. invariance.

6 Characterizing the Boston Mechanisms

This section presents the main results of this paper. It turns out that respect of preference rankings is a crucial property of the Boston mechanisms in the sense that this axiom, together with more standard ones, characterizes the mechanisms. In fact, we provide alternative characterizations. Respect of preference rankings is one of the axioms for each of the characterizations while different sets of other axioms appear in alternative characterizations, which suggests that this property is the main content of the Boston mechanisms.

6.1 Main Result: Characterization

Theorem 1 *A mechanism φ is the Boston mechanism induced by some priority profile $\succ$, i.e., $\varphi = \psi^\succ$, if and only if φ respects preference rankings and satisfies consistency, resource monotonicity, and r.r. invariance.*

As pointed out in Remark 1, no two distinct priority profiles induce the same Boston mechanism. Therefore, the representation of the mechanism in Theorem 1 in terms of a Boston mechanism is unique.

The following examples show that the axioms in Theorem 1 are independent when $|I| \geq 3$ and $|C| \geq 2$ (in Appendix G we study all other cases and show that the axioms are not independent when $|I| \leq 2$ or $|C| = 1$).

Example 2 A mechanism violating only respect of preference rankings: Fix $i \in I$ and $c \in C$. Consider the mechanism such that

1. if $q_c \geq 2$ and $cP_i\emptyset$, then assign c to i and $\emptyset$ to every other student, and

2. otherwise, assign $\emptyset$ to every student.

This mechanism satisfies consistency, resource monotonicity, and r.r. invariance, but does not respect preference rankings. To show that the mechanism violates respect of preference rankings, first note that i is assigned c in Case (1) above by assumption, so $i \in I_c$. Next, suppose that $q_c = 1$ and $cP_i\emptyset$. Then $\varphi_i[P; q] = \emptyset$ by assumption of Case (2) and, moreover, no student different from i is matched to c even though $cP_i\emptyset$. Thus respect of preference rankings is violated.

Example 3 A mechanism violating only consistency: Suppose that $|I| \geq 3$: Let $I = \{1, \ldots, n\}$ with $n \geq 3$. Fix a school $c \in C$ and define priority orders $\succ_c$ and $\succ'_c$ by

$\succ_c$	$\succ'_c$
1	1
2	3
3	2
4	4
$\vdots$	$\vdots$
n	n
$\emptyset$	$\emptyset$

For all $d \neq c$, let

$\succ_d$
1
2
3
4
$\vdots$
n
$\emptyset$

Consider the mechanism φ defined by

$$\varphi[P; q] = \begin{cases} \psi^{\succ'_c, \succ_{-c}}[P; q] & \text{if } q_c = 1, \\ \psi^{\succ}[P; q] & \text{otherwise.} \end{cases}$$

17

It can be verified that φ respects preference rankings and satisfies resource monotonicity and r.r. invariance. To see that consistency is violated, consider a quota profile q with $q_c = 2$ and preference profile P where c is top-ranked by every student. Students 1 and 2 are matched to c at $\varphi[P;q]$, but student 3 is matched to c at $\varphi[P_1^\emptyset, P_{-1}; q_c - 1, q_{-c}]$. Therefore φ does not satisfy consistency.

Example 4 A mechanism violating only resource monotonicity: Suppose that $|C| \geq 2$ and $|I| \geq 2$: Let $I = \{1, \ldots, n\}$. For each $d \in C$, define priority orders $\succ_d, \succ'_d, \succ''_d \in \Pi$ by

$\succ_d$	$\succ'_d$	$\succ''_d$
1	2	$\emptyset$
2	1	$\vdots$
3	3	
$\vdots$	$\vdots$	
n	n	
$\emptyset$	$\emptyset$	

Fix some school $c \in C$ and define mechanism φ by, for any problem $[P;q]$,

$$\varphi[P;q] = \begin{cases} \psi^{\succ''_c, \succ_{-c}}[P;q] & \text{if } q_c \geq 2, \\ \psi^{\succ''_c, \succ'_{-c}}[P;q] & \text{otherwise.} \end{cases}$$

It can be shown that φ respects preference rankings and satisfies consistency and r.r. invariance. On the other hand, it violates resource monotonicity. To see this point consider preference and quota profiles P and q where there is a school $e \in C \setminus c$ (such a school exists since $|C| \geq 2$) that is top-ranked by every student at P and $q_c = 2, q_e = 1$. At this profile, student 2 is not matched to her most preferred school e. If q_c is reduced to 1, student 2 is matched to her most preferred school e, violating resource monotonicity.

Example 5 A mechanism violating only r.r. invariance: Suppose that $|C| \geq 2$ and $|I| \geq 2$:

18

Let $I = \{1, ..., n\}$ for some $n \geq 2$. Fix $c \in C$. Define priority orders $\succ_c, \succ'_c \in \Pi$ such that

$\succ_c$	$\succ'_c$
1	2
2	3
3	4
4	$\vdots$
$\vdots$	n
n	1
$\emptyset$	$\emptyset$

Fix some priority order profile $\succ_{-c} \in \Pi^{|C|-1}$ for other schools. Fix a school $d \neq c$ (such a school exists since $|C| \geq 2$). Define mechanism φ such that for any problem $[P; q]$,

$$\varphi[P; q] = \begin{cases} \psi^{\succ_c, \succ_{-c}}[P; q] & \text{if } P_1(d) < P_1(\emptyset), \\ \psi^{\succ'_c, \succ_{-c}}[P; q] & \text{otherwise.} \end{cases}$$

It can be shown that φ respects preference rankings and satisfies consistency and resource monotonicity. On the other hand, it violates r.r. invariance. To see this, consider $P_1, P'_1 \in \mathcal{P}$ satisfying

P_1	P'_1
c	c
d	$\emptyset$
$\emptyset$	d
$\vdots$	$\vdots$

Let $P_{-1} \in \mathcal{P}^{|I|-1}$ be such that $P_2(c) = 1$. Let q be a quota vector with $q_c = 1$. Then, $\varphi_1[P; q] = \psi_1^{\succ_c, \succ_{-c}}[P; q] = c$. Thus, (P'_1, P_{-1}) is a rank-respecting monotonic transformation of P at $\varphi[P; q]$, yet $\varphi_1[P'_1, P_{-1}; q] = \psi_1^{\succ'_c, \succ_{-c}}[P'_1, P_{-1}; q] = \emptyset$, violating r.r. invariance.

6.2 Single-Unit Supply

We consider an environment in which q_c is fixed at 1 for all $c \in C$. In this setup, we denote a problem by its preference profile P and the matching produced under a mechanism φ for this problem by $\varphi[P]$.

All the concepts and notation are adjusted to account for this change in the model. For example, the set I_c is defined as $I_c = \{i \in I : \varphi_i[P] = c \text{ for some } P\}$ for the fixed quota profile $q = (1, 1, ..., 1)$.

Since we fix the quota profile in this setup, we do not allow q to vary in the statement of the axioms. As a consequence, respect of preference rankings does not imply individual rationality in this environment unlike in the environment with variable quotas. To see this point, recall that $I_\emptyset = I$ in the environment with variable quotas because every student is assigned $\emptyset$ if there is no supply of school seats, but $I_\emptyset$ does not need to coincide with I if the quota vector is fixed. If $i \notin I_\emptyset$, respect of preference rankings does not require that i is assigned $\emptyset$ even if i is matched to an unacceptable school. Thus individual rationality does not follow from respect of preference rankings.

While the assumption that each school has a quota of one may not hold in the context of student placement in schools, it may be satisfied in problems such as housing or office allocation where each object is arguably unique. Moreover, this restriction enables a simpler characterization of the Boston mechanisms as stated below.

Theorem 2 *Fix the quota profile at q with $q_c = 1$ for all $c \in C$. A mechanism φ is the Boston mechanism induced by some priority profile $\succ$, i.e., $\varphi = \psi^\succ$, if and only if φ respects preference rankings and satisfies individual rationality, population monotonicity, and r.r. invariance.*

The following examples show that the axioms in Theorem 2 are independent when $|I| \geq 3$ and $|C| \geq 2$ (in Appendix G we study all other cases and show that the axioms are not independent when $|I| \leq 2$ or $|C| = 1$).

Example 6 A mechanism violating only respect of preference rankings: Suppose that $|C| \geq 2$ and $|I| \geq 2$. Fix a school $c \in C$ and students $1, 2 \in I$ (such a school and students exist by assumption). Consider a mechanism such that

$$\varphi_i[P] = \begin{cases} c & \text{if } i = 1 \text{ and } cP_1\emptyset \text{ or} \\ & \quad i = 2, \emptyset P_1 c \text{ and } cP_2\emptyset, \\ \emptyset & \text{otherwise.} \end{cases}$$

It can be verified that φ satisfies individual rationality, population monotonicity and r.r. invariance. On the other hand, the mechanism violates respect of preference rankings. To see this point first

observe that $2 \in I_c$ by the construction of the mechanism. Consider a preference profile P such that

P_1	P_2	P_i $(i \neq 1, 2)$
d	c	$\emptyset$
c	$\vdots$	$\vdots$
$\vdots$		

where d is a school different from c (such a school exists since $|C| \geq 2$). By the definition of the mechanism φ, it follows that $c\, P_2\, \emptyset = \varphi_2[P]$ although $2 \in I_c$ and there is no student $i \neq 2$ with $P_i(c) \leq P_2(c)$, violating respect of preference rankings.

Example 7 A mechanism violating only individual rationality: Fix $c \in C$ and $i \in I$. Consider a mechanism φ such that $\varphi_i[P] = c$ and $\varphi_j[P] = \emptyset$ for all $j \neq i$ at every preference profile P. The mechanism trivially respects preference rankings and satisfies population monotonicity and r.r. invariance. Mechanism φ violates individual rationality since for any preference profile P such that $\emptyset\, P_i\, c$, we have $\emptyset\, P_i\, c = \varphi_i[P]$.

Example 8 A mechanism violating only population monotonicity: Suppose that $|I| \geq 3$ and $|C| \geq 2$: Let $I = \{1, 2, 3, \ldots, n\}$. For each $c \in C$, define priority orders $\succ_c, \succ'_c \in \Pi$ such that

$\succ_c$	$\succ'_c$
1	1
2	3
3	2
$\emptyset$	$\emptyset$
$\vdots$	$\vdots$

Let $\succ = (\succ_c)_{c \in C}$ and $\succ' = (\succ'_c)_{c \in C}$. Define mechanism φ by,

$$\varphi[P] = \begin{cases} \psi^{\succ}[P] & \text{if } P_1(\emptyset) = 1, \\ \psi^{\succ'}[P] & \text{otherwise.} \end{cases}$$

Mechanism φ respects preference rankings and satisfies individual rationality and r.r. invariance. On the other hand, φ violates population monotonicity. To see this point consider a preference profile P such that student 1 top-ranks a school $c \in C$ and all other students top-rank $d \in C \setminus c$ (such schools c and d exist by the assumption $|C| \geq 2$). Then $\varphi_3[P] = d$ by definition of the mechanism φ. However, by definition of φ it follows that $\varphi_3[P_1^{\emptyset}, P_{-1}] \neq d$, thus violating population monotonicity.

Example 9 A mechanism violating only r.r. invariance: Suppose that $|I| \geq 2$ and $|C| \geq 2$: The mechanism presented in Example 5 satisfies respect of preference rankings, individual rationality, and population monotonicity, while violating r.r. invariance.

7   Conclusion

This paper provided two characterizations of the Boston mechanisms. We introduced two new axioms, respect of preference rankings and rank-respecting invariance. Two alternative characterizations were given, and these two axioms played key roles in both characterizations.

Although one might suspect that assuming respect of preference rankings is tantamount to requiring the mechanism to be the Boston mechanism, our analysis shows that other properties are needed for the Boston mechanism as well. In particular, the results suggest that rank-respecting invariance is useful for our understanding of strategic properties of the Boston mechanisms: As Maskin monotonicity is equivalent to group strategy-proofness in our setup, our analysis suggests that the extent of manipulability of the Boston mechanisms is captured as the difference between Maskin monotonicity and rank-respecting invariance.

One modeling decision we made is to allow a student to be unqualified for some schools as is the case in certain school districts. Accordingly, our main results characterize a broad class of Boston mechanisms in which a student can be unacceptable to some schools. If we impose the requirement that every student be qualified to all schools, then our analysis can be modified to characterize a subclass of the Boston mechanisms in which every student is acceptable to every school. This class of mechanisms may be appealing because these mechanisms are Pareto efficient, but the restriction excludes some applications. Whether this new set of axioms is independent is unknown and left for future research.

Another modeling decision we made is to focus our attention to deterministic mechanisms. In practical school choice problems, random mechanisms are often employed in order to achieve fairness among students. For that purpose, the Boston mechanism is often used with respect to weak priorities. The mathematical structure of random mechanisms are very different from deterministic ones, and it appears to be difficult to modify our technique to characterize the Boston mechanisms with weak priorities. However, the analysis of the Boston mechanisms in such a situation is of practical interest, and it seems to be an interesting direction of future research.

In this paper we refrained from recommending a specific mechanism. Rather, our objective was to provide a basic understanding of the Boston mechanisms and to clarify the mechanisms' advantages and disadvantages. The respect of preference rankings axiom implies a certain efficiency property, which clarifies a sense in which the Boston mechanisms take welfare into account if students report preferences truthfully. On the other hand, we also noted that this axiom may make the mechanism vulnerable to strategic misreporting of preferences. These two aspects are at the heart of the ongoing debate about the Boston mechanisms. One could interpret our characterizations as confirming the relevance of the debate, as we show that the respect of preference rankings is the representative feature of the Boston mechanisms. However, our results also imply that this interpretation needs some qualification. There exist mechanisms different from the Boston mechanisms that satisfy respect of preference rankings, and a Boston mechanism is obtained only in combination with other axioms, most notably rank-respecting invariance. Thus our paper shows that the literature's focus on the respect of preference rankings (or its implications) is justified, but with a qualification.

A   Appendix: Proof of Proposition 1

Fix a problem $[P; q]$ and construct sets $(I_c)_{c \in C}$, utility functions $(u_i)_{i \in I}$ that are consistent with the constraints given in the statement of the proposition and representing the preferences of the students. Let the optimal solution of the linear program be given as $Z = [z_{i,c}]_{i \in I, c \in C \cup \{\emptyset\}}$, which generates total welfare w. Observe that $z_i, c = 1$ implies $i \in I_c$. To the contrary of the claim, suppose Z is not equivalent to the outcome of the Boston mechanism. Then under Z, some student i is assigned a seat at $d \in C \cup \{\emptyset\}$ for $P_i(c) < P_i(d)$ for some school c with $i \in I_c$ such that either:

- a seat at c is empty: Then, as $u_i(c) > u_i(d)$, the matching that matches i with c and keeps all the other matches as in Z has a higher welfare weight, $w + u_i(c) - u_i(d)$, than Z, contradicting the fact that Z solves the linear program;

  or

- some student $j \neq i$ is honored a seat at c with either $P_i(c) < P_j(c)$, or $P_i(c) = P_j(c)$ and $\succ_c (i) <\succ_c (j)$: In either case, $u_i(c) > 2u_i(d)$ and $u_i(c) > 2u_j(c)$.Then the matching that assigns student i to c and leaves student j unmatched, and otherwise keeps the matches as in

Z generates the total welfare $w + u_i(c) - u_i(d) - u_j(c) > w$. Thus, Z cannot be the solution of the optimization problem, a contradiction.

Thus, the outcome of the optimization problem is unique and should coincide with the outcome of the Boston mechanism for $[P; q]$.

B   Appendix: Proof of Proposition 2

For contradiction, assume that matching μ respects preference rankings but is not constrained Pareto efficient. Since μ respects preference rankings, there exists no school c and student i such that $i \in I_c$, $|\mu_c| < q_c$ and $cP_i\mu_i$. This and the assumption that μ is not constrained Pareto efficient imply that there exists a sequence of students $i_1, i_2, \ldots, i_n$ such that $i_k \in I_{\mu_{i_{k+1}}}$, and $\mu_{i_{k+1}} P_{i_k} \mu_{i_k}$ or equivalently $P_{i_k}(\mu_{i_{k+1}}) < P_{i_k}(\mu_{i_k})$ for all $k = 1, \ldots, n$ (with the convention that $n + 1 = 1$). Since μ respects preference rankings, $P_{i_{k+1}}(\mu_{i_{k+1}}) \le P_{i_k}(\mu_{i_{k+1}})$ for all $k = 1, \ldots, n$. Combining these inequalities, we obtain

$$P_{i_1}(\mu_{i_1}) \le P_{i_n}(\mu_{i_1}) < P_{i_n}(\mu_{i_n}) \le P_{i_{n-1}}(\mu_{i_n}) < P_{i_{n-1}}(\mu_{i_{n-1}}) \le \cdots \le P_{i_1}(\mu_{i_2}) < P_{i_1}(\mu_{i_1}),$$

a contradiction.

C   Appendix: Proof of Lemma 1

Suppose that φ satisfies resource monotonicity and consistency, and fix P, q, and i arbitrarily. If $\varphi_i[P; q] = \emptyset$, then consistency implies

$$\varphi_j[P_i^\emptyset, P_{-i}; q] = \varphi_j[P; q] \text{ for all } j \ne i \text{ and all } P_i^\emptyset \in \mathcal{P}^\emptyset.$$

Suppose $\varphi_i[P; q] \ne \emptyset$. Then, by consistency, $\varphi_j \left[P_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1; q_{-\varphi_i[P;q]} \right] = \varphi_j [P; q]$ for all $j \ne i$ and all $P_i^\emptyset \in \mathcal{P}^\emptyset$. Hence, resource monotonicity implies that for all $j \ne i$

$$\varphi_j \left[P_i^\emptyset, P_{-i}; q \right] R_j \varphi_j \left[P_i^\emptyset, P_{-i}; q_{\varphi_i[P;q]} - 1, q_{-\varphi_i[P;q]} \right] = \varphi_j [P; q].$$

The above displayed relations show that φ satisfies population monotonicity.

D   Appendix: Proof of Proposition 3

Assume that mechanism φ respects preference rankings and satisfies Maskin monotonicity and non-wastefulness. Fix three distinct students $1, 2, 3$, and two distinct schools c, d (such students and schools exist because $|I| \geq 3$ and $|C| \geq 2$ by assumption). Let quota q be such that $q_c = q_d = 1$ and consider student preferences

P_1	P_1'	P_2	P_3	P_i	$(i \neq 1, 2, 3)$
c	d	c	c	$\emptyset$	
d	c	d	d	$\vdots$	
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		

Under preference profile P, since φ is non-wasteful, there exist one student who is assigned c and one student who is assigned d among students $1, 2$, and 3. Without loss of generality, assume $\varphi_1[P; q] = \emptyset, \varphi_2[P; q] = c$, and $\varphi_3[P; q] = d$. On the other hand, by respect of preference rankings and non-wastefulness, $\varphi_1[P_1', P_{-1}; q] = d$. This is a contradiction to Maskin monotonicity, since $(P_1', P_{-1}) \, m.t. \, P \, at \, \varphi[P; q]$ but $\varphi_1[P_1', P_{-1}; q] = d \neq \emptyset = \varphi_1[P; q]$.

E   Appendix: Proof of Theorem 1

It is straightforward to see that the Boston mechanism for an arbitrary priority profile $\succ$ satisfies all the axioms in the statement. Thus, we show the converse.

Based on Lemma 1, we invoke population monotonicity of φ in various parts of the proof. For any $c \in C$ and $I' \subseteq I$, let $\mathcal{P}^{|I|} \langle c; I' \rangle$ be the set of preference profiles such that all students in I' rank c as the first choice and all other students rank $\emptyset$ as the first choice. We prove the theorem in two parts.

E.1   Part 1: Construction of Priority Profile $\succ$

For each $c \in C$, construct $\succ_c$ as follows: Let $q_c = 1$ and $q_{c'} = 0$ for all $c' \in C \setminus \{c\}$. Fix some $P^{(1)} \in \mathcal{P}^{|I|} \langle c, I \rangle$ and let the top-priority student under $\succ_c$, denoted by i_c^1, be the student in $\varphi_c[P^1; q]$ if any (who is unique, if existent, since $q_c = 1$). Iteratively continue, such that: For each $\ell \geq 2$, fix

some $P^{(\ell)} \in \mathcal{P}^{|I|} \left\langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\} \right\rangle$ and let the ℓth priority student under $\succ_c$, denoted by i_c^ℓ, be the student in $\varphi_c[P^{(\ell)}; q]$ if any (who is unique, if existent, since $q_c = 1$). If $\varphi_c[P^{(\ell)}; q] = \emptyset$ for any $\ell \geq 1$, then order all students in $I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$ so that $\emptyset \succ_c i$ for all $i \in I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$. This procedure defines a priority order $\succ_c$. It is an implication of the following claim that the construction of $\succ_c$ is independent of the choice of preference $P^{(\ell)} \in \mathcal{P}^{|I|} \left\langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\} \right\rangle$ in each step ℓ:

Claim 1 *Let $I' \subseteq I$. Suppose $\varphi_c[P; q] = i$ where $P \in \mathcal{P}^{|I|} \langle c, I' \rangle$ and $q_c = 1, q_{c'} = 0$ for all $c' \neq c$. Then $\varphi_c[P'; q'] = i$ for all q' with $q'_c = 1$ and $P' \in \mathcal{P}^{|I|} \langle c, I'' \rangle$ with $\{i\} \subseteq I'' \subseteq I'$.*

Proof. We will prove the claim in three steps:

Proof Step 1:

First, we show the claim when $q' = q$ and $I'' = I'$. Let $P \in \mathcal{P}^{|I|} \langle c, I' \rangle$ and $\varphi_i[P; q] = c$. Assume $P'_j \neq P_j$ for some $j \in I$ and $P'_k = P_k$ for all $k \neq j$ without loss of generality. Two cases are possible for the identity of j, where Case 2 has also two sub-cases:

Case 1 $j = i$:

Then, since $\varphi_i[P; q] = c$ and c is top ranked both at P_i and P'_i, P' m.t. P at $\varphi[P; q]$ and

$$U_k\left(P, \varphi[P; q]\right) = U_k\left(P', \varphi[P; q]\right),$$
$$V_k\left(P, \varphi[P; q]\right) = V_k\left(P', \varphi[P; q]\right)$$

for all $k \in I$. Thus P' r.r.m.t. P at $\varphi[P; q]$. Since φ satisfies r.r. invariance, we conclude that $\varphi[P'; q] = \varphi[P; q]$ and hence $\varphi_i[P'; q] = c$.

Case 2-(i) $j \neq i$ and $\varphi_j[P'; q] \neq c$:

For any $P_j^\emptyset \in \mathcal{P}^\emptyset$, since φ satisfies population monotonicity, $\varphi_i[P_j^\emptyset, P_{-j}; q] R_i \varphi_i[P; q] = c$. Since P_i top-ranks c, we obtain

$$\varphi_i[P_j^\emptyset, P_{-j}; q] = c. \tag{1}$$

Suppose for contradiction that $\varphi_i[P'; q] \neq c$. Then, since φ respects preference rankings, $i \in I_c$, and $\varphi_j[P; q] \neq c$, there exists $i' \neq i, j$ such that $\varphi_{i'}[P'; q] = c$. Thus $P_{i'} = P'_{i'}$ top-ranks c as φ respects preference rankings (otherwise i' receives $\emptyset$ by individual rationality) and, by population monotonicity,

$$\varphi_{i'}[P_j^\emptyset, P'_{-j}; q] = c \tag{2}$$

26

Relations (1) and (2) contradict each other since $i' \neq i$, $q_c = 1$, and $P'_{-j} = P_{-j}$.

Case 2-(ii) $j \neq i$ and $\varphi_j[P'; q] = c$:

Then, since both P_j and P'_j top-rank c (otherwise j receives $\emptyset$ by individual rationality), $P \, m.t. \, P'$ at $\varphi[P'; q]$ and

$$U_k\left(P, \varphi[P'; q]\right) = U_k\left(P', \varphi[P'; q]\right),$$
$$V_k\left(P, \varphi[P'; q]\right) = V_k\left(P', \varphi[P'; q]\right)$$

for all $k \in I$. Thus $P \, r.r.m.t. \, P'$ at $\varphi[P'; q]$. Since φ satisfies r.r. invariance we conclude $\varphi_j[P; q] = c$, a contradiction to $j \neq i$, $\varphi_i[P; q] = c$ and $q_c = 1$.

Proof Step 2:

Given the preceding argument, population monotonicity of φ implies that the claim holds for all cases when $q' = q$ and $\{i\} \subseteq I'' \subseteq I'$, noting that c is the top-ranked school at P_i.

Proof Step 3: Finally, we will show that the claim holds for all q' with $q'_c = 1$ and $P' \in \mathcal{P}^{|I|} \langle c, I'' \rangle$ with $\{i\} \subseteq I'' \subseteq I'$. Thus assume that q' satisfies $q'_c = 1$ and $P' \in \mathcal{P}^{|I|} \langle c, I'' \rangle$ with $\{i\} \subseteq I'' \subseteq I'$. By Proof Step 2,

$$\varphi_i[P'; q] = c. \tag{3}$$

Since φ satisfies resource monotonicity and $q'_c \geq q_c$ for all $c \in C$ (as $q'_c = q_c = 1$ and $q'_{c'} \geq 0 = q_{c'}$ for all $c' \neq c$),

$$\varphi_i[P'; q'] R'_i \varphi_i[P'; q]. \tag{4}$$

Relations (3) and (4) imply

$$\varphi_i[P'; q'] R'_i c. \tag{5}$$

Since c is top ranked under P'_i by assumption, relation (5) implies

$$\varphi_i[P'; q'] = c,$$

completing the proof. ∎

E.2   Part 2: Proof that $\varphi = \psi^{\succ}$

Let $\succ = (\succ_c)_{c \in C}$ be the priority order profile constructed in Part 1. For any given preference profile P and quota profile q, we will show that $\varphi[P; q] = \psi^{\succ}[P; q]$. Construct the following student sets and quotas:

For any $c \in C \cup \{\emptyset\}$, define

$$I_c(0) = \emptyset,$$

$$J_c(0) = \emptyset,$$

$$q_c(0) = q_c.$$

For any $\ell \geq 1$, given $(I_c(0), J_c(0), q_c(0))_{c \in C \cup \{\emptyset\}}, \ldots, (I_c(\ell-1), J_c(\ell-1), q_c(\ell-1))_{c \in C \cup \{\emptyset\}}$, recursively define

$$I_c(\ell) = \left\{ i \in I_c \setminus \left(\bigcup_{d \in C \cup \{\emptyset\}} \bigcup_{\ell'=1}^{\ell-1} J_d(\ell') \right) \; : \; P_i(c) = \ell \right\},$$

$$J_c(\ell) = \{ i \in I_c(\ell) \; : \; \varphi_i[P; q] = c \},$$

$$q_c(\ell) = q_c(\ell-1) - |J_c(\ell-1)|,$$

$q_c(\ell)$ denotes the number of seats remaining after assigning seats to students who rank c as the $(\ell-1)^{\text{st}}$ choice or higher. $I_c(\ell)$ is the set of qualified students who rank c as their ℓ^{th} choice and have not received any higher-ranked school. $J_c(\ell)$ is the set of students in $I_c(\ell)$ who receive seats at c under $\varphi[P; q]$. Let $\succ_{\emptyset} \in \Pi$ be an arbitrary priority order. Individual rationality of φ (following from resource monotonicity of φ) implies $I_{\emptyset} = I$, and it suffices to show the following claim.

Claim 2 *For all $\ell \geq 1$, $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ for all $c \in C \cup \{\emptyset\}$, where*

$$\arg\max_{I_c(\ell), q_c(\ell)} \succ_c := \{ i \in I_c(\ell) \; : \; |\{ j \in I_c(\ell) \; : \; j \succeq_c i \}| \leq q_c(\ell) \},$$

is the set of (at most) $q_c(\ell)$ students who have the highest priorities at c among those in $I_c(\ell)$.

Proof. Let $c \in C \cup \{\emptyset\}$. Fix $\ell \geq 1$. If $|I_c(\ell)| \leq q_c(\ell)$, then $\arg\max_{I_c(\ell), q_c(\ell)} \succ_c = I_c(\ell)$. Then $J_c(\ell) = I_c(\ell)$ because φ respects preference rankings.[17] Hence the conclusion $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ holds.

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[17]To see this, assume for contradiction that $J_c(\ell) \neq I_c(\ell)$. Then, since $J_c(\ell) \subseteq I_c(\ell)$ by definition, there exists

Thus, assume $|I_c(\ell)| > q_c(\ell)$. Suppose for contradiction that the conclusion $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ does not hold. Since φ respects preference rankings, it follows that $|J_c(\ell)| = q_c(\ell)$ and hence, there exist $i \in \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ such that $\varphi_i[P;q] \neq c$ and $j \in I_c(\ell) \setminus \left(\arg\max_{I_c(\ell), q_c(\ell)} \succ_c\right)$ such that $\varphi_j[P;q] = c$.

Let $P' = (P_i, P_j, P^{\emptyset}_{-i,j})$ for some $P^{\emptyset}_{-i,j} \in \left(\mathcal{P}^{\emptyset}\right)^{|I|-2}$. Let $q' = (q'_{c'})_{c' \in C}$ be defined by

$$q'_{c'} = q_{c'} - |\{k \in I \setminus \{i,j\} : \varphi_k[P;q] = c'\}|$$

for each $c' \in C$. Note that $q'_c = 1$ since $i \notin J_c(\ell)$, $j \in J_c(\ell)$, and all other capacities of c are allocated to students in $J_c(1), \ldots, J_c(\ell)$ because $|J_c(\ell)| = q_c(\ell)$. By construction of $I_c(\ell)$, since $i \in I_c(\ell)$, she has not been matched to a higher-choice school than c. Thus, $cP_i\varphi_i[P;q]$. Since φ is consistent,

$$\varphi_j[P';q'] = \varphi_j[P;q] = c,$$
$$\varphi_i[P';q'] = \varphi_i[P;q] \text{ and hence, } cP_i\varphi_i[P';q'].$$

Consider preference P''_i, P''_j such that c is top ranked at P''_i and P''_j and relative rankings of all other schools are unchanged from P'_i and P'_j, respectively. Then $P'' := (P''_i, P''_j, P'_{-i,j})$ $r.r.m.t.$ P' at $\varphi[P';q']$. Thus, since φ satisfies r.r. invariance, $\varphi_i[P'';q'] \neq c$. This is a contradiction since Claim 1 and the assumption that i has a higher priority than j at c imply $\varphi_i[P'';q'] = c$. $\blacksquare$

Claim 2 completes the proof of the Theorem.

F   Appendix: Proof of Theorem 2

The Boston mechanism for an arbitrary priority profile clearly satisfies all the axioms in the statement. Let q be the quota vector with $q_c = 1$ for all $c \in C$, which is fixed throughout the current analysis.

F.1   Part 1: Construction of Priority Profile $\succ$

For each $c \in C$, construct $\succ_c$ as follows: Fix some $P^{(1)} \in \mathcal{P}^{|I|}\langle c, I\rangle$ and let the top-priority student under $\succ_c$, denoted by i^1_c, be the student in $\varphi_c[P^{(1)}]$ (who is unique, if existent, since $q_c = 1$).

$i \in I_c(\ell)$ who is not in $J_c(\ell)$. Then, since i is not assigned to any of her ℓ most preferred objects, respect of preference rankings implies that the entire capacity of c should be allocated to students who rank c as one of their ℓ most preferred objects. However, this is a contradiction because by the inductive assumption, not all the capacities of c has been assigned to those students.

Iteratively continue, such that: For each $\ell \geq 2$, fix some $P^{(\ell)} \in \mathcal{P}^{|I|} \langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\} \rangle$ and let the ℓth priority student under $\succ_c$, denoted by i_c^ℓ, be the student in $\varphi_c[P^{(\ell)}]$ (who is unique, if existent, since $q_c = 1$). If $\varphi_c[P^{(\ell)}] = \emptyset$ for any $\ell \geq 1$, then order all students in $I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$ so that $\emptyset \succ_c i$ for all $i \in I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\}$. This procedure defines a priority order $\succ_c$.[18] It is an implication of the following claim that the construction of $\succ_c$ is independent of the choice of preference $P^{(\ell)} \in \mathcal{P}^{|I|} \langle c, I \setminus \{i_c^1, \ldots, i_c^{\ell-1}\} \rangle$ in each step ℓ:

Claim 3 *Let $I' \subseteq I$. Suppose $\varphi_c[P] = i$ where $P \in \mathcal{P}^{|I|} \langle c, I' \rangle$. Then $\varphi_c[P'] = i$ for all $P' \in \mathcal{P}^{|I|} \langle c, I'' \rangle$ with $\{i\} \subseteq I'' \subseteq I'$.*

   Proof. The proof is omitted since it is identical to Steps 1 and 2 of the proof of Claim 1. ∎

F.2   Part 2: Proof that $\varphi = \psi^\succ$

Let $\succ = (\succ_c)_{c \in C}$ be the priority order profile constructed in Part 1. For any given preference profile P, we will show that $\varphi[P] = \psi^\succ[P]$. For each $c \in C \cup \{\emptyset\}$ and $\ell = 1, 2, \ldots$, define $I_c(\ell)$, $J_c(\ell)$, and $q_c(\ell)$ as in Part 2 of the Proof of Theorem 1. Recall that $I_c(\ell)$ is the set of students who rank c as their ℓ^{th} choice and have not received any higher-ranked school (except those who cannot be matched to c under any preference profile), $J_c(\ell)$ is the set of students in $I_c(\ell)$ who receive seats at c under $\varphi[P]$, and $q_c(\ell) \in \{0, 1\}$ denotes the number of seats in c remaining after assigning seats to students who rank c as the $(\ell - 1)^{\text{st}}$ choice or higher. Let $\succ_\emptyset \in \Pi$ be an arbitrary priority order. Individual rationality of φ implies $I_\emptyset = I$, and it suffices to show the following claim.

Claim 4 *For all $\ell \geq 1$, $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ for all $c \in C \cup \{\emptyset\}$, where*

$$\arg\max_{I_c(\ell), q_c(\ell)} \succ_c := \{i \in I_c(\ell) \ : \ |\{j \in I_c(\ell) \ : j \succeq_c i\}| \leq q_c(\ell)\},$$

is the set of (at most) $q_c(\ell)$ students who have the highest priorities at c among those in $I_c(\ell)$.

   Proof. Let $c \in C \cup \{\emptyset\}$. Fix $\ell \geq 1$. If $|I_c(\ell)| \leq q_c(\ell)$, then $\arg\max_{I_c(\ell), q_c(\ell)} \succ_c = I_c(\ell)$. Then $J_c(\ell) = I_c(\ell)$ because φ respects preference rankings. Hence the conclusion $J_c(\ell) = \arg\max_{I_c(\ell), q_c(\ell)} \succ_c$ holds.

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[18]Note that the construction of $\succ_c$ is similar to the one in the proof of Theorem 1. The only difference is that $q_{c'} = 1$ for every $c' \in C$ here, while $q_{c'} = 0$ for every $c' \neq c$ in the proof of Theorem 1.

Thus, assume $|I_c(\ell)| > q_c(\ell)$. Suppose for contradiction that the conclusion does not hold. Since φ respects preference rankings, it follows that $|J_c(\ell)| = 1$ and hence, there exist $i \in \arg\max_{I_c(\ell)} \succ_c$ such that $\varphi_i[P] \neq c$ and $j \in I_c(\ell) \setminus \left(\arg\max_{I_c(\ell), q_c(\ell)} \succ_c\right)$ such that $\varphi_j[P] = c$.

Consider preference profile P' such that (1) at P'_k for every student $k \in I_c(\ell)$, c is top ranked and relative rankings of all other schools are unchanged from P_k, and (2) preferences of all other students are unchanged. Since $q_c = 1$, it follows that P' $r.r.m.t.$ P at $\varphi[P]$. Thus, since φ satisfies r.r. invariance, $\varphi_j[P'] = c$.

Let $P'' = (P'_i, P'_j, P^\emptyset_{-i,j})$ for some $P^\emptyset_{-i,j} \in \left(\mathcal{P}^\emptyset\right)^{|I|-2}$. Since φ satisfies population monotonicity,

$$\varphi_j[P''] R'_j \varphi_j[P'] = c.$$

This and the assumption that c is top ranked at P'_j imply

$$\varphi_j[P''] = c.$$

Since $q_c = 1$, the above relation implies

$$\varphi_i[P''] \neq c,$$

which is a contradiction since Claim 3 and the construction of $\succ$ that gives i a higher priority than j at c imply $\varphi_i[P''] = c$. $\blacksquare$

Claim 4 completes the proof of the Theorem.

G   Appendix: Independence of Axioms for Remaining Cases

The main text has presented examples showing that the axioms in the characterizations are independent for all but a few values of $|I|$ and $|C|$. This section completes the investigation by considering all other cases.

G.1   Axioms for Theorem 1

A mechanism violating only consistency: Suppose $|I| \leq 2$: If $|I| = 1$, then consistency is vacuously satisfied by any mechanism. If $|I| = 2$, then respect of preference rankings implies consistency. To see this first note that, for any $i \in I$ and $j \neq i$, $\varphi_i[P; q]$ is the most preferred school in $\{c \in C \cup \{\emptyset\} : i \in I_c, q_c - \mathbf{1}_{\varphi_j[P;q]=c} \geq 1\}$ by respect of preference rankings, where $\mathbf{1}_{\varphi_j[P;q]=c} = 1$ if $\varphi_j[P;q] = c$ and

0 otherwise. By inspection, respect of preference rankings implies that $\varphi_i[P_i, P_j^{\emptyset}; (q_c - \mathbf{1}_{\varphi_j[P;q]=c})_{c \in C}]$ is the most preferred school in $\{c \in C \cup \{\emptyset\} : i \in I_c, q_c - \mathbf{1}_{\varphi_j[P;q]=c} \geq 1\}$, showing consistency. Thus, there is no mechanism that violates consistency while respecting preference rankings.

A mechanism violating only resource monotonicity: Suppose that $|I| = 1$ or $|C| = 1$: If $|I| = 1$, then respect of preference rankings implies resource monotonicity. To see this point observe that the unique agent, denoted i, receives her most preferred school in $\{c \in C : i \in I_c, q_c \geq 1\}$. Since this set is increasing in each q_c in the set inclusion sense, resource monotonicity follows.

If $|C| = 1$, then respect of preference rankings, consistency, and r.r. invariance imply resource monotonicity. To show this first recall that respect of preference rankings implies individual rationality (since $I_{\emptyset} = I$ and $q_{\emptyset} = \infty$). Now suppose, for contradiction, a mechanism φ satisfies respect of preference rankings, consistency, and r.r. invariance, while violating resource monotonicity. Then there exists a student $i \in I$, preference profile P, and a quota q_c of the unique school c such that

$$\varphi_i[P; q_c - 1] = c, \tag{6}$$

$$\varphi_i[P; q_c] = \emptyset, \tag{7}$$

$$c \, P_i \, \emptyset. \tag{8}$$

By relationships (6) – (8) and respect of preference rankings, $|\varphi_c[P; q_c]| = q_c$, and hence there exists a student $j \neq i$ such that

$$\varphi_j[P; q_c] = c, \tag{9}$$

$$\varphi_j[P; q_c - 1] = \emptyset. \tag{10}$$

By consistency of φ and relation (9), it follows that

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = \varphi_i[P; q_c]. \tag{11}$$

Relations (7) and (11) imply

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = \emptyset. \tag{12}$$

Meanwhile relation (10) implies that $(P_j^{\emptyset}, P_{-j}) \, r.r.m.t. \, P$ at $\varphi[P; q_c - 1]$. Thus by r.r. invariance of φ,

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = \varphi_i[P; q_c - 1]. \tag{13}$$

Then, by relations (6) and (13), we obtain

$$\varphi_i[P_j^{\emptyset}, P_{-j}; q_c - 1] = c. \tag{14}$$

Relations (12) and (14) contradict each other, showing that φ is resource monotonic.

A mechanism violating only r.r. invariance: Suppose that $|C| = 1$ or $|I| = 1$: If $|C| = 1$, then consistency, resource monotonicity, and respect of preference rankings imply r.r. invariance. To see this point first recall that these properties imply individual rationality and population monotonicity. Let φ be a mechanism for that these axioms hold and for each $i \in I$ let

$$\frac{P_i' \quad P_i''}{\begin{array}{cc} c & \emptyset \\ \emptyset & c \end{array}}$$

Fix a preference profile P arbitrarily. Since φ satisfies individual rationality,

$$\varphi_i[P; q] = c \Rightarrow P_i = P_i' \tag{15}$$

Moreover, if $\varphi_i[P; q]$ is the top-ranked school for i at P_i, then the only monotonic transformation of P_i at $\varphi_i[P; q]$ is P_i itself. By this fact and (15), if $\tilde{P}$ is a monotonic transformation of P at $\varphi[P; q]$, then for any $i \in I$, either

1. $\tilde{P}_i = P_i$, or

2. $P_i = P_i'$, $\varphi_i[P; q] = \emptyset$, and $\tilde{P}_i = P_i''$.

For any i such that Case 1 above applies, population monotonicity of φ and Cases 1 and 2 imply $\varphi_i[\tilde{P}; q] \, R_i \varphi_i[P; q]$. So, if $\varphi_i[P; q] = c$, then $\varphi_i[\tilde{P}; q] = c$. Since φ respects preference rankings, it follows that $\varphi_i[P; q] = \emptyset$ implies $\varphi_i[\tilde{P}; q] = \emptyset$. Thus we conclude $\varphi_i[P; q] = \varphi_i[\tilde{P}; q]$. For any i such that Case 2 above applies, individual rationality of φ implies $\varphi_i[\tilde{P}; q] = \emptyset = \varphi_i[P; q]$. Therefore we conclude $\varphi[\tilde{P}; q] = \varphi[P; q]$, showing r.r. invariance.

If $|I| = 1$, then respect of preference rankings implies r.r. invariance. To show this, let i be the unique student in I and $C_i = \{c \in C \cup \{\emptyset\} : i \in I_c, q_c \geq 1\}$. Since i is the unique student, respect of preference rankings imply that $\varphi_i[P_i; q]$ is the school that is top ranked by P_i within C_i. Any monotonic transformation P_i' of P_i at $\varphi_i[P_i; q]$ leaves the top-ranked school in C_i unchanged (namely $\varphi_i[P_i; q]$), so $\varphi_i[P_i'; q] = \varphi_i[P_i; q]$ by respect of preference rankings. This shows that φ satisfies r.r. invariance.

G.2   Axioms for Theorem 2

A mechanism violating only respect of preference rankings: Suppose that $|C| = 1$ or $|I| = 1$:
If $|C| = 1$, then any mechanism φ that satisfies individual rationality and r.r. invariance respects preference rankings. To see this, let $C = \{c\}$ and assume for contradiction that φ does not respect preference rankings. Then there exists $i \in I_c$ (so $\varphi_i[P] = c$ for some P) such that $cP_i'\varphi_i[P']$ for some P' where either $\varphi_c[P'] = \emptyset$ or $\varphi_c[P'] = j$ with $P_i'(c) < P_j'(c)$. The latter is a contradiction to individual rationality of φ, because $P_i'(c) < P_j'(c)$ and $|C| = 1$ imply $\emptyset P_j' c$. Thus assume $\varphi_c[P'] = \emptyset$. Since $q_c = 1$ and $C = \{c\}$, it follows that $\varphi_j[P] = \varphi_j[P'] = \emptyset$ for all $j \neq i$. Also note that $P_i = P_i'$ because $\varphi_i[P] = c$ implies $cP_i\emptyset$ by individual rationality of φ, $cP_i'\emptyset$ from before, and $|C| = 1$. Hence $(P_i, P_{-i}^\emptyset)\,r.r.m.t.P$ at $\varphi[P]$ and $(P_i, P_{-i}^\emptyset)\,r.r.m.t.P'$ at $\varphi[P']$ for any $P_{-i}^\emptyset \in (P^\emptyset)^{|I|-1}$. Thus r.r. invariance implies $c = \varphi_i[P] = \varphi_i[P_i, P_{-i}^\emptyset] = \varphi_i[P'] = \emptyset$, a contradiction.

If $|I| = 1$, then r.r. invariance implies respect of preference rankings. To see this point suppose for contradiction that a mechanism φ violates respect of preference rankings while satisfying r.r. invariance. Then, for the unique student i, there exist P_i, P_i' such that

$$\varphi_i[P_i']P_i\varphi_i[P_i]. \tag{16}$$

Then a preference P_i'' such that

$$\begin{array}{c} P_i'' \\ \hline \varphi_i[P_i'] \\ \varphi_i[P_i] \\ \vdots \end{array}$$

satisfies $P_i''\,r.r.m.t.\,P_i$ at $\varphi[P_i]$ and $P_i''\,r.r.m.t.\,P_i'$ at $\varphi[P_i']$. By r.r. invariance we obtain

$$\varphi[P_i] = \varphi[P_i''] = \varphi[P_i'],$$

a contradiction to relation (16).

A mechanism violating only population monotonicity: Suppose that $|C| = 1$ or $|I| \leq 2$:
If $|I| = 1$, then population monotonicity is vacuously satisfied. If $|I| = 2$, then individual rationality and respect of preference rankings imply population monotonicity. To see this point let $I = \{1, 2\}$ and $P_2^\emptyset \in \mathcal{P}^\emptyset$. Consider an arbitrary preference profile P. Since φ is individually rational,

$\varphi_1[P] \, R_1 \, \emptyset$. If $\varphi_1[P] = \emptyset$, then individual rationality of φ implies that $\varphi_1[P_1, P_2^\emptyset] \, R_1 \, \emptyset = \varphi_1[P]$, satisfying the conclusion of population monotonicity. Thus suppose $\varphi_1[P] \neq \emptyset$ and, for contradiction, that $\varphi_1[P] \, P_1 \, \varphi_1[P_1, P_2^\emptyset]$. Then, by respect of preference rankings, $\varphi_2[P_1, P_2^\emptyset] = \varphi_1[P]$. However, $\emptyset \, P_2^\emptyset \, \varphi_1[P]$ since $P_2^\emptyset$ top-ranks $\emptyset$, which contradicts individual rationality of φ. By a symmetric argument the conclusion of population monotonicity holds for matchings of student 2, showing that population monotonicity holds.

If $|C| = 1$, then individual rationality and r.r. invariance imply population monotonicity. To see this point first note that $\varphi_i[P] \, R_i \, \emptyset$ for all i and P since φ is individually rational. Hence if $\varphi_i[P] = \emptyset$, then $\varphi_i[P_j^\emptyset, P_{-j}] R_i \emptyset = \varphi_i[P]$ for any $i \in I, j \neq i$ and $P_j^\emptyset \in \mathcal{P}^\emptyset$, thus the conclusion of population monotonicity holds. So suppose $\varphi_i[P] = c \neq \emptyset$. Then, since $|C| = 1$ and $q_c = 1$, $\varphi_k[P] = \emptyset$ for all $k \neq i$. Thus, for any $j \neq i$ and $P_j^\emptyset \in \mathcal{P}^\emptyset$, we have $(P_j^\emptyset, P_{-j}) \, r.r.m.t. \, P$ at $\varphi[P]$. By r.r. invariance of φ, we obtain $\varphi_i[P_j^\emptyset, P_{-j}] = \varphi_i[P]$, showing that the conclusion of population monotonicity holds. These arguments show the claim.

A mechanism violating only r.r. invariance: Suppose that $|I| = 1$ or $|C| = 1$: If $|I| = 1$ or $|C| = 1$, then the paragraph on r.r. invariance in Section G.1 shows that there exists no mechanism that satisfies respect of preference rankings, individual rationality, and population monotonicity, and yet violates r.r. invariance.[19]

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[19]The paragraph in Section G.1 supposes, in addition to respect of preference rankings, resource monotonicity and consistency instead of individual rationality and population monotonicity. However, consistency and resource monotonicity are used only to obtain individual rationality and population monotonicity. Therefore the example is valid in the current context.

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