REVIEWS

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Fearless Symmetry: Exposing the Hidden Patterns of Numbers. By Avner Ash and Robert Gross. Princeton University Press, Princeton, 2006, xxx + 272 pp., ISBN 0-691-12492-2, $24.95.

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What sort of subject is suitable for a popular book in mathematics? There are classic broad surveys such as Courant and Robbins [**1**] and well chosen compendia such as Newman [**2**]. Recently there have been books organized around particular numbers such as π, e, i, Euler's constant γ, the golden ratio ϕ, and 0, as well as others focused on problems or events, such as Hilbert's Problems, the Riemann Hypothesis, or the proof of Fermat's Last Theorem. While occasioned by particular mathematical phenomena, these books are as much about mathematical history and biography as they are about actual mathematics. As we all realize, it's quite difficult to write a serious treatment of a mathematical topic for an untrained audience. Of course, physicists such as Stephen Hawking and Brian Greene have shown that it is possible to grapple with deep and abstract ideas underlying modern science, but they enjoy the advantage that the ultimate basis for their subject is the physical universe, at least somewhat known to their readers.

How about a popular book on group representations and reciprocity laws, which unashamedly concerns itself with pure mathematics and not with the people who have created it? Not possible? Well, I believe that Ash and Gross have managed it, though probably not for their intended audience consisting of the large group of people who have studied calculus. But their book is very carefully thought out and offers to the smaller group who have studied some abstract mathematics an opportunity to learn about some of the fundamental ideas and principles driving modern mathematics. This group, by the way, includes most mathematicians, since much of this material is not part of the standard graduate training of many mathematicians. I, for one, learned a lot by reading this book, and I recommend the book to you with considerable enthusiasm. This collection of ideas is one of the foundations on which Wiles's proof of Fermat's Last Theorem is based, and one of the last chapters of the book is concerned with sketching this connection. The authors are established experts in algebraic number theory, and their book starts with a Foreword by Barry Mazur. They are mathematicians trying to explain their art to the scientifically literate world.

Let me briefly describe a few of the principal subjects of the book. The fundamental object is the field of algebraic numbers $\boldsymbol{Q}^{\text{alg}}$, which consists of all complex numbers that satisfy some polynomial equation with integer coefficients. This number system includes all rationals, all algebraic combinations of nth roots of rationals for various positive integers n, together with many other numbers (such as the roots of the equation $x^5 + x - 1 = 0$) that are not expressible using radicals. Thus, nearly all familiar numbers are algebraic, although it was an achievement of the nineteenth century (Hermite, Lindemann) to show that some particular numbers such as π and e are not algebraic and of the twentieth (Gelfond, Schneider) to prove that numbers of the form a^b, where a is rational (but not 0 or 1) and b is irrational, are never algebraic. $\boldsymbol{Q}^{\text{alg}}$ is the correct domain on which to work if one wishes to study many properties of the ordinary integers, such properties being, of course, the principal concern of classical number theory.

The group of all permutations of $\boldsymbol{Q}^{\text{alg}}$ that preserve addition and multiplication is called the absolute Galois group $G_{\boldsymbol{Q}}$. This group of transformations admits the Galois group of each polynomial with integer coefficients as a quotient group, so it is a large and complicated object. A principal thread of modern number theory is to try to learn as much about this mysterious group as possible. By the way, $G_{\boldsymbol{Q}}$ is my candidate for the most fundamental mathematical object of which most students and too many mathematicians have never heard! This book offers a gentle way to close this intellectual gap.

To study $G_{\boldsymbol{Q}}$, various tools have been developed, and one of the major goals of the Ash/Gross book is to describe some of these tools. The principal one is Galois representations, i.e., morphisms from the absolute Galois group $G_{\boldsymbol{Q}}$ into groups of matrices or permutations. In particular, if we fix a number system R, such as the integers $\boldsymbol{Z}$, the complex numbers $\boldsymbol{C}$, or a finite field $\boldsymbol{F}_p$, where p is a prime, and let $GL(n, R)$ denote the group of all invertible n by n matrices with entries in R, we can consider linear representations of $G_{\boldsymbol{Q}}$, which are morphisms from $G_{\boldsymbol{Q}}$ into $GL(n, R)$. Since the images are matrices, these Galois representations provide a way to do computations that offer partial information about the structure of $G_{\boldsymbol{Q}}$. The authors are deliberately vague about what "number system" means, since they need this concept quite early in the book: they declare it to be a set of numbers on which the usual laws of algebra hold. They have in mind a field or occasionally an integral domain. This indefiniteness might confuse some readers, since characteristic p arithmetic may not coincide with their concept of the usual rules of algebra. Also crucial are the characters of these representations, defined by simply taking the traces of the image matrices. It has always struck me as strange that most mathematics undergraduates learn about groups but don't even encounter the definition of a group character, while undergraduate chemists learn something about group characters in physical chemistry but may not see even the definition of a group! Other useful tools that this book discusses to a greater or lesser extent are elliptic curves, modular forms, and étale cohomology. If you have always wondered what these words mean, this book will enlighten you.

The final main theme of the book is generalized reciprocity laws. Quadratic reciprocity, a law first proven by Gauss, relates the solvability of the equation $x^2 = p$ in $\boldsymbol{F}_q$ to that of the equation $x^2 = q$ in $\boldsymbol{F}_p$, where p and q are both primes. With appropriate insight, this classical fact can be restated and generalized using the machinery of Galois representations. For each prime p the absolute Galois group $G_{\boldsymbol{Q}}$ contains a set of elements known as Frobenius elements. Any Galois representation r comes with a set S of bad primes (those which are ramified for r) so that if p is not in S, the character of r evaluated at each Frobenius element for p results in the same number. The sequence of these numbers as the prime p ranges over the unramified primes gives information about $G_{\boldsymbol{Q}}$ and r. The same sequence of numbers can sometimes be produced by what the authors term "Black Boxes," which can be built using other algebraic, analytic, or topological tools. One example of a Black Box output is the number of solutions to certain systems of equations modulo various primes. Another is the collection of Fourier coefficients of a modular form corresponding to the primes. A generalized reciprocity law is a (conjectured or proven) statement about the equality of the numbers produced by the Black Box and those obtained from evaluating the character defined by a Galois representation on the Frobenius elements associated to various primes p.

As you can see from this brief description, the mathematics discussed in this book is far from simple and tends to be abstract rather than concrete. How can the authors possibly make this material at all accessible to an untrained audience? They have a strategy which (I think) almost works. Their approach is highly descriptive with simple examples (and exercises!) inserted at appropriate places. Definitions are made in the simplest possible context with awkward cases deferred to brief appendices or simply omitted as beyond the scope of the book. Short chapters are devoted to a single concept, such as Frobenius elements or the restriction morphism that relates the absolute Galois group G_Q to the Galois group $G(f)$ of a particular polynomial f. Another very helpful device is the "Road Map" at the beginning of each chapter, which describes the goal and contents of the chapter. There are some clever rhetorical devices, such as a playlet, really a dialog, to help the reader appreciate just what it means for a mapping to be an element of the absolute Galois group. There are flashes of some real literary style throughout the book.

The authors are quite happy to state facts without proving them, but when possible they offer simple examples to make what they are discussing concrete. A few theorems, however, are carefully argued on the basis of assumed facts. For example, there is a nice treatment of the two-torsion of an elliptic curve, consisting of a sequence of three examples of increasing sophistication and then an honest proof (based on stated facts). The authors introduce concepts in the order in which the concepts are needed for purposes of the exposition. Thus, for example, the very first chapter in the book is devoted to the concept of a representation with some concrete examples such the counting number attached to a finite set to capture the notion of its cardinality or the even/odd designation attached to counting numbers to register their parity. Groups are discussed before modular arithmetic, and elliptic curves before matrices. Such ordering seems strange to mathematicians, since it does not respect depth of the concepts, but the choice makes good sense for a popular book. Another device which works to lighten the development is the inclusion of discussion on general topics such as the nature of a mathematical definition or the ways in which notation guides thinking. These digressions are interesting also to mathematicians. For example, the one about the nature of conjectures offers the Mertens conjecture as an example of a conjecture believed true for some time but eventually proven false (in 1985 by Odlyzko and te Riele). It asserts that if $M(x)$ is the sum of the values of the Möbius function over all n between 1 and x, then $|M(x)| < \sqrt{x}$ for all $x > 1$. Had it turned out to be true, it would have implied the Riemann Hypothesis, a conjecture which many of us still believe.

The book is organized into three parts. Part One concentrates on basic concepts: the idea of a representation, groups, equations and the varieties which they define, modular arithmetic, and a description of classical quadratic reciprocity. This material is quite accessible and is described very clearly. For instance, one of the basic examples of a group is $SO(3)$, presented (without matrices) as rotations of a physical globe. The discussion points out that even though this group is noncommutative it is true that for any two rotations f and g, the angular measure of the rotations $f \circ g$ and $g \circ f$ will be the same. Later this result is proven as an application of characters of representations. Part Two deals with Galois theory, elliptic curves, group representations, characters, and Frobenius elements. The authors make much of this material surprisingly clear as well, but there is a steep increase in sophistication once Frobenius enters. Part Three gets to generalized reciprocity laws, shows how quadratic reciprocity can be viewed as an example of such a law, discusses some ways of producing and studying Galois representations, and comments on connections with the proof of Fermat's Last Theorem.

Beautifully produced by Princeton University Press, the book benefits from good choices regarding (relatively large) type face and spacing to make the text as digestible as possible for a general reader. The writing style is attractive and fluent (although too many theorems are labeled ”amazing”). It contains a good bibliography of related books and papers at various levels, including Penrose’s tour de force [**3**], which deserves to be better known within the mathematical community. I noticed very few misprints or errors. Occasionally the examples do not quite fit the exemplified fact (as happens, for instance, for representations distinguishing between conjugacy classes on page 169).

This book is about serious, modern, abstract mathematics described in a fashion that mathematicians will recognize as the way they think and talk about their subject and not the way they usually write about it. In this respect it offers a window into the mathematical world. It is no surprise, therefore, that one needs a fair amount of tenacity to read this book. My guess is that most readers without some graduate training in mathematics will get stuck at some point, probably towards the end of Part Two. Getting stuck is not a bad thing—after all, it is the common fate of anyone learning real mathematics. However far a reader gets into this book, he or she will learn a great deal about the nature and content of modern number theory.

REFERENCES

1. R. Courant and H. Robbins, *What Is Mathematics? An Elementary Approach to Ideas and Methods*, Oxford University Press, London, 1941.
2. J. R. Newman, *The World of Mathematics*, Simon and Schuster, New York, 1956.
3. R. Penrose, *The Road to Reality: A Complete Guide to the Laws of the Universe*, Knopf, New York, 2005.

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Symmetry and the Monster. By Mark Ronan. Oxford University Press, Oxford, England, 2006, vii + 255 pp. ISBN 0-19-280722-6, $27.

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Glory be to God for sporadic things! Without them, we can build great pyramids and towering skyscrapers. But if a mathematician is a person who stands awestruck before the beauty and mystery of numbers and shapes and then asks why, would there be any mathematicians if the mathematical landscape were not dappled with exceptions? Could there have been a Pythagorean cult without the pentagram and its Golden irrationality? How dull would the Platonic solids seem without the dodecahedron and the icosahedron—not to mention their 4-dimensional hyper-kin. And would we have modern algebra without the intractable quintic, leading us back by Kleinian recirculation to the icosahedron and environs?

It is this spirit of awe and adventure which is marvelously evoked by Mark Ronan in this enthralling little book. I couldn’t put it down, even though I knew how the story ended. For indeed, to a large extent, this book is an adventure tale with a cast of colorful characters and a Monster lurking at the heart of the labyrinth. At this level, Ronan does a superb job of capturing the excitement of mathematical discovery, as it emerges in a social network of letters and visits, phone calls and lectures, inspired guesses and