Experimental Mathematics

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/uexm20

Lattice-Packing by Spheres and Eutactic Forms

Avner Ash & Robert Gross

To cite this article: Avner Ash & Robert Gross (2022) Lattice-Packing by Spheres and Eutactic Forms, Experimental Mathematics, 31:1, 302-308, DOI: 10.1080/10586458.2019.1624660

To link to this article: https://doi.org/10.1080/10586458.2019.1624660

Published online: 26 Jun 2019.

EXPERIMENTAL MATHEMATICS
2022, VOL. 31, NO. 1, 302–308
https://doi.org/10.1080/10586458.2019.1624660

Lattice-Packing by Spheres and Eutactic Forms

Avner Ash and Robert Gross

Mathematics Department, Boston College, Chestnut Hill, MA, USA

ABSTRACT
We consider a semi-random walk on the space X of lattices in Euclidean n-space which attempts to maximize the sphere-packing density function Φ. A lattice (or its corresponding quadratic form) is called "sticky" if the set of directions in X emanating from it along which Φ is infinitesimally increasing has measure 0 in the set of all directions. Thus the random walk will tend to get "stuck" in the vicinity of a sticky lattice. We prove that a lattice is sticky if and only if the corresponding quadratic form is semi-eutactic. We prove our results in the more general setting of self-adjoint homogeneous cones. We also present results from our experiments with semi-random walks on X. These indicate some idea about the landscape of eutactic lattices in the space of all lattices.

KEYWORDS
sphere-packing; eutactic lattices

2010 MATHEMATICS SUBJECT CLASSIFICATION
Primary 11H31; secondary 37B30; 82C41

1. Introduction

A "lattice-packing" is a packing of $\mathbb{R}^n$ (equipped with the usual Euclidean inner product) by non-overlapping spheres of maximum-possible equal radius centered at the points of a lattice Λ. The density of the packing is a continuous function $\Phi(\Lambda)$ on the space of lattices in $\mathbb{R}^n$. An important problem is to find the densest lattice packing in each dimension, i.e., the maximum of Φ. The only dimensions in which the densest possible packing is known are $n \leq 8$ and $n = 24$ [Cohn et al. 17]. It is expected that in large dimensions (perhaps even for $n = 9$), the densest possible sphere packing will not be a lattice packing.

Voronoi [Voronoi 08] proved that the local maxima of Φ occur at lattices that are perfect and eutactic (we will define "eutactic" and "semi-eutactic" below, and the concept of perfection is not needed for this paper). These local maxima are called "extreme lattices." In [Ash 77], the first author proved that Φ is a topological morse function on the space of lattices in $\mathbb{R}^n$. Its critical points are exactly the eutactic lattices. This generalizes Voronoi's theorem. Also, the results of [Ash 77] imply immediately that for fixed n, there are only a finite number of eutactic and semi-eutactic lattices in $\mathbb{R}^n$.

In recent years, a number of researchers have used quasi-random methods to look for the densest lattice. Marcotte and Torquato [Marcotte and Torquato 13] combined such methods with linear programing to create a very impressive algorithm that managed to re-discover the densest known lattice packings for $n \leq 19$. The reader may refer to their paper for references to earlier work along these lines.

In this paper, we report on work that uses a quasi-random walk on the space of lattices, beginning at a randomly chosen "seed" lattice, designed always to increase Φ. Our random walk generally gets "stuck" at lattices that are not particularly dense, although for $n \leq 8$, it sometimes attains the densest lattice. This paper gives a theoretical explanation of these phenomena plus an interpretation of our data in terms of eutactic lattices. We are led to ask some questions about the distribution of eutactic lattices within the space of all lattices.

For $n = 3$, we obtained good evidence that the "sticky" lattices are exactly the semi-eutactic ones. For $n = 3$, every semi-eutactic lattice is eutactic. Our main theorem implies that we can expect the sticky lattices to be exactly the semi-eutactic ones.

Definition 1.1. Let ψ be a real-valued function on a Riemannian orbifold X. We say $x \in X$ is *sticky with respect to* ψ if and only if

$$\lim_{r \to 0} \frac{\mathrm{vol}\left(\left\{y \in B_r(x) \mid \psi(y) > \psi(x)\right\}\right)}{\mathrm{vol}(B_r(x))} = 0,$$

CONTACT Robert Gross gross@bc.edu Mathematics Department, Boston College, Maloney Hall, Chestnut Hill, MA, USA.
Supplemental data for this article can be accessed on the https://doi.org/10.1080/10586458.2019.1624660.

where $B_r(x)$ denotes the ball of radius r with center at x.

This definition can be converted into a quantitative measure of the "degree of stickiness" of any point $x \in X$ with respect to ψ, assuming that the limit of the left-hand side exists at x.

It's easy to see that if ψ is a differentiable morse function on a differentiable Riemannian manifold X, then the only points that are sticky with respect to ψ are the local maxima. However, if ψ is only a topological morse function, then other points may also be sticky with respect to ψ.

We will say that a lattice is eutactic or semi-eutactic if its associated quadratic form is eutactic or semi-eutactic.

Theorem 1.2. *Let Φ be the sphere-packing density function on the space X of lattices in $\mathbb{R}^n$. A lattice is sticky with respect to Φ if and only if it is semi-eutactic.*

With probability 1, a point that is sticky with respect to ψ behaves like a sink for a random process on X that moves along paths of increasing ψ. Our main theorem thus explains why our algorithm generally sticks at or very near to semi-eutactic lattices. (More accurately, the lattices on which our algorithm gets stuck were observed to have the same or nearly the same value of Φ as a eutactic lattice. Presumably they are in fact equal or close to eutactic lattices, but it would have been overly expensive to check that.)

The concepts of eutacticity and the function Φ can be generalized to all self-adjoint homogeneous cones, of which the cone of positive-definite symmetric real matrices is just one kind. (See, for instance, [Ash et al. 10, ch. 2].) The theorem below applies to all self-adjoint homogeneous cones and works with a more stringent definition of stickiness than the one given above. We will explain how each of the concepts we use in working with self-adjoint homogeneous cones may be written explicitly in the case of the space of quadratic forms.

Because our random walk gets stuck at semi-eutactic lattices, we can view our experimental results as saying something about how many semi-eutactic lattices there are in $\mathbb{R}^n$ as a function of Φ. We formulate a conjecture about this in the last section of the paper.

2. Definitions

We follow the notation in [Ash 77], where any definitions not given below may be found. Our definitions and proofs apply not only to the space of positive definite symmetric $n \times n$ real matrices, but to all self-homogeneous cones.

Let V be a real vector space of dimension N and C a self-adjoint homogeneous cone contained in V. Denote the closure of C in V by $\bar{C}$. When we say C is self-adjoint, it is with reference to some fixed inner product on V. In fact, there is a Jordan algebra structure on V (determined by the cone C and the choice of a basepoint in C), and we take the inner product (z, w) to be the trace of Jordan multiplication by $z \circ w$ on V (where $\circ$ denotes the Jordan multiplication).

For the purpose of studying the sphere-packing density function on the space of lattices, we take V to be the space of all real symmetric $n \times n$ matrices and C the subcone of positive definite ones. We call this our "running example." In our running example, the Jordan algebra structure is given by $A \circ B = (AB + BA)/2$ where A and B are symmetric matrices. The inner product of A and B equals $\frac{n+1}{2}\mathrm{Tr}(AB)$, where Tr denotes the usual trace of a square matrix, not the Jordan trace of multiplication by an element in V.

The space of lattices in $\mathbb{R}^n$ is isomorphic to $C/\mathrm{GL}(n, \mathbb{Z})$. A lattice Λ maps to a point in C by sending it to the Gram matrix (e.g., [Conway and Sloane 93, p. 4]) of a basis of Λ. We remove the dependence on a choice of basis by taking the quotient of C by $\mathrm{GL}(n, \mathbb{Z})$. It is convenient (and equivalent) to view the sphere-packing density Φ as a $\mathrm{GL}(n, \mathbb{Z})$-invariant function on C rather than as a function on lattices.

Every self-adjoint homogeneous cone C comes equipped with a canonical normalized positive function $\phi(x)$ such that $\phi(x)\, dx$ is a G-invariant measure (where G is the real Lie group of automorphisms of C). See [Ash et al. 10, p. 39] for more details. The function ϕ is called the "characteristic function" of the cone. In our running example, $\phi(z) = \det(z)^{-(n+1)/2}$.

To apply the results of [Ash 77], we must fix an admissible lattice L in V. (For the definition of "admissible," see [Ash 77, p. 1042].) In our running example, we take all symmetric matrices with integer coefficients on the diagonal and half-integers elsewhere. Given $z \in C$, define its minimal vectors to be the members of the set $M(z)$ of $a \in \bar{C} \cap L - \{0\}$ such that (z, a) is minimized. Define $m(z)$ to be this minimum, which is a positive number. In fact, the set of minimal vectors $M(z)$ is a finite subset of the boundary of C.

In our running example, Barnes and Cohn [Barnes and Cohn 75] proved that every $a \in M(z)$ is of the form $a = vv^t$ for some nonzero integral column vector $v \in \mathbb{R}^n$, and up to a constant μ not depending on z, the minimum $m(z)$ is the length of v with respect to the metric given by z, i.e., $v^t zv$. From now on, when dealing with our running example, we will renormalize $m(z)$ by dividing it by μ. This brings our definition of Φ into conformity with the usual definition of density of a lattice packing.

The "packing" function $F(z)$ that we shall use from now on is the one used in [Ash 77], namely

$$F(z) = m(z)^{-N}\phi^{-1}(z).$$

Let V_n be the volume of a sphere of radius 1 in dimension n, and then in our running example

$$\Phi(z) = \frac{V_n}{2^n}\left(\frac{1}{F(z)}\right)^{\frac{1}{n+1}}.$$

Therefore, we will minimize F in order to maximize Φ.

The main theorem of [Ash 77] is that F is a topological morse function whose critical points are precisely the eutactic elements of C. Here is the definition of "eutactic":

Definition 2.1. Let $z \in C$ and $z^{-1} \in C$ be its inverse in the Jordan algebra. Let $M(z) = \{a_1, ..., a_p\}$. Then z is said to be *eutactic* if

$$z^{-1} = \sum_{i=1}^{p} \alpha_i a_i$$

for some choice of positive α_i.

In our running example, the Jordan inverse of a positive definite matrix is the same as its ordinary matrix inverse.

We also need the following definitions which are not in [Ash 77]. For more discussion of these concepts, see [Martinet 03, ch. 3].

Definition 2.2.

1. z is *semi-eutactic* if

$$z^{-1} = \sum_{i=1}^{p} \alpha_i a_i$$

for some choice of nonnegative α_i.

2. z is *weakly eutactic* if

$$z^{-1} = \sum_{i=1}^{p} \alpha_i a_i$$

for some choice of real α_i.

Note that each category includes the previous one.

Definition 2.3. Let F be a positive function on C. $z \in C$ is *F-sticky* iff there exists a finite union of hyperplanes H and a neighborhood U of 0 in V such that $z + U \subset C$ and for all $h \in U - H$, $F(z+h) > F(z)$. A "hyperplane" means a co-dimension 1 subspace of V through the origin.

Note that "F-sticky" implies "sticky with respect to F^{-1}" as defined in definition 1.1, because the inequalities are reversed in the two definitions. We can make the definition of F-sticky because we can take advantage of the linear structure of V.

3. Theorem on stickiness

Theorem 3.1. *Let $z \in C$. Then z is F-sticky if and only if z is semi-eutactic.*

Proof. Without loss of generality we multiply z by a positive scalar so that $m(z) = 1$.

The key fact is the corollary [Ash 77, p. 1047], which implies the following:

Lemma 3.2. *There is a positive number κ, depending only on z, such that*

$$\phi^{-1}(z+h) = \kappa\big(1 + (z^{-1}, h) + \text{h.o.t.}\big)$$

where h.o.t. *stands for "higher order terms in h."*

Setting $h=0$, we see that $\kappa = \phi^{-1}(z) = F(z)$, because $m(z) = 1$.

Let K_z be the hyperplane orthogonal to z^{-1}. Then $K_z = H_z - z$ where H_z is defined on p. 1046 of [Ash 77]:

$$H_z = \{v \in V | (z^{-1}, v - z) = 0\}.$$

In the Jordan algebra, $z \circ z^{-1} = \text{id}$. Therefore, $(z, z^{-1}) = N$, because it is the trace of Jordan multiplication by the identity on V.

We need the following result:

Lemma 3.3. *If we add a small $h \in V$ to z, then $M(z+h) \subset M(z)$.*

By "small," we mean "contained in an open ball of radius ϵ centered at 0," where ϵ is chosen small enough for whatever context we are in.

Proof. Without loss of generality, $m(z) = 1$. Because $z \in C$, we know that $(z, w) > 0$ for all $w \in \bar{C} - \{0\}$. Let E_R denote the exterior of the open ball of radius R about 0 in V. Let B be a compact neighborhood of 0 in V such that $z + B \subset C$. Then

$$\min\Big\{(x, y) | x \in z + B, y \in E_R \cap \bar{C}\Big\} = A(R) > 0,$$

and $A(R)$ is a linear function of R. It follows that

- There exists ϵ between 0 and 1 such that $(z, \ell) > 1 + \epsilon$ for all $\ell \in \bar{C} \cap L - \{0\} - M(z)$.

- There exists $R > 0$ such that $(z+h, \ell) > 2$ for all $h \in B$ and $\ell \in E_R \cap \bar{C}$.
- Shrinking B if necessary, $|(h, \ell)| < \epsilon/2$ for all $h \in B$ and $\ell \in \bar{C} - E_R$.

With these values of ϵ and R, we see that $(z + h, \ell) > 1 + \epsilon/2$ if $h \in B$ and $\ell \in \bar{C} \cap L - \{0\} - M(z)$. Therefore, $M(z+h) \subset M(z)$ if $h \in B$. □

We continue with the proof of Theorem 3.1. We have that $F(h+z) = m(z+h)^{-N}\phi^{-1}(z+h)$. Let us suppose that a_1 is a minimal vector for $z+h$. Then $m(z+h) = (z+h, a_1) = 1 + (h, a_1)$, so to first order

$$\begin{aligned}\kappa^{-1}F(h+z) &= \kappa^{-1}(1+(h,a_1))^{-N}\phi^{-1}(z+h)\\ &= (1-N(h,a_1))\left(1+\left(z^{-1},h\right)\right) + \text{h.o.t.}\end{aligned}$$

We obtain that to first order, $F(h+z) = F(z)(1-N(h,a_1) + (z^{-1},h)) + \text{h.o.t.}$, using the fact that $\kappa = F(z)$. This proves:

Lemma 3.4. *If h is small, and if a_1 is a minimal vector for $z+h$, then the sign of $F(z+h) - F(z)$ is the same as the sign of*

$$f(h) := -N(h, a_1) + \left(z^{-1}, h\right).$$

(If two numbers are both 0, we say they have the same sign.)

If z is weakly eutactic, then

$$z^{-1} = \sum_{i=1}^{p} \alpha_i a_i$$

for some real α_i. Because a_i are minimal vectors, we have $(z, a_i) = 1$ for all i. Therefore, $N = (z, z^{-1}) = \sum_i \alpha_i (z, a_i) = \sum_i \alpha_i$. Restating, if z is weakly eutactic, then

$$\sum_{i=1}^{p} \alpha_i = N.$$

For any $z \in V$, define $\mathcal{H}(z)$ to be the union of K_z and all the hyperplanes orthogonal to $a_i - a_j$ for all pairs $i \neq j$. If $h \notin \mathcal{H}(z)$, then $(z+h, a_i)$ attains pairwise distinct values, so that $M(z+h)$ has only one vector in it.

We continue the proof of Theorem 3.1. First, assume z is semi-eutactic, so that $\alpha_i \geq 0$ for all i. Let $h \notin \mathcal{H}(z)$ so that $M(z+h)$ has only one vector in it, which we may assume to be a_1. Then

$$f(h) = -\sum \alpha_i (h, a_1) + \left(h, \sum \alpha_i a_i\right) = \sum \alpha_i (h, a_i - a_1). \tag{3–1}$$

Because a_1 is the only minimal vector of $z+h$, we have $(z+h, a_i) > (z+h, a_1)$. Because all the a_i are minimal vectors of z, this implies $(h, a_i - a_1) > 0$ for all $i \neq 1$.

C consists of squares of invertible elements [Ash 10, Theorem 2.13]. So by [Ash 10, Corollary 2.8], z^{-1} is in C, and so we know that at least two of the α_i are not equal to 0, because each a_i lies in the boundary of C. Therefore $f(h) > 0$ and so $F(z+h) > F(z)$. We have proved that if z is semi-eutactic, then z is F-sticky.

We now prove the reverse implication. To show that a point z is not F-sticky, it suffices to find a cone A (with vertex at 0 and open in V) such that for h sufficiently small and $h \in A$, we get $f(h) < 0$. This implies that z is not F-sticky, because there is a set of directions with positive measure emanating from z in which F is decreasing.

For the following arguments, remember that C is an open convex cone. Suppose first that z is weakly eutactic but not semi-eutactic. Then, after re-ordering the indices, for some $q \leq p$, there is an expression

$$z^{-1} = \sum_{i=1}^{q} \alpha_i a_i$$

where now the a_i run over a subset of the minimal vectors of z (possibly all of them), the $\{a_i\}$ are linearly independent, all the $\alpha_i \neq 0$, and $\alpha_1 < 0$. As noted above, there are at least two terms in the sum, because the minimal vectors are in the boundary of C and z^{-1} is in C.

Lemma 3.5. *Let W be a finite-dimensional real vector space with an inner product $(\cdot,\cdot)$. Let $w_1, \ldots, w_m$ be linearly independent vectors in W, and $\lambda_1, \ldots, \lambda_m \in \mathbb{R}$. Then there exists $w \in W$ such that $(w_i, w) = \lambda_i$ for all i.*

Proof. Extend $w_1, \ldots, w_m$ to a basis. Then w is an appropriate linear combination of elements of the dual basis. □

Because $a_1, \ldots, a_q$ are linearly independent, $a_1 - a_2, a_3 - a_2, \ldots, a_q - a_2$ are also linearly independent. Therefore, using Lemma 3.5, we may choose v_0 such that $(v_0, a_1 - a_2) = 1 > 0$ and for each $j \geq 3$, $(v_0, a_j - a_2) = |\alpha_1|/3q|\alpha_j| > 0$.

Therefore, if v is in a sufficiently small open ball B centered at v_0, we have that

- $(v, a_1 - a_2) > 0$.
- For each $j \geq 3$, $(v, a_j - a_2) > 0$.
- We can bound above the ratios: $(v, a_j - a_2)/(v, a_1 - a_2) < |\alpha_1|/2q|\alpha_j|$.

Because these things remain true when v is replaced by λv for any positive λ, it follows that for any v in the cone A over B, $(v, a_1 - a_2) > 0$ and for each $j \geq 3, (v, a_j - a_2) > 0$ and $(v, a_j - a_2)/(v, a_1 - a_2) < |\alpha_1|/\ 2q|\alpha_j|$.

Now assume that h is sufficiently small so that $z + h \in C$ and $M(z+h) \subset M(z)$, and let $h \in A$. Then $(z+h, a_1 - a_2) > 0$ and for each $j \geq 3, (z + h, a_j - a_2) > 0$ and the ratios are bounded: $(h, a_j - a_2)/\ (h, a_1 - a_2) < |\alpha_1|/2q|\alpha_j|$.

In particular, a_2 is the unique minimal vector of $z + h$. Therefore, from equation (3–1) with a_2 in place of a_1, we obtain:

$$f(h) = \sum_{i=1}^{q} \alpha_i (h, a_i - a_2)$$

and the sign of $f(h)$ is the same as the sign of $F(z+h) - F(h)$.

Dividing both sides by the positive number $(h, a_1 - a_2)$ we obtain

$$\frac{f(h)}{(h, a_1 - a_2)} = \alpha_1 + \sum_{j=3}^{q} \alpha_j \frac{(h, a_j - a_2)}{(h, a_1 - a_2)}.$$

Each term in the sum is less than $|\alpha_1|/2q$ in absolute value, so the whole sum is less than $|\alpha_1|/2$ in absolute value. Because $\alpha_1 < 0$, we obtain that if $h \in A$ is sufficiently small, then $f(h) < 0$ and z is not sticky.

The remaining possibility is that z is not weakly eutactic. Then

$$z^{-1} = \sum \alpha_i a_i + \sum \beta_j b_j,$$

where a_i are minimal vectors of z, b_j are non-minimal vectors in V (that may not even be in the lattice L but are still in $\bar{C}$), and $\{a_i, b_j\}$ are linearly independent, all the α_i and β_j are nonzero, and there is at least one b_j. Let a small $\varepsilon > 0$ be given (in particular less than 1). There exists $y_0 \in V$ orthogonal to all the a_i and all the b_j for $j \neq 1$ and with $\beta_1(y_0, b_1) = -2\varepsilon$. Then for a small enough open ball B_0 centered at y_0, we have $h \in B_0$ implies $|(h, \alpha_i a_i)| < \varepsilon^2$ for all i and $|(h, N a_i)| < \varepsilon^2$ for all i, and $|(h, \beta_j b_j)| < \varepsilon^2$ for all $j \neq 1$ and $(h, \beta_1 b_1) < -\varepsilon$.

We continue to require h so small that the minimum of $z + h$ is still given by one of the a_i's. Then by Lemma 3.4, $f(h) = -N(h, a_i) + (h, z^{-1})$ for some i. If ε is sufficiently small and $h \in B_0$, the dominant term on the right-hand side is $(y_0, \beta_1 b_1)$, and $f(h) < 0$.

Because $f(h)$ is a linear function of h, we conclude that if h is in the cone over B_0, we still have $f(h) < 0$. So z is not F-sticky. □

4. Data analysis

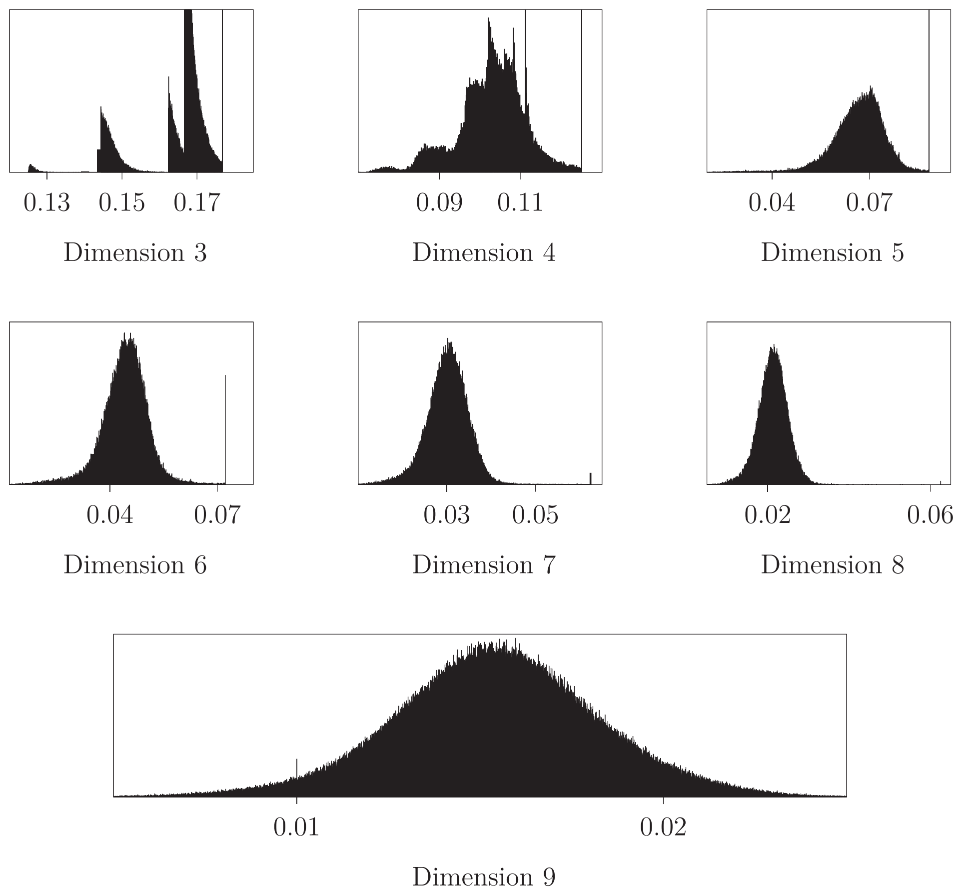
We performed our experiments for various values of n from 2 to 16. We report here on n from 2 to 9, because the results for n larger than 9 were not qualitatively different from those for $n = 9$.

Our programs used several libraries: gmp [gmp 18], NTL [Shoup 18], and fplll [The FPLLL Development Team 16]. NTL has its internal representation for arbitrary precision integers, while fplll and gmp use a different representation. We wrote simple functions to translate between the two.

The analysis above uses a symmetric matrix to define a sphere packing. The library fplll instead uses the standard inner product, and, given a basis of a lattice, computes the shortest vector in that lattice with respect to the standard inner product. The basis of that lattice must be in the form of integer vectors.

If the integers are too small, those vectors will typically not be able to approximate good lattice packings very well (typically, the best lattice packings have irrational bases). If the integers are too large, the computer programs will run too slowly. We compromised by starting with a seed matrix randomly chosen (using a random number generator from gmp), containing integer entries between $-N$ and N for an integer $N \approx 1000$. The columns of this matrix are used as a basis for the lattice. We used entries with a mean of 0. We computed the shortest vector in the lattice using the shortest_vector function of fplll, which in turn gave us the density of the lattice packing corresponding to a matrix. We call this density the "efficiency" of the matrix. We think of this matrix as the "parent."

We then altered each entry in the matrix randomly, initially by adding -1, 0, or 1, and then computed the efficiency of the "child" matrix. If the child matrix gives a denser packing than the parent, it becomes the new parent, and we repeat. After five tries, if none of the five children are more efficient than the "parent," we double the possible modification size, so that now we change each entry randomly by -2, ..., 2. We continue doubling the possible modification whenever five children are no denser than the parent, and reset the maximum modification to 1 if a child is denser than its parent and becomes the new parent. Every 300 steps without any improvement in density, we double all of the entries in the matrix. This has the effect of making each step we take relatively smaller, allowing us to try to thread our way through a bottleneck. If there are no improvements after 1000 trials, the algorithm ceases. To prevent programs running forever, we also capped the total number of steps,



typically at 10^6. This cap was seldom reached, never more than 100 times in 10^5 or 10^6 experiments. In other words, the random walk almost always got stuck somewhere.

We present our results in the form of histograms. In each, the horizontal axis shows Φ/V_n, the so-called "center density" of the lattice. The vertical axis is a count of the number of times that our program halted at a lattice of the given density. For most dimensions, the program was run 10^5 times. In dimension 9, the program was run 10^6 times in the hope that at least one such run would produce a packing with density equal to or greater than the best known packing in that dimension.

In the histogram for dimension 3, we have cut off the top of the picture, so that we can use a scale that shows the structure near all five eutactic points. The thick spike around 0.17 actually continues upwards, thinning into a spike above 0.166. If we included all of this spike and the spike at 0.177, the part of the histogram to the left becomes almost invisible. Similarly two of the spikes in dimension 4 and one spike in dimension 5 have been cut off at the top.

Our algorithms nearly always found the densest packing in dimension 2. We therefore omit this histogram, because it is essentially a single peak at the densest packing.

From now on, the term "density" refers to the center density of a given lattice packing. In dimension 3, we can see that the algorithm can get "stuck" near 0.125, corresponding to the identity matrix, and also near 0.144, 0.162, 0.166, and finally 0.176777, the densest packing and the thin line at the right of the histogram. These are the lattice-packing densities of the 5 $GL_3(\mathbb{Z})$-orbits of eutactic forms in dimension 3. In dimensions 4 and higher, there are far more eutactic lattices, and so it is harder to spot them as different densities at which our algorithm halted.

The lattices at which our algorithm sticks, which are very likely to be semi-eutactic lattices or close approximations thereto, clearly have a unimodal distribution. This does not appear to be a normal distribution, because the tails are too thick. Our histograms are saying something about the number of semi-eutactic lattices with a given sphere-packing density.

We are led to formulate the following conjecture.

Conjecture 4.1. *Let $n \geq 9$. Let M_n be the absolute maximum of Φ in dimension n. Let x be a positive real variable, and let $f_n(x)$ be the real-valued function such* *that*

$$\int_a^b f_n(x)\, dx$$

is the number of semi-eutactic lattices in $\mathbb{R}^n$ *with sphere-packing density in the interval* $[M_n a, M_n b]$ *divided by the number of all semi-eutactic lattices in* $\mathbb{R}^n$*. With this normalization,* $f_n(x)$ *is supported on* $[0, 1]$ *and has total area 1. Then* $f_n(x)$ *is unimodal, and*

$$f := \lim_{n\to\infty} f_n$$

exists.

It would be interesting to make this conjecture more precise by predicting the exact shape of the limiting graph (as n tends to infinity) of $f(x)$, or at least its mean and standard deviation, but we do not have sufficient data for doing so.

Perhaps, the conjecture should take account of the size of the basin of attraction of each semi-eutactic lattice with respect to our algorithm. Because there are huge numbers of semi-eutactic lattices when n is large, we guess that these sizes would average out so as not to have an effect on our conjecture.

We believe that the number of inequivalent semi-eutactic lattices grows at least exponentially with n. The growth rate of the number of semi-eutactic or eutactic lattices seems not to be known. See [Bacher 18] for a proof that the number of perfect lattices grows at least exponentially with n. However, it seems to be likely that for large n most of the perfect lattices will not be eutactic. For example, Riener [Riener 06] shows that for $n = 8$, there are 20,916 inequivalent perfect forms, of which 2408 are eutactic (and hence extreme) and an additional 28 are semi-eutactic but not eutactic. On the other hand, we expect that there are vastly more eutactic lattices than extreme forms, since the former occur as critical points with varying indices of the sphere-packing function whereas the extreme forms only have maximal index.

Acknowledgements

We wish to thank Roland Bacher, Jacques Martinet, Gabriele Nebe, and Christophe Soulé for help in tracking down recent results about the number of eutactic lattices in various dimensions. We thank the referee of an earlier version of this paper for many helpful comments.

References

[Ash 77]. A. Ash. "On Eutactic Forms." *Canad. J. Math.* 29:5 (1977), 1040–1054. https://cms.math.ca/openaccess/cjm/v29/cjm1977v29.1040-1054.pdf. DOI: 10.4153/CJM-1977-101-2.

[Ash 10]. A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai. *Smooth Compactifications of Locally Symmetric Varieties*, Cambridge Mathematical Library. Second Edition. Cambridge: Cambridge University Press, 2010. With the Collaboration of Peter Scholze. DOI: 10.1017/CBO9780511674693.

[Bacher 18]. R. Bacher. "On the Number of Perfect Lattices." Available online (https://arxiv.org/abs/1704.02234), 2018.

[Barnes and Cohn 75]. E. S. Barnes and M. J. Cohn. "On the Inner Product of Positive Quadratic Forms." *J. London Math. Soc.* 12:1 (1975), 32–36. DOI: 10.1112/jlms/s2-12.1.32.

[Cohn 17]. H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. Viazovska. "The Sphere Packing Problem in Dimension 24." *Ann. Math.* 185:3 (2017), 1017–1033. DOI: 10.4007/annals.2017.185.3.8.

[Conway and Sloane 93]. J. H. Conway and N. J. A. Sloane. *Sphere Packings, Lattices and Groups*, Grundlehren Der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Vol. 290, Second Edition. New York: Springer-Verlag, 1993. With additional contributions by E. Bannai, R. E. Borcherds, J. Leech, S. P. Norton, A. M. Odlyzko, R. A. Parker, L. Queen, and B. B. Venkov. DOI: 10.1007/978-1-4757-2249-9.

[The FPLLL Development Team 16]. The FPLLL Development Team. "fplll, a Lattice Reduction Library." Available online (https://github.com/fplll/fplll), 2016.

[gmp 18]. gmp. "The GNU Multiple Precision Library." Available online (https://gmplib.org), 2018.

[Marcotte and Torquato 13]. É. Marcotte and S. Torquato. "Efficient Linear Programming Algorithm to Generate the Densest Lattice Sphere Packings." *Phys. Rev. E.* 87 (2013), 063303–063301–9. DOI 10.1103/PhysRevE.87.063303.

[Martinet 03]. J. Martinet. *Perfect Lattices in Euclidean Spaces*. Grundlehren Der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Vol. 327. Berlin: Springer-Verlag. 2003. DOI: 10.1007/978-3-662-05167-2.

[Shoup 18]. V. Shoup. "NTL: A Library for Doing Number Theory." Available online (http://www.shoup.net/ntl), 2018.

[Riener 06]. C. Riener. "On Extreme Forms in Dimension 8." *J. Théor. Nombres Bordeaux.* 18:3 (2006), 677–682. English, with English and French summaries).

[Voronoi 08]. G. Voronoi. "Nouvelles applications des paramètres continus à la théorie des formes quadratiques. Premier mémoire. Sur quelques propriétés des formes quadratiques positives parfaites." *J. Reine Angew. Math.* 133 (1908), 97–102. (French). DOI: 10.1515/crll.1908.133.97.