MATH1007
Homework 9
Answers

1. On Monday morning, the price of a share of a particular stock is \$6. Suppose that the price increases by the same percentage amount on Monday, Tuesday, Wednesday, Thursday, and Friday, and at the end of the day on Friday, the price of the stock is \$10. What was the percentage increase each day of the week?

Answer: Suppose that the percentage increase is r, and let $R = 1 + r$. Then we have $6R^5 = 10$, or $R^5 \approx 1.6667$. That gives $R \approx 1.1076$, and therefore $r \approx 10.76\%$.

2. Suppose in a geometric sequence, $P_5 = 7$ and $P_{10} = 6$. What is P_3? Compute the answer to 4 decimal places.

Answer: If R is the common ratio, then we have $P_{10} = P_5 R^5$, and so then $7R^5 = 6$. Solving gives $R \approx 0.9696$. We also have $P_5 = R^2 P_3$, and therefore $P_3 \approx 7.4452$.

3. Compute the Banzhaf power of each player in the following voting systems:
(*a*) $[7 : 4, 3, 2, 1]$.
(*b*) $[8 : 4, 3, 2, 1]$.
(*c*) $[9 : 4, 3, 2, 1]$.

Answer: Here's a list of winning coalitions with the critical players underlined:
(*a*) $\{\underline{P_1}, \underline{P_2}\}$ $\{\underline{P_1}, \underline{P_2}, P_3\}$ $\{\underline{P_1}, \underline{P_2}, P_4\}$ $\{\underline{P_1}, \underline{P_3}, \underline{P_4}\}$ $\{\underline{P_1}, P_2, P_3, P_4\}$. Therefore, $\beta_1 = \frac{5}{10}$, $\beta_2 = \frac{3}{10}$, $\beta_3 = \frac{1}{10}$ and $\beta_4 = \frac{1}{10}$.
(*b*) $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$ $\{\underline{P_1}, \underline{P_2}, \underline{P_4}\}$ $\{\underline{P_1}, \underline{P_2}, P_3, P_4\}$. Therefore, $\beta_1 = \frac{3}{8}$, $\beta_2 = \frac{3}{8}$, $\beta_3 = \frac{1}{8}$, and $\beta_4 = \frac{1}{8}$.
(*c*) $\{\underline{P_1}, \underline{P_2}, \underline{P_3}\}$ $\{\underline{P_1}, \underline{P_2}, \underline{P_3}, P_4\}$. Therefore, $\beta_1 = \frac{2}{6}$, $\beta_2 = \frac{2}{6}$, $\beta_3 = \frac{2}{6}$, and $\beta_4 = 0$.

4. Compute the Banzhaf power of each player in the voting system $[6 : 5, 1, 1, 1, 1, 1]$. Hint: Compute the power of P_1, and use that to compute the power of the other 5 players.

Answer: First, notice that P_1 has veto power, so she is always underlined in every winning coalition. There are 5 winning coalitions with 2 players, and as always, in a 2-player coalition, both players are critical: $\{\underline{P_1}, \underline{P_2}\}$ $\{\underline{P_1}, \underline{P_3}\}$ $\{\underline{P_1}, \underline{P_4}\}$ $\{\underline{P_1}, \underline{P_5}\}$ $\{\underline{P_1}, \underline{P_6}\}$.

How many 3-player coalitions are there? We can pick any 2 of the last 5 players to get a winning coalition with 3 players. There are $\binom{5}{2} = 10$ of these. Just to reassure you, here is the list:

$$\{\underline{P_1}, P_2, P_3\} \quad \{\underline{P_1}, P_2, P_4\} \quad \{\underline{P_1}, P_2, P_5\} \quad \{\underline{P_1}, P_2, P_6\} \quad \{\underline{P_1}, P_3, P_4\}$$
$$\{\underline{P_1}, P_3, P_5\} \quad \{\underline{P_1}, P_3, P_6\} \quad \{\underline{P_1}, P_4, P_5\} \quad \{\underline{P_1}, P_4, P_6\} \quad \{\underline{P_1}, P_5, P_6\}$$

That's another 10 critical points for P_1.

How many 4-player coalitions are there? We can pick 3 of the last 5 players, so there are $\binom{5}{3} = 10$ of them. Yet 10 more critical points for P_1. How about 5-player coalitions?

There are $\binom{5}{4} = 5$ of them, with another 5 critical points for P_1. Finally, there is the grand coalition, which contributes 1 more critical point for P_1.

In total, P_1 has $5 + 10 + 10 + 5 + 1 = 31$ critical points. Each other player has 1 critical point. Therefore, $\beta_1 = \frac{31}{36}$, and $\beta_2 = \beta_3 = \beta_4 = \beta 5 = \beta_6 = \frac{1}{36}$.

5. There are 100 members of the United States Senate. How many ways are there to pick a committee consisting of 3 of them?

Answer: This is precisely $\binom{100}{3} = 161700$.

6. There are 20 female Senators and 80 male Senators. How many ways are there to pick a committee consisting of

(*a*) 3 women?
(*b*) 3 men?
(*c*) 2 women and 1 man?
(*d*) 2 men and 1 woman?

What is the relationship between your answers to this problem and to the previous problem?

Answer: (*a*) This is $\binom{20}{3} = 1140$. (*b*) This is $\binom{80}{3} = 82160$. (*c*) This is a more interesting question. There are $\binom{20}{2} = 190$ ways to pick the 2 women, and then there are 80 choices for the man, so in total there are $190 \cdot 80 = 15200$ to pick the committee this way. (*d*) Now there are $\binom{80}{2} = 3160$ ways to pick the 2 men, and 20 choices for the woman, so that in all there are $3160 \cdot 20 = 63200$ ways.

The point of the problem is that the sum of these 4 numbers must be the answer to the previous problem $1140 + 82160 + 15200 + 63200 = 161700$.

7. The Venusian War Council consists of 3 little green men, and 3 giant rabbits. In order to declare war on their enemies, at least 2 of the little green men must vote yes, and at least 2 of the giant rabbits must vote yes.

(*a*) How many winning coalitions are there?
(*b*) What is the Banzhaf power of each of the 6 members of the Venusian War Council?
(*c*) Explain why it is *not* possible to describe the Venusian War council as a weighted voting system.

Answer: (*a*) We need to do this systematically. Count the number of coalitions with 4 members. That must be 2 little green men, and 2 giant rabbits. There are 3 ways to pick the 2 little green men, and 3 ways to pick the giant rabbits, so in all there are 9 coalitions with 4 members.

Next, count the coalitions with 3 little green men and 2 giant rabbits: there are 3 of them. Now count the coalitions with 2 little green men and 3 giant rabbits: there are also 3 of them. Finally, count the number of coalitions with 6 members: 1. In all, there are $9 + 3 + 3 + 1 = 16$ winning coalitions. (You could get here much quicker if you realize that the answer is $4 \cdot 4$.)

(*b*) Because the situation is completely symmetric, the Banzhaf power of each member is $\frac{1}{6}$.

(*c*) Suppose that we could describe the system as $[q : m, m, m, r, r, r]$, where m is the weight of a little green man and r is the weight of a giant rabbit. We must have $2m+2r \geqslant q$, because 2 little green men and 2 giant rabbits form a winning coalition. We also must have $m + 3r < q$ and $3m + r < q$, because one man and three rabbits is not a winning coalition, and neither is three men and one rabbit. But if $m + 3r < q$ and $3m + r < q$, we can add to get $4m + 4r < 2q$. Now, divide by 2 to get $2m + 2r < q$, which contradicts $2m + 2r \geqslant q$.

8. The Martian War Council consists of 3 little green women, and a giant squid. In order to declare war on their enemies, at least 2 of the little green women must vote yes, and the giant squid must also vote yes.

(*a*) What is the Banzhaf power of each of the 4 members of the Martian War Council?

(*b*) Is it possible to describe the Martian War Council using a weighted voting system? If so, find the weights; if not, explain why it is not possible.

Answer: (*a*) Call the players S, W_1, W_2, and W_3. The winning coalitions are

$$\{\underline{S}, \underline{W_1}, \underline{W_2}\} \quad \{\underline{S}, \underline{W_1}, \underline{W_3}\} \quad \{\underline{S}, \underline{W_2}, \underline{W_3}\} \quad \{\underline{S}, W_1, W_2, W_3\}$$

Therefore, the power of the squid is $\frac{4}{10} = \frac{2}{5}$, and each of the women has power $\frac{2}{10} = \frac{1}{5}$.

(*b*) Suppose that the system can be described by $[q : s, w, w, w]$. We must have $3w < q$, $s + 2w \geqslant q$, and $s + w < q$. A bit of trial and error gives one solution: $[4 : 2, 1, 1, 1]$.

9. To try to find an election system which avoids some of the problems associated with plurality voting, someone suggests the following complicated system:

(*i*) If there is a Condorcet winner (that is, if there is a winner of all head-to-head match-ups), then that person is the winner.

(*ii*) If there is no Condorcet winner, then use Borda count to determine the winner.

Here is a preference schedule:

	Number of voters		
	7	**6**	**2**
First choice	P	D	W
Second choice	D	W	P
Third choice	W	P	D

(*a*) Who is the winner using the system outlined above?

(*b*) Suppose that candidate W (a losing candidate) withdraws from the election. Who wins the resulting election between P and D?

(*c*) Comment on this situation.

Answer: (*a*) We first try to find a Condorcet winner and fail: W beats P 8-7, P beats D 9-6, and D beats W 13-2. Therefore, we fall back on the Borda count, and compute that P has 31 points, D has 34 points, and W has 25 points. Therefore, the winner is D.

(b) If W withdraws, then we see that P beats D, and therefore P is a Condorcet candidate and wins the election.

(c) This example shows that the method above violates IIA. Removing a losing candidate changed the winning candidate.

10. A weighted voting system with 3 players has exactly 3 winning coalitions: $\{P_1, P_2, P_3\}$, $\{P_1, P_2\}$, and $\{P_1, P_3\}$.

(a) Find the critical players in each winning coalition.

(b) Find the Banzhaf power distribution of this system.

(c) Find the Shapley–Shubik distribution of this system.

Answer: (a) A bit of thought lets you underline the critical players in each winning coalition: $\{\underline{P_1}, P_2, P_3\}$, $\{\underline{P_1}, \underline{P_2}\}$, and $\{\underline{P_1}, \underline{P_3}\}$.

(b) Therefore, $\beta_1 = \frac{3}{5}$, $\beta_2 = \frac{1}{5}$, and $\beta_3 = \frac{1}{5}$.

(c) There are $3! = 6$ ordered coalitions, and we underline the pivotal player in each one.

$$\langle P_1, \underline{P_2}, P_3\rangle \quad \langle P_1, \underline{P_3}, P_2\rangle \quad \langle P_2, \underline{P_1}, P_3\rangle \quad \langle P_2, P_3, \underline{P_1}\rangle \quad \langle P_3, P_2, \underline{P_1}\rangle \quad \langle P_3, \underline{P_1}, P_2\rangle$$

We have $\sigma_1 = \frac{4}{6}$, and $\sigma_2 = \sigma_3 = \frac{1}{6}$.