Rob Gross
Homework 12
Mathematics 2216.01
Due October 14, 2022

1. Define a sequence of real numbers with the definitions

$$x_1 = 1$$
$$x_n = \sqrt{x_{n-1} + 1}$$

On the examination, we showed that $x_n \leq 2$ for all positive integers n.

(a) Show using induction that $x_n \leq x_{n+1}$ for all positive integers n.

(b) A theorem from real analysis now tells us that $\lim_{n\to\infty} x_n$ exists. (The theorem states that a sequence of numbers that is increasing and bounded must have a limit.)

Suppose that $\lim_{n\to\infty} x_n = L$. What is L? State your answer in terms of radicals. *Hint*: Start with the equation $x_n = \sqrt{x_{n-1} + 1}$ and compute $\lim_{n\to\infty}$ of that equation.

2. Prove that if k and n are integers, with $n \geq 2$ and $k \geq 0$, then

$$F_n F_{n+k} - F_{n-1} F_{n+k+1} = (-1)^{n+1} F_{k+1}.$$

Note: One challenge is deciding whether to use induction on k or n.

3. Find three complex numbers z so that $z^3 = i$ in two different ways:

(a) First, write $(a + bi)^3 = i$, expand using the binomial theorem, and solve for the real numbers a and b.

(b) Second, use de Moivre's Theorem or Euler's Formula, whichever you prefer.