Mathematics 3310.01
Homework 1
Due September 7, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Prove using induction that $2^n < n!$ if n is an integer and $n \geq 4$.

2. Prove using induction that $n < 1.5^n$ if n is an integer and $n \geq 2$.

3. Let d be the greatest common divisor of 21124 and 30055. Use the Euclidean algorithm to find d, and to find integers a and b so that $21124a + 30055b = d$.

4. Let a, b, and c be positive integers. Suppose that $a|bc$ and $(a,b)|c$. Prove that $a|c^2$. Here, as usual, (a,b) is our notation for the greatest common divisor of a and b.

5. Suppose that R is a commutative ring with no zero divisors. Suppose that $r \in R$ solves the equation $r^2 = 1$. Show that $r = 1$ or $r = -1$.

6. In $\mathbf{F}_3 = \mathbf{Z}/3\mathbf{Z}$, the equation $x^2 + 1 \equiv 0$ has no solution. Just as we have done working with $\mathbf{R}$, we can invent a solution to this congruence. Let's call it α, so as not to confuse it with the imaginary number i, and we will use the rule that $\alpha^2 \equiv -1 \equiv 2$ when evaluating expressions. Let R be the ring

$$\{a + b\alpha \mid a, b \in \mathbf{F}_3\}$$

There are 9 elements in R, including 0, 1, and 2.
 (a) Write down the other 6 elements.
 (b) Write out the addition and multiplication tables for R.
 (c) Show that every non-zero element of R is a unit. This shows (no more justification needed) R is a field with 9 elements.

7. Suppose that we repeat the previous exercise in $\mathbf{F}_2 = \mathbf{Z}/2\mathbf{Z}$. Even though there *is* a solution to $x^2 + 1 \equiv 0$ in $\mathbf{F}_2$, suppose that we didn't notice and we invent a solution β and use the rule $\beta^2 \equiv -1 \equiv 1$ when evaluating expressions. Let S be the ring

$$\{a + b\beta \mid: a, b \in \mathbf{F}_2\}$$

Then $S = \{0, 1, \beta, 1 + \beta\}$.
 (a) Write out the addition and multiplication tables for S.
 (b) What are the units in S? What are the zero divisors?
 (c) Is S a field?