Mathematics 3310.01
Homework 2
Due September 14, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Suppose that R and S are commutative rings, and $f : R \to S$ is a homomorphism. Suppose that a is a unit of R. Prove that $f(a)$ is a unit of S, and in fact prove that $f(a^{-1}) = f(a)^{-1}$.

2. Use the previous problem to show that the only homomorphism $f : \mathbf{Q} \to \mathbf{Q}$ is the identity function $f(x) = x$.

3. Suppose that d and m are integers larger than 1, and $d|m$. Show that the homomorphism $f : \mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ defined by $f(k) = [k]_d$ induces a homomorphism $\overline{f} : \mathbf{Z}/m\mathbf{Z} \to \mathbf{Z}/d\mathbf{Z}$ defined by $f([k]_m) = [k]_d$.

4. Let R be a commutative ring, and $M_2(R)$ be the set of 2×2 matrices with entries in R. Show that the function $f : R \to M_2(R)$ defined by

$$f(r) = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

is a homomorphism.

5. Suppose that F is a field with finitely many elements. Show that F has characteristic p for some prime p.

6. Suppose that r and s are relatively prime integers, each at least 2. Suppose that the order of a in $\mathbf{Z}/r\mathbf{Z}$ is d, and the order of a in $\mathbf{Z}/s\mathbf{Z}$ is e. Show that the order of a in $\mathbf{Z}/rs\mathbf{Z}$ is the least common multiple of d and e.

7. Use the previous problem to find the order of 2 in $\mathbf{Z}/77\mathbf{Z}$.

8. What is the order of 2^{10} in $\mathbf{Z}/77\mathbf{Z}$?

9. Find a numerical example in which r and s are not relatively prime, and the conclusion of problem 6 is false.

10. Let $m = 2^{15} - 1 = 32767$. Show

   (a) The order of 2 in $\mathbf{Z}/m\mathbf{Z}$ is 15.

   (b) 15 does not divide $m - 1$.

Explain why you now can conclude that m is not prime, even though we do not know any factors of m.