Mathematics 3310.01
Homework 5
Due October 12, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Let $p(x) \in \mathbf{Z}/6\mathbf{Z}[x]$ be the polynomial $4x^3 + 3x$. Find another polynomial $q(x) \in \mathbf{Z}/6\mathbf{Z}[x]$ so that $p(x) \not\equiv q(x) \pmod 6$, but $p(n) \equiv q(n) \pmod 6$ for all integers n.

2. Let G be the group U_{45}, the positive integers which are less than 45 and relatively prime to 45, with group operation multiplication. You may assume without checking that there are 24 elements in G.

(a) Let H_1 be the subgroup generated by 4. There are 6 elements in H_1. Find them, and find the order of each of the 4 cosets in G/H_1.

(b) Let H_2 be the subgroup generated by 11. There are 6 elements in H_2. Find them, and find the order of each of the 4 cosets in G/H_2.

3. Find an element $x \in \mathbf{Z}/105\mathbf{Z}$ so that

$$x \equiv 2 \pmod 3$$
$$x \equiv 4 \pmod 5$$
$$x \equiv 3 \pmod 7$$

or explain why no such element x exists.

4. As usual, write $\mathbf{F}_2$ as a shorter symbol for $\mathbf{Z}/2\mathbf{Z}$. Let R be the ring $\mathbf{F}_2 \times \mathbf{F}_2$. The ring R has 4 elements.

(a) Write the addition table for R.

(b) Write the multiplication table for R.

(c) Is R a field?

(d) Let G be the group $(R, +, 0)$. (In other words, forget that you can multiply elements of R, and just consider adding them.) List the order of each of the 4 elements in G.

5. We can define a ring homomorphism $f : \mathbf{Z}/12\mathbf{Z} \to \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/6\mathbf{Z}$ with the formula $f([n]_{12}) = ([n]_2, [n]_6)$. You do not need to check that this is a ring homomorphism.

(a) How many elements are in the image of f? (The *image* of a function $f : A \to B$ is defined to be the set $f(A)$.)

(b) Compute the kernel of f.