Mathematics 3310.01
Homework 6
Due October 19, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Suppose that G_1 is a group with a subgroup H_1, G_2 is a group with a subgroup H_2, and $f : G_1 \to G_2$ is a group homomorphism. Prove or give a counterexample to each of these two statements:

(*a*) $f(H_1)$ is a subgroup of G_2.

(*b*) $f^{-1}(H_2)$ is a subgroup of G_1.

Remember that if $f : X \to Y$ is any function, with $A \subset X$ and $B \subset Y$, then

$$f(A) = \{f(a) : a \in A\}$$
$$f^{-1}(B) = \{x \in X \ : f(x) \in B\}$$

Remember that f^{-1} is typically *not* a function, but it still makes sense to take the inverse image of a set.

To give a counterexample, you must give specific groups G_1 and G_2, a specific group homomorphism f, and a specific subgroup that makes the given statement false.

2. Let $f(x) = x^4 + 2x^2 + 3x + 2 \in \mathbf{F}_7[x]$, and $d(x) = 3x^3 + x + 5 \in \mathbf{F}_7[x]$. Find the quotient q and remainder r so that $f = qd + r$ with $r = 0$ or $\deg(r) < 3$, and, of course, $q, r \in \mathbf{F}_7[x]$. You can write your work by hand on a separate sheet of paper.

3. Let $f(x) = x^4 + 2x^2 + 3x + 2 \in \mathbf{F}_{11}[x]$, and $d(x) = 3x^3 + x + 5 \in \mathbf{F}_{11}[x]$. Find the quotient q and remainder r so that $f = qd + r$ with $r = 0$ or $\deg(r) < 3$, and, of course, $q, r \in \mathbf{F}_{11}[x]$. You can write your work by hand on a separate sheet of paper.

4. Let $f(x) = x^4 + 2x^2 + 3x + 2 \in \mathbf{F}_{13}[x]$, and $g(x) = 3x^3 + x + 5 \in \mathbf{F}_{13}[x]$. Use the Euclidean algorithm to find d, the monic greatest common divisor of f and g. You can write your work by hand on a separate sheet of paper.

5. Suppose that m and n are positive integers. Show that the greatest common divisor in $\mathbf{Q}[x]$ of $x^m - 1$ and $x^n - 1$ is $x^d - 1$, where $d = (m, n)$. HINT: Surprisingly, this can be done by induction.