Mathematics 3310.01
Homework 7
Due October 26, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Let R be the ring $\mathbf{F}_2 \times \mathbf{F}_2$, and let S be the ring $\mathbf{Z}/4\mathbf{Z}$.
   (a) Show that there is no ring homomorphism $f : R \to S$.
   (b) Find a ring homomorphism $g : S \to R$.

2. Now consider $G = \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ just as a group, with group operation addition, and $H = \mathbf{Z}/4\mathbf{Z}$ again with group operation addition.
   (a) Find a nontrivial group homomorphism $f : G \to H$.
   (b) Find a nontrivial group homomorphism $f : H \to G$.

"Nontrivial" means a homomorphism other than the one that maps every element in the domain to the identity element in the codomain. In this case, that means finding a homomorphism that does not send every element of the domain to 0.

3. Suppose that m and n are positive relatively prime integers larger than 1. We know that we can find integers x and y so that $mx + ny = 1$. Show that we can choose those integers so that $|x| < n$ and $|y| < m$.

4. Let F be a field. Suppose that $f, g \in F[x]$ are irreducible polynomials. We know that we can find $a, b \in F[x]$ so that $af + bg = 1$. Show that we can choose a and b so that $\deg(a) < \deg(g)$ and $\deg(b) < \deg(f)$.

5. Let $R = \mathbf{Z}/21\mathbf{Z}[x]$. Because $\mathbf{Z}/21\mathbf{Z}$ is a commutative ring with zero divisors, we know that R might have different properties from $F[x]$ when F is a field.
   (a) Find a and b nonzero elements of $\mathbf{Z}/21\mathbf{Z}$ so that $ab = 0$.
   (b) Show that in R, $x^2 - (a + b)x$ has two different factorizations into irreducible polynomials.
   (c) Show that in R, we can find an irreducible polynomial p so that $p|fg$, $p \nmid f$, and $p \nmid g$.

6. Factor $x^4 + 1$ into irreducible factors in
   (a) $\mathbf{R}[x]$
   (b) $\mathbf{C}[x]$.

7. Using mostly trial and error, find all irreducible polynomials in $\mathbf{F}_2[x]$ of degree less than or equal to 4.