Mathematics 3310.01
Homework 9
Due November 16, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Suppose that D is an integral domain. Remember that in an integral domain, if $ab = ac$ and $a \neq 0$, then $b = c$. That property will be helpful in these next few problems.

Let

$$M = \{(d_1, d_2) \in D \times D : d_2 \neq 0\}.$$

Define a relation on M by saying that if $(d_1, d_2), (d_3, d_4) \in M$, then $(d_1, d_2) \sim (d_3, d_4)$ if and only if $d_1 d_4 = d_2 d_3$. Show that this relation is an equivalence relation.

2. Now define two different binary operations on M:

$$(d_1, d_2) \oplus (d_3, d_4) = (d_1 d_4 + d_2 d_3, d_2 d_4).$$
$$(d_1, d_2) \otimes (d_3, d_4) = (d_1 d_3, d_2 d_4).$$

You may assume without checking that these operations are commutative.

(a) Show that $(0, 1)$ is an identity element for $\oplus$.
(b) Show that $(1, 1)$ is an identity element for $\otimes$.

Suppose that $(d_1, d_2), (d_3, d_4), (e_1, e_2), (e_3, d_4) \in M$, and $(d_1, d_2) \sim (e_1, e_2)$ and $(d_3, d_4) \sim (e_3, e_4)$.

(c) Show that $(d_1, d_2) \oplus (d_3, d_4) \sim (e_1, e_2) \oplus (e_3, e_4)$.
(d) Show that $(d_1, d_2) \otimes (d_3, d_4) \sim (e_1, e_2) \otimes (e_3, e_4)$.

3. Suppose that R_1 and R_2 are commutative rings, and $f : R_1 \to R_2$ is a ring homomorphism. Prove or give a counterexample:

(a) If f is surjective and R_1 is an integral domain, then R_2 is an integral domain.
(b) If f is injective and R_1 is an integral domain, then R_2 is an integral domain.

A counterexample means finding specific rings R_1 and R_2 and clearly explaining the homomorphism f.

4. Suppose that R is a commutative ring and that I and J are ideals of R. Define

$$I + J = \{i + j : i \in I, j \in J\}.$$

Show that $I + J$ is an ideal of R.

5. In $\mathbf{Z}$, take any two positive integers m and n. Let $I = (m)$, the ideal consisting of multiples of m, and $J = (n)$. Define $I + J$ as in the last problem, and let $d = (m, n)$. Show that $I + J = (d)$.