Mathematics 3310.01
Homework 10
Due November 30, 2018

Please remember that if your submission is longer than one page, you must use a stapler or paper clip.

1. Let R be a commutative ring, and suppose that I and J are ideals of R. Prove that $I \cap J$ is an ideal of R.

2. Let m and n be positive integers greater than 1. Let k be the least common multiple of m and n. Show that $(m) \cap (n) = (k)$. Here, as usual, (m) means the ideal containing all multiples of m.

3. We have seen that if I and J are ideals of a commutative ring R, there are two ways to construct new ideals: $I + J$ and $I \cap J$. There is another useful way to construct a new ideal. The set consisting of products of all elements of I with all elements of J is not closed under addition, so it is not an ideal. Instead, we define

$$IJ = \{i_1 j_1 + i_2 j_2 + \cdots + i_k j_k : i_1, i_2, \ldots, i_k \in I, j_1, j_2, \ldots, j_k \in J\}$$

Show that IJ is an ideal.

4. On a previous homework, we decomposed $\frac{1}{(x^2+x+1)(x^2+2x+1)}$ into partial fractions. Use that partial fraction decomposition to compute

$$\int \frac{dx}{(x^2+x+1)(x^2+2x+1)}.$$

5. Write a multiplication table for $\mathbf{F}_2[x]/(x^3 + x^2 + 1)$, a field with 8 elements. You can omit the row and column corresponding to multiplication by 0. Rather than using x, write β for the element of the field that satisfies $\beta^3 + \beta^2 + 1 = 0$.

6. Write a multiplication table for $\mathbf{F}_3[x]/(x^2 + 1)$, a field with 9 elements. You can omit the row and column corresponding to multiplication by 0. Rather than using x, write γ for the element of the field that satisfies $\gamma^2 + 1 = 0$.

7. Continuing the previous problem, let K be the field with 9 elements $\{a + b\gamma : a, b \in \mathbf{F}_3\}$. Factor the polynomial $x^2 + x + 2$ using the elements of K.

8. The polynomial $x^6 + 1$ is *not* irreducible in $\mathbf{F}_2[x]$. Find a factorization of that polynomial, and use that factorization to show that $\mathbf{F}_2[x]/(x^6 + 1)$ is not an integral domain.