MATH4410
Homework 3
Due February 26, 2021
Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EST on February 26. Please name your file hw03-lastname-firstname.pdf. For example, my solution file is hw03-gross-robert.pdf.

I will try to acknowledge receipt of each e-mail.

1. Consider the initial value problem

$$y' = y^{1/3}, \qquad y(0) = 0. \tag{1}$$

(*a*) Show that $y = -(\frac{2}{3}t)^{3/2}$ solves the differential equation.

(*b*) Fix any positive real number s, fix a choice of plus or minus sign, and define

$$\phi(t) = \begin{cases} 0 & 0 \le t \le s \\ \pm\left[\frac{2}{3}(t-s)\right]^{3/2} & s \le t \end{cases}$$

Show that $\phi(t)$ is continuous.

(*c*) Show that $\phi(t)$ is differentiable. The only nonobvious part is that you must show that $\phi'(s)$ is defined.

(*d*) Show that $\phi(t)$ solves the initial value problem (1).

2. Define

$$g(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $g(t)$ is continuous. Therefore our theorems tell us that there is a unique solution (defined for all real t) to the initial value problem

$$y' + 2y = g(t), \qquad y(0) = A$$

for any constant A. Find the solution, which of course will depend on A. Be sure to find the solution for negative values of t as well as positive ones.

3. Suppose that $\alpha(t)$ and $\beta(t)$ are continuous functions, and we want to solve the differential equation

$$y' + \alpha(t)y = \beta(t)y^k \tag{2}$$

for some constant $k \neq 0$. Make the substitution $z = y^{1-k}$, and show that this results in a linear differential equation for z with continuous coefficients. This is fortunate, because there are many real-world problems that lead to (2).

4. Sometimes a constant equilibrium solution has the property that solutions lying on one side of the equilibrium solution tend to approach that solution, while solutions on the other side of the equilibrium solution depart from that solution. Such an equilibrium is called *semistable*.

Consider the differential equation

$$\frac{dy}{dt} = k(1-y)^2, \quad y(0) = y_0,$$

for some positive constants k and y_0. Solve the differential equation. Show that $y = 1$ is a semistable solution, by showing that if $y_0 \leq 1$, then $\lim_{t\to\infty} y(t) = 1$, while if $y_0 > 1$, then $\lim_{t\to\infty} y(t) \neq 1$.

5. In many settings, a differential equation of the form $y' = f(a, y)$ is a useful model, with a parameter a that might be any real number. The behavior of the solution can vary as a changes. In particular, the number and type of equilibrium solutions can change as a changes.

Consider the differential equation $y' = a - y^2$, with initial condition $y(0) = y_0$.

- (a) Suppose first that $a < 0$. Write $a = -b^2$ with $b > 0$. Solve the differential equation. Show that there are no critical points, and no equilibrium solutions, by showing that $\lim_{t\to\infty} y(t)$ is not defined for any y_0.
- (b) Now suppose that $a = 0$. Solve the differential equation. Find all equilibrium solutions, and classify each as asymptotically stable, semistable, or unstable.
- (c) Suppose that $a > 0$, and write $a = b^2$ with $b > 0$. Solve the differential equation. Find all equilibrium solutions, and classify each as asymptotically stable, semistable, or unstable.

The change in behavior at $a = 0$ is called a *bifurcation*. Bifurcations are classified, and this particular bifurcation is called a *saddle-node* bifurcation.