MATH4410

Homework 4

Due March 12, 2021

Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EST on March 12. Please name your file hw04-lastname-firstname.pdf. For example, my solution file is hw04-gross-robert.pdf. Please use hyphens and not underlines, please use all lower-case letters, and please do not leave any spaces in the name of your file. You will have 5 points added to your homework score if you follow these instructions.

I will try to acknowledge receipt of each e-mail.

1. Find the general solution of $y'' + 2y' + 5y = 0$, and find the specific solution satisfying $y(0) = y'(0) = 0$.

2. Find the general solution of $y'' + 2y' + 5y = e^{-t}$, and find the specific solution satisfying $y(0) = y'(0) = 0$.

3. Find the general solution of $y'' + 2y' + 5y = \sin 2t$, and find the specific solution satisfying $y(0) = y'(0) = 0$.

4. Find the general solution of $y'' + 2y' + 5y = e^{-t}\sin 2t$, and find the specific solution satisfying $y(0) = y'(0) = 0$.

5. The differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

is called *exact* if it can be written in the form

$$[P(x)y']' + [f(x)y]' = 0,$$

where $f(x)$ is to be determined in terms of $P(x)$, $Q(x)$, and $R(x)$. If the equation is exact, integration leads to the differential equation

$$P(x)y' + f(x)y = C,$$

which in turn can be solved because it is a first-order linear differential equation.

Show that if a second-order differential equation is exact, then

$$P''(x) - Q'(x) + R(x) = 0.$$

6. On the examination, we studied the behavior of the equilibria of the first-order differential equation

$$y' = ay - by^2 - q, \qquad y(0) = C,$$

where a and b are positive constants, and q varies. Now it's time to solve the equation. The formula varies depending on the size of q, but in all cases you can express y as a function of t.

It is helpful to remember that if the quadratic polynomial $Ax^2 + Bx + C$ has roots α and β, then $Ax^2 + Bx + C = A(x - \alpha)(x - \beta)$.

(*a*) Suppose that $q < \dfrac{a^2}{4b}$. Show that the polynomial $ar - br^2 - q$ has 2 real roots. Call those roots r_1 and r_2, with $r_1 < r_2$. You can solve the differential equation using partial fractions to do the integration, and then write your answer explicitly solving for y as a function of t, in terms of r_1 and r_2.

Then show that if $C > r_1$, then $\lim_{t\to\infty} y(t) = r_2$, and if $C < r_1$, then $y(t)$ is undefined for sufficiently large values of t.

(b) Suppose that $q = \dfrac{a^2}{4b}$. Show that the polynomial $ar - br^2 - q$ has 1 double root, which we will call r. You can solve the differential equation, and then write your answer explicitly solving for y as a function of t, in terms of r.

Then show that if $C > r$, $\lim_{t\to\infty} y(t) = r$, while if $C < r$, then $y(t)$ is undefined for sufficiently large values of t.

(c) Suppose that $q > \dfrac{a^2}{4b}$. Show that the polynomial $ar - br^2 - q$ has no real roots. You can solve the differential equation by completing the square. Write y as a function of t, and show that $y(t)$ is undefined for sufficiently large values of t for any value of C.