MATH4410

Homework 6

Due March 26, 2021

Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on March 26. Please name your file hw06-lastname-firstname.pdf. For example, my solution file is hw06-gross-robert.pdf. Please use hyphens and not underlines, please use all lower-case letters, and please do not leave any spaces in the name of your file. You will have 5 points added to your homework score if you follow these instructions.

I will try to acknowledge receipt of each e-mail.

1. Find the general solution of

$$y'' + 9y = \sec^2 3t.$$

2. Solve

$$y^{(4)} + 9y^{(3)} + 28y'' + 36y' + 16y = 0, \qquad y(0) = y'(0) = y''(0) = 0, \quad y^{(3)}(0) = 1$$

3. Sometimes a differential equation of the form $y'' + p(t)y' + q(t)y = 0$ can be turned into one with constant coefficients by means of a change of independent variable. In particular, consider

$$t^2 \frac{d^2y}{dt^2} + \alpha t \frac{dy}{dt} + \beta y = 0,$$

where α and β are real constants.

(a) Let $x = \log t$. Compute $\dfrac{dy}{dt}$ and $\dfrac{d^2y}{dt^2}$ in terms of $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

(b) Show that the original differential equation becomes

$$\frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0.$$

This differential equation can be solved, and then the substitution $x = \log t$ gives the general solution of the original equation.

(c) Solve $t^2y'' + ty' + y = 0$.

4. This problem shows one more way to find a second fundamental solution of the second-order differential equation $ay'' + by' + cy = 0$ once we have found one.

Suppose that $ax^2 + bx + c = a(x - r_1)^2$. Let $L[y] = ay'' + by' + cy$, and let r be a variable.

(a) Verify that

$$L[e^{rt}] = a(r - r_1)^2 e^{rt}. \tag{1}$$

(b) Now, differentiate (1) with respect to r, and interchange the order of differentiation with respect to r and t and verify the following computation:

$$L[te^{rt}] = L\left[\frac{\partial}{\partial r} e^{rt}\right] = \frac{\partial}{\partial r} L[e^{rt}] = ate^{rt}(r - r_1)^2 + 2ae^{rt}(r - r_1). \tag{2}$$

The right-hand side of (2) is 0 when $r = r_1$. Why does this show that $te^{r_1 t}$ is a second solution of $L[y] = 0$?