MATH4410
Homework 7
Due April 9, 2021
Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on April 9. Please name your file hw07-lastname-firstname.pdf. For example, my solution file is hw07-gross-robert.pdf. Please use hyphens and not underlines, please use all lower-case letters, and please do not leave any spaces in the name of your file. You will have 5 points added to your homework score if you follow these instructions.

I will try to acknowledge receipt of each e-mail.

1. One model that requires a fourth-order differential equation is a mass suspended from a spring suspended from a second mass suspended from a second spring. With a specific set of constants, a model of such a system is given by

$$u_1'' + 5u_1 = 2u_2 \tag{1}$$

$$u_2'' + 2u_2 = 2u_1 \tag{2}$$

where $u_1$ and $u_2$ are the displacements of the two masses.

($a$) Solve equation (1) for $u_2$ and substitute into the second equation to derive a single fourth order equation $u_1^{(4)} + 7u_1'' + 6u_1 = 0$.

($b$) Find the general solution of this fourth-order differential equation.

($c$) Suppose we have initial conditions

$$u_1(0) = 1 \qquad u_1'(0) = 0 \qquad u_2(0) = 2 \qquad u_2'(0) = 0$$

Use equation (1) to determine the values of $u_1''(0)$ and $u_1^{(3)}(0)$. After that, show that $u_1(t) = \cos t$ and then that $u_2(t) = 2\cos t$.

($d$) Suppose instead we have initial conditions

$$u_1(0) = -2 \qquad u_1'(0) = 0 \qquad u_2(0) = 1 \qquad u_2'(0) = 0$$

Now show that the solution corresponding to these initial conditions is $u_1(t) = -2\cos\sqrt{6}t$ and $u_2(t) = \cos\sqrt{6}t$.

The first solution corresponds to the two masses vibrating in phase with frequency 1. The second solution corresponds to the two masses moving out of phase (one moves up at the same time as the other moves down) with frequency $\sqrt{6}$. It turns out that the general solution consists of a linear combination of these two solutions.

2. Abel's Theorem (giving the value of the Wronskian, up to a constant, in terms of the coefficients of the second-order linear differential equation) generalizes to higher order differential equations. Here's how the argument goes for the third-order differential equation $y''' + p_1(t)y'' + p_2(t)y' + p_3(t)y = 0$. Suppose that the solutions of that differential equation are $y_1$, $y_2$, and $y_3$.

($a$) Write $W$ for the Wronskian, which is

$$W(y_1, y_2, y_3) = \det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}.$$

Show that

$$W' = \det \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{bmatrix}$$

You should assume that the derivative of a determinant is computed by summing the determinants obtained by successively differentiating each of the rows of the original determinant.

($b$) We can express $y_1'''$, $y_2'''$, and $y_3'''$ in terms of lower derivatives, because $y_1$, $y_2$, and $y_3$ solve the differential equation. Substitute those values into the determinant for $W'$.

($c$) Multiply the first row of the determinant for $W'$ by $p_3(t)$ and the second row by $p_2(t)$ and add the results to the third row. Conclude that $W' = -p_1(t)W$.

($d$) Show that

$$W = c \exp\left[\int -p_1(t)\, dt\right].$$

3. The gamma function $\Gamma(s)$ is defined by

$$\Gamma(s) = \int_0^\infty e^{-x} x^{s-1}\, dx.$$

You may assume that this integral converges provided that $s > 0$.

($a$) Show that $\Gamma(1) = 1$.

($b$) Show that $\Gamma(s+1) = s\Gamma(s)$ if $s > 0$.

($c$) If $n$ is a positive integer, show that $\Gamma(n) = (n-1)!$.

($d$) In many formulas, the function $n!$ can be interpreted for non-integer values of $n$ by replacing $n!$ with $\Gamma(n+1)$. In particular, show that

$$\mathscr{L}(t^p) = \frac{\Gamma(p+1)}{s^{p+1}}$$

if $p > 0$ is any real number, not necessarily an integer.

4. Solve

$$y'' - 2y' + 2y = 0, \qquad y(0) = 0, \quad y'(0) = 1$$

by using Laplace transforms.

5. Solve

$$y'' + k^2 y = \cos 3t, \qquad y(0) = 1, \quad y'(0) = 0, \quad k^2 \neq 9$$

by using Laplace transforms.

6. Let

$$g(t) = \begin{cases} 1 & 0 \le t \le \pi \\ e^{\pi - t} & t \ge \pi \end{cases}$$

and consider the initial value problem

$$y'' + y = g(t), \qquad y(0) = 0, \quad y'(0) = 0$$

Write $g(t)$ in terms of step functions, and solve the initial value problem using Laplace transforms.