MATH4410
Homework 9
Due April 23, 2021
Your Name Here

Your answers must be in the form of a typed PDF file, and must be e-mailed to me by 5PM EDT on April 23. Please name your file hw09-lastname-firstname.pdf. For example, my solution file is hw09-gross-robert.pdf. Please use hyphens and not underlines, please use all lower-case letters, and please do not leave any spaces in the name of your file. You will have 5 points added to your homework score if you follow these instructions.

I will try to acknowledge receipt of each e-mail.

1. A previous homework problem considered two different solutions y_1 and y_2 of the same homogeneous linear second-order differential equation, and drew a conclusion about when the functions can be 0. This problem shows that there is also a statement that can be made relating solutions of two different differential equations.

Consider two different differential equations

$$y'' + \alpha(x)y = 0 \tag{1}$$

and

$$y'' + \beta(x)y = 0. \tag{2}$$

where $\beta(x) > \alpha(x)$. (That inequality means that for every real number x, $\beta(x) > \alpha(x)$.) Suppose that $\phi(x)$ solves equation (1) and $\psi(x)$ solves equation (2). Suppose as well that there are real numbers a and b so that $\phi(a) = \phi(b) = 0$ and $\phi(x) > 0$ if $a < x < b$. We wish to show that $\psi(k) = 0$ for some k between a and b.

Suppose that $\psi(x) \neq 0$ for $a < x < b$.

(a) Show that

$$(\psi\phi' - \phi\psi')' = (\beta - \alpha)\phi\psi. \tag{3}$$

(b) Suppose first that $\psi(x) > 0$ for $a < x < b$. Integrate equation (3) to show that

$$\psi(b)\phi'(b) - \psi(a)\phi'(a) > 0. \tag{4}$$

(c) Show that $\phi'(b) < 0$ and $\phi'(a) > 0$. Explain why this contradicts equation (4).

(d) Explain why there is also a contradiction if we suppose that $\psi(x) < 0$ for $a < x < b$.

(e) Why does this show that $\psi(k) = 0$ for some k between a and b?

2. Solve the initial value problem

$$\mathbf{x}' = \begin{bmatrix} -7 & 4 \\ -6 & 7 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 100e^t \\ 100 \end{bmatrix}, \qquad \mathbf{x}(0) = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$$

3. There is a version of Abel's theorem about the Wronskian that applies to systems of linear differential equations.

(a) Suppose that $\mathbf{P}(t)$ is a 2×2 matrix. Suppose that $\mathbf{v}$ and $\mathbf{w}$ form a fundamental set of solutions to the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$. Remember that the Wronskian $W(\mathbf{v}, \mathbf{w})$ is the determinant of the 2×2 matrix $\mathbf{X}$ whose first column is $\mathbf{v}$ and whose second column is $\mathbf{w}$. We can apply the same rule about differentiating a determinant as on an earlier problem set: the derivative W' is a sum of two determinants, one computed after differentiating the first row of $\mathbf{X}$, and the other computed after differentiating the second row of $\mathbf{X}$. In other words, if

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix},$$

and $W = \det(\mathbf{X})$, then

$$W' = \det \begin{bmatrix} x'_{11} & x'_{12} \\ x_{21} & x_{22} \end{bmatrix} + \det \begin{bmatrix} x_{11} & x_{12} \\ x'_{21} & x'_{22} \end{bmatrix}$$

You do not need to check this.

Show that $W' = (p_{11} + p_{22})W$, where p_{11} and p_{22} are the two functions on the diagonal of the matrix $\mathbf{P}$.

(*b*) Now suppose that $\mathbf{P}(t)$ is a 3×3 matrix. Reasoning similarly, show that $W' = (p_{11} + p_{22} + p_{33})W$, where p_{11}, p_{22}, and p_{33} are the three functions on the diagonal of the matrix $\mathbf{P}$.

4. For any reasonably nice function $f(t)$ with a Laplace transform $F(s)$, it is not hard to show that $\lim_{s\to\infty} F(s) = 0$. Use that fact (without proving it) to show that $\lim_{s\to\infty} sF(s) = f(0)$. What other assumptions do you need to make to draw this conclusion?

5. We now know these formulæ:

$$\begin{aligned} \mathscr{L}(\sin t) &= \frac{1}{s^2+1} \\ \mathscr{L}(\cos t) &= \frac{s}{s^2+1} \\ \mathscr{L}(t \sin t) &= \frac{2s}{(s^2+1)^2} \\ \mathscr{L}(t \cos t) &= \frac{s^2-1}{(s^2+1)^2} \end{aligned}$$

We can derive the first two directly, and the second two follow by using the formula $\mathscr{L}(tf(t)) = -F'(s)$ from last week's homework. You do not need to check these equations.

These are not the most useful forms for computing the inverse Laplace transform. By combining these expressions in different ways, compute

(*a*) $\mathscr{L}^{-1}\left(\dfrac{1}{(s^2+1)^2}\right)$.

(*b*) $\mathscr{L}^{-1}\left(\dfrac{s}{(s^2+1)^2}\right)$.

(*c*) $\mathscr{L}^{-1}\left(\dfrac{s^2}{(s^2+1)^2}\right)$.

(*d*) $\mathscr{L}^{-1}\left(\dfrac{s^3}{(s^2+1)^2}\right)$.