Mathematics 102
Examination 1
Answers

1. *(16 points)* (a) State the Intermediate Value Theorem.

(b) Use the Intermediate Value Theorem to explain why the equation $\sin x = x^2 - 1$ has a solution with x between 0 and π.

Answer: The Intermediate Value Theorem states: "If $f(x)$ is continuous on the closed interval $[a, b]$, with $f(a) \neq f(b)$, and N is any number between a and b, then there is a number c between a and b so that $f(c) = N$."

For the second part of the problem, let $f(x) = \sin x - x^2 + 1$. We easily compute that $f(0) = 1$, and $f(\pi) = -\pi^2 + 1 < 0$. Therefore, between 0 and π, there is a number c so that $f(c) = 0$, which is the same as $\sin c = c^2 - 1$. The number c is approximately 1.4096, incidentally.

2. *(10 points)* Find the equation of the tangent line to the graph of $y = 3x^2 + 4$ at the point $(2, 16)$.

Answer: We must start by computing the slope of the tangent line, using the formula $\lim\limits_{x\to a} \frac{f(x)-f(a)}{x-a}$:

$$m = \lim_{x\to 2} \frac{3x^2 + 4 - 16}{x - 2} = \lim_{x\to 2} \frac{3x^2 - 12}{x - 2} = \lim_{x\to 2} 3x + 6 = 12.$$

The equation of the tangent line can then be found by the usual methods to be $y = 12x - 8$.

3. *(54 points)* Compute the following limits. Be sure to justify your answers. If a limit does not exist, but equals ∞ or $-\infty$, you must say so in order to get full credit. As usual, $[x]$ refers to the greatest-integer function.

$$(a)\ \lim_{x\to 0} \frac{(3+x)^2 - 9}{x} \qquad (b)\ \lim_{x\to 3^+} \frac{x^2+2}{x-5} \qquad (c)\ \lim_{x\to 3^-} \frac{x}{[x]}$$

$$(d)\ \lim_{x\to 6^+} [2x + 0.99] \qquad (e)\ \lim_{x\to 4^-} \frac{x}{x-4} \qquad (f)\ \lim_{x\to 5} \frac{x^2+9}{x-5}$$

$$(g)\ \lim_{x\to 0} \frac{\sqrt{x^2+4} - 2}{x^2} \qquad (h)\ \lim_{x\to \frac{\pi}{2}^-} \tan x \qquad (i)\ \lim_{x\to 2} \frac{x^3 - 4x^2 + 7x - 6}{x - 2}$$

Answer: We have

$$\lim_{x\to 0} \frac{(3+x)^2 - 9}{x} = \lim_{x\to 0} \frac{9 + 6x + x^2 - 9}{x} = \lim_{x\to 0} \frac{6x + x^2}{x} = \lim_{x\to 0} 6 + x = 6.$$

For (b), we can use continuity:

$$\lim_{x\to 3^+} \frac{x^2+2}{x-5} = \frac{\lim\limits_{x\to 3^+} x^2 + 2}{\lim\limits_{x\to 3^+} x - 5} = \frac{11}{-2} = -5.5.$$

For (c), we can use limit laws, recalling that $\lim\limits_{x\to 3^-} [x] = 2$:

$$\lim_{x\to 3^-} \frac{x}{[x]} = \frac{\lim\limits_{x\to 3^-} x}{\lim\limits_{x\to 3^-} [x]} = \frac{3}{2} = 1.5.$$

For (d), we need to reason carefully. If x is close enough to 6 and larger than 6, then $2x$ will be larger than 12, and smaller than 12.01, and then $2x + 0.99$ will also be larger than 12 and smaller than 13; therefore, $\lim\limits_{x\to 6^+} [2x + 0.99] = 12$.

For (e), the first and simplest answer is that the limit does not exist: the numerator is close to 4, while the denominator gets closer and closer to 0. But we can do better: because the denominator is always negative, and the numerator is positive, we can conclude that $\lim_{x\to 4^-} \frac{x}{x-4} = -\infty$.

For (f), the limit does not exist. We cannot say anything further, because if $x > 5$, then the denominator will be positive; if $x < 5$, then the denominator will be negative. The numerator is always positive. Therefore, the limit is neither ∞ nor $-\infty$.

For (g), we use some algebra:

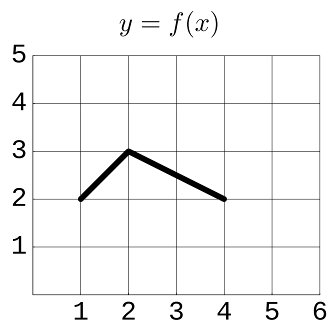
$$\lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} = \lim_{x\to 0} \frac{\sqrt{x^2+4}-2}{x^2} \cdot \frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2} = \lim_{x\to 0} \frac{x^2+4-4}{x^2\left(\sqrt{x^2+4}+2\right)}$$
$$= \lim_{x\to 0} \frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} = \lim_{x\to 0} \frac{1}{\sqrt{x^2+4}+2} = \frac{1}{4}.$$

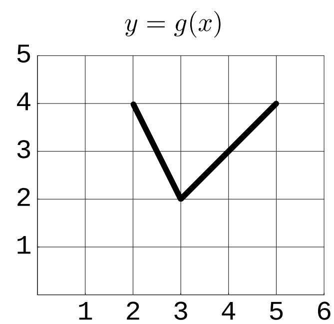
For (h), the simplest way to proceed is to recall how the graph of $\tan x$ looks. The value $x = \frac{\pi}{2}$ is not in the domain of $\tan x$, but we can say more: $\lim_{x\to \frac{\pi}{2}^-} \tan x = \infty$.

Finally, for (i), we can divide the numerator by the denominator:

$$\lim_{x\to 2} \frac{x^3-4x^2+7x-6}{x-2} = \lim_{x\to 2} \frac{(x-2)(x^2-2x+3)}{x-2} = \lim_{x\to 2} x^2-2x+3 = 3.$$

4. *(20 points)* The first graph below is the graph of $y = f(x)$, and the second is the graph of $y = g(x)$:





Find numbers A, B, C, and D so that $g(x) = Af(B(x+C)) + D$.

Answer: To begin, note that the width of the two graphs is identical: the first one goes from 1 to 4, and the second from 2 to 5. Therefore, there is no horizontal shrinking or stretching, so $B = 1$.

Next, note that the graph is shifted to the right by 1, so $C = -1$. So we know that $g(x) = Af(x-1)+D$.

Next, we see that the graph of $g(x)$ is stretched vertically by a factor of 2, but also is inverted. That means that $A = -2$. So we see that $g(x) = -2f(x-1) + D$.

One way to compute D is by substitution. Plug in $x = 2$. We know that $g(2) = 4$, and we also know that $f(1) = 2$. Substituting gives $4 = (-2)(2) + D$, so $D = 8$.

It's a good idea to check other values of x to verify that $g(x) = -2f(x-1) + 8$, such as $x = 3$, and $x = 5$.

Grade	Number of people
97	2
93	1
92	1
91	1
90	1
87	1
85	1
77	5
76	1
75	1
72	1
70	2
66	1
63	1
58	1

Mean: 79.38
Standard deviation: 10.90