Mathematics 102
Examination 2
October 22, 2004
Answers

1. *(10 points)* (a) State the definition of $f'(x)$, the derivative of $f(x)$.
(b) Let $f(x) = (x+1)^3 + 4$. Using only the definition of the derivative and basic theorems about limits, compute $f'(4)$.

Answer:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}.$$

In this case, we have

$$\begin{aligned} f'(4) &= \lim_{h\to 0} \frac{(5+h)^3 + 4 - 129}{h} = \lim_{h\to 0} \frac{125 + 75h + 15h^2 + h^3 - 125}{h} \\ &= \lim_{h\to 0} \frac{75h + 15h^2 + h^3}{h} = \lim_{h\to 0} 75 + 15h + h^2 = 75. \end{aligned}$$

2. *(30 points)* Compute $\frac{dy}{dx}$ for each of the following functions.

$(a)\ \sqrt{x^2+1}$	$(b)\ \tan(2x)$	$(c)\ (x^3+2)^4$
$(d)\ x\sin 3$	$(e)\ \sin(3x)$	$(f)\ 3\sin x$

Answer: We have (omitting the details to save space)

$$\begin{aligned} \frac{d}{dx}\left(\sqrt{x^2+1}\right) &= \frac{x}{\sqrt{x^2+1}} \\ \frac{d}{dx}\tan(2x) &= 2\sec^2 2x \\ \frac{d}{dx}\left((x^3+2)^4\right) &= 12x^2(x^3+2)^3 \\ \frac{d}{dx}(x\sin 3) &= \sin 3 \\ \frac{d}{dx}(\sin 3x) &= 3\cos 3x \\ \frac{d}{dx}(3\sin x) &= 3\cos x \end{aligned}$$

3. *(5 points)* State the Intermediate Value Theorem.

Answer: If $f(x)$ is a continuous function defined on a closed interval $[a, b]$, with $f(a) \neq f(b)$, and M is any number between $f(a)$ and $f(b)$, then there is a number c between a and b so that $f(c) = M$.

4. *(10 points)* Let $f(x) = |x-2| + 3$. For which values of x is $f'(x)$ undefined? At which values of x is the function $f(x)$ differentiable? At those values for which the derivative exists, compute $f'(x)$ and simplify your answer as much as possible.

Answer: One way to do this problem is to use the formula, derived on homework, that $\frac{d}{dx}|x| = \frac{x}{|x|}$. That formula tells us that $f'(x) = \frac{x-2}{|x-2|}$. From this formula, it is immediately apparent that $f'(x)$ is undefined when $x = 2$. For all other values of x, it is defined, and we can further compute that

$$f'(x) = \begin{cases} 1 & \text{if } x > 2, \\ -1 & \text{if } x < 2. \end{cases}$$

Another way to see this same answer is to recall that the graph of $f(x)$ is just the graph of $y = |x|$, shifted to the right by 2 and upwards by 3. That picture allows the computation of $f'(x)$ quickly.

5. *(10 points)* Consider the graph defined by the equation $x^4 + xy + y^5 = 19$. Find the equation of the tangent line to this graph at the point $(2, 1)$.

Answer: The easiest way to do this problem is by using implicit differentiation. We have

$$\begin{aligned} x^4 + xy + y^5 &= 19 \\ \frac{d}{dx}\left(x^4 + xy + y^5\right) &= \frac{d}{dx}(19) \\ 4x^3 + y + x\frac{dy}{dx} + 5y^4\frac{dy}{dx} &= 0 \\ 4x^3 + y + \left(x + 5y^4\right)\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{4x^3 + y}{x + 5y^4} \end{aligned}$$

Therefore, when $x = 2$ and $y = 1$, we have $\frac{dy}{dx} = -\frac{33}{7}$, and so an equation for the tangent line is given by $(y - 1) = -\frac{33}{7}(x - 2)$. That simplifies to $7y - 7 = -33x + 66$, or $33x + 7y = 73$.

6. *(20 points)* Compute the following limits.

$$\lim_{x\to 0}\frac{\cos x - 1}{x^2} \qquad \lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2} - h\right)}{2h} \qquad \lim_{y\to 0}\frac{\sin y}{\sin 2y} \qquad \lim_{x\to 2^-}[94x - 1]$$

Remember that as usual, $[x]$ refers to the greatest integer function.

Answer: We have

$$\begin{aligned} \lim_{x\to 0}\frac{\cos x - 1}{x^2} &= \lim_{x\to 0}\frac{\cos x - 1}{x^2}\cdot\frac{\cos x + 1}{\cos x + 1} = \lim_{x\to 0}\frac{\cos^2 x - 1}{x^2(\cos x + 1)} \\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x^2(\cos x + 1)} = \lim_{x\to 0} - \left(\frac{\sin x}{x}\right)\cdot\left(\frac{\sin x}{x}\right)\cdot\left(\frac{1}{\cos x + 1}\right) = -1\cdot 1\cdot\frac{1}{2} = -\frac{1}{2}. \end{aligned}$$

$$\lim_{h\to 0}\frac{\cos\left(\frac{\pi}{2} - h\right)}{2h} = \lim_{h\to 0}\frac{\sin h}{2h} = \lim_{h\to 0}\frac{1}{2}\left(\frac{\sin h}{h}\right) = \frac{1}{2}.$$

$$\lim_{y\to 0}\frac{\sin y}{\sin 2y} = \lim_{y\to 0}\frac{\sin y}{2\sin y\cos y} = \lim_{y\to 0}\frac{1}{2\cos y} = \frac{1}{2}.$$

For the last one, we see that $[94x - 1]$ is never quite as large as $188 - 1 = 187$ if x does not get to be as large as 2. Therefore, $\lim_{x\to 2^-}[94x - 1] = 186$.

7. *(15 points)* Let n be a positive integer. In class, we derived the formula

$$\frac{d}{dx}\left(\sin^n x\cos nx\right) = n\sin^{n-1} x\cos(n+1)x.$$

Find a similar formula for

$$\frac{d}{dx}\left(\sin^n x\sin nx\right).$$

Be sure to simplify your answer as much as possible.

Answer: We have

$$\begin{aligned} \frac{d}{dx}\left(\sin^n x\sin nx\right) &= \frac{d}{dx}\left(\sin^n x\right)\sin nx + \sin^n x\frac{d}{dx}\left(\sin nx\right) \\ &= n\sin^{n-1} x\cos x\sin nx + \sin^n x(n\cos nx) \\ &= n\sin^{n-1} x\left(\cos x\sin nx + \sin x\cos nx\right) = n\sin^{n-1} x\sin(n+1)x. \end{aligned}$$

Grade	Number of people
90	1
89	1
87	1
85	1
84	1
82	1
80	1
77	1
73	2
71	1
70	1
69	1
65	2
63	1
55	1
52	1
41	1
37	1
25	1

Mean: 68.24
Standard deviation: 17.35