Mathematics 102
Examination 3
Answers

1. *(5 points)* State the Intermediate Value Theorem.

Answer: If $f(x)$ is a continuous function on a closed interval $[a, b]$, and $f(a) \neq f(b)$, and N is any number between $f(a)$ and $f(b)$, then there is a number c between a and b so that $f(c) = N$.

2. *(10 points)* Find the maximum and minimum values of the function $f(x) = 3x^2 + 24x + 11$ if $0 \leq x \leq 1$.

Answer: We first find critical points. We have $f'(x) = 6x+24$, so the only solution to the equation $f'(x) = 0$ is $x = -4$, which is not in the interval. Therefore, the extreme values must occur at $x = 0$ and $x = 1$, and some computation shows that the minimum value is $f(0) = 11$ and the maximum value is $f(1) = 38$.

3. *(15 points)* (a) State the Mean Value Theorem.

(b) If a and b are numbers so that $-\frac{\pi}{2} < a < b < \frac{\pi}{2}$, show that

$$\tan b - \tan a \geq b - a.$$

Hint: Apply the Mean Value Theorem to the function $\tan x$.

Answer: If $f(x)$ is a continuous function on the closed interval $[a, b]$ which is differentiable on the open interval (a, b), then there is a number c between a and b so that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Apply the theorem to the function $f(x) = \tan x$, which is continuous and differentiable on $[a, b]$ because of the given inequality, and we can conclude that there is a number c between a and b so that

$$\frac{\tan b - \tan a}{b - a} = \sec^2 c.$$

We know that $\sec^2 x \geq 1$ for any number x, so we can conclude that

$$\frac{\tan b - \tan a}{b - a} \geq 1,$$

which is the desired inequality.

4. *(10 points)* Use a linear approximation to give a fractional approximation of $\sqrt{82}$.

Answer: The point here is that it is easy to compute $\sqrt{81}$. We compute the tangent line to the graph of $y = \sqrt{x}$ when $x = 81$. The derivative evaluated at 81 is $\frac{1}{18}$, and so the linear approximation is $y - 9 = \frac{1}{18}(x - 81)$. Substitute $x = 82$ and solve for y, and we get $\sqrt{82} \approx 9\frac{1}{18}$. This approximation is accurate to 3 decimal places, incidentally.

5. *(10 points)* Show that the equation $x^{49} + x^{25} + x + 1 = 0$ has exactly one real solution.

Answer: There are two things that we need to show: the equation has at least one solution, and it has no more than one solution. Let $f(x) = x^{49} + x^{25} + x + 1$. Then $f(-1) = -2$, and $f(0) = 1$, so the Intermediate Value Theorem tells us that there is a number α between -1 and 0 with $f(\alpha) = 0$. That tells us that the given equation has at least one solution.

Suppose now that the equation has another solution β. Then Rolle's Theorem says that there is a number γ between α and β so that $f'(\gamma) = 0$. But $f'(x) = 49x^{48} + 25x^{24} + 1$, and the smallest value of $f'(x)$ is 1 (because the first two terms are never smaller than 0). Therefore, there cannot be more than one solution.

6. *(15 points)* Compute the following limits. Be sure to justify your answers. If a limit does not exist, but equals ∞ or $-\infty$, you must say so in order to get full credit. As usual, $[x]$ refers to the greatest-integer function.

$$\lim_{y\to 0}\frac{\tan y}{y} \qquad \lim_{x\to-\infty}\frac{2x^2+3}{5x^3} \qquad \lim_{x\to 3^+}[x]-[-x]$$

Answer: We have

$$\lim_{y\to 0}\frac{\tan y}{y}=\lim_{y\to 0}\frac{\sin y}{y\cos y}=\lim_{y\to 0}\frac{\sin y}{y}\cdot\frac{1}{\cos y}=1\cdot 1=1.$$

For the next one,

$$\lim_{x\to-\infty}\frac{2x^2+3}{5x^3}=\lim_{x\to-\infty}\frac{2\frac{x^2}{x^3}+\frac{3}{x^3}}{5}=\lim_{x\to-\infty}\frac{2\cdot 0+0}{5}=0.$$

For the last one, we have

$$\lim_{x\to 3^+}[x]-[-x]=\lim_{x\to 3^+}[x]-\lim_{x\to 3^+}[-x]=3-(-4)=7.$$

7. *(10 points)* Suppose that a rectangle has area A. In terms of the number A, what is the minimum length of the diagonal of the rectangle?

Answer: Suppose that the length of the rectangle is x, and the width is y. We want to minimize $D=\sqrt{x^2+y^2}$, given that $xy=A$. Eliminate y from the equation for D, and we have the job of minimizing $D=\sqrt{x^2+\frac{A^2}{x^2}}=\frac{\sqrt{x^4+A^2}}{x}$.

We compute

$$D'=\frac{x\cdot\frac{2x^3}{\sqrt{x^4+A^2}}-\sqrt{x^4+A^2}}{x^2}.$$

Rather than simplify, let's just solve the equation $D'=0$ by setting the numerator to 0:

$$\begin{aligned}\frac{2x^4}{\sqrt{x^4+A^2}}&=\sqrt{x^4+A^2}\\ 2x^4&=x^4+A^2\\ x^4&=A^2\\ x&=\sqrt{A}\end{aligned}$$

Therefore, $y=\sqrt{A}$ also, and the minimum length of the diagonal is $\sqrt{x^2+y^2}=\sqrt{2A}$.

8. *(25 points)* Let $y=\dfrac{x^2+x-7}{x+8}$. Use the first and second derivatives to figure out the intervals on which y is increasing, decreasing, concave up, and concave down. Find all local extrema and inflection points. Find all vertical and horizontal asymptotes. Show that $y=x-7$ is a slant asymptote.

Answer: We start by computing y':

$$y'=\frac{(x+8)(2x+1)-(x^2+x-7)}{(x+8)^2}=\frac{2x^2+17x+8-x^2-x+7}{(x+8)^2}=\frac{x^2+16x+15}{(x+8)^2}.$$

The critical points occur when $x^2+16x+15=0$ and when $x+8=0$, and those equations have the solutions $x=-15$, $x=-1$, and $x=-8$. We can rewrite the derivative formula as $y'=(x+1)(x+15)/(x+8)^2$, which helps in determining the sign of y'. If $x<-15$, $y'>0$; if $-15<x<-8$, $y'<0$; if $-8<x<-1$, $y'<0$, and if $x>-1$, $y'>0$. We therefore have local extrema at $x=-15$ and $x=-1$; the first is a local maximum and the second is a local minimum.

Continuing, we compute y'':

$$y'' = \frac{(x+8)^2(2x+16) - (x^2+16x+15)(2)(x+8)}{(x+8)^4} = \frac{(x+8)(2x+16) - (x^2+16x+15)(2)}{(x+8)^3}$$
$$= \frac{2x^2+32x+128-2x^2-32x-30}{(x+8)^3} = \frac{98}{(x+8)^3}.$$

Therefore, we see that $y'' < 0$ if $x < -8$ and $y'' > 0$ if $x > -8$. There are no inflection points.

We can collect all of this information as follows:

Interval	Behavior
$x < -15$	Increasing, concave down
$-15 < x < -8$	Decreasing, concave down
$-8 < x < -1$	Decreasing, concave up
$-1 < x$	Increasing, concave up

We also compute

$$\lim_{x\to\infty} \frac{x^2+x-7}{x+8} = \lim_{x\to\infty} \frac{x+1-\frac{7}{x}}{1+\frac{8}{x}} = \infty$$

and similarly

$$\lim_{x\to-\infty} \frac{x^2+x-7}{x+8} = \lim_{x\to-\infty} \frac{x+1-\frac{7}{x}}{1+\frac{8}{x}} = -\infty.$$

Therefore, there are no horizontal asymptotes. The vertical asymptote is $x = -8$, and there we have

$$\lim_{x\to-8^+} \frac{x^2+x-7}{x+8} = +\infty \qquad \lim_{x\to-8^-} \frac{x^2+x-7}{x+8} = -\infty.$$

Finally, to show that $y = x+7$ is a slant asymptote, we compute

$$\lim_{x\to\infty} \frac{x^2+x-7}{x+8} - (x-7) = \lim_{x\to\infty} \frac{x^2+x-7}{x+8} - \frac{(x-7)(x+8)}{x+8} = \lim_{x\to\infty} \frac{x^2+x-7}{x+8} - \frac{x^2+x-56}{x+8}$$
$$= \lim_{x\to\infty} \frac{49}{x+8} = 0,$$

and similarly as $x \to -\infty$.

Grade	Number of people
73	2
71	1
69	1
67	1
66	2
65	1
62	2
60	2
59	1
58	2
57	1
51	1
50	1
49	1
35	1
32	1

Mean: 59.19
Standard deviation: 10.71