Mathematics 102
Examination 4
Answers

1. *(10 points)* State the Extreme Value Theorem.

Answer: A continuous function $f(x)$ attains its minimum and maximum values on the closed interval $[a, b]$.

2. *(42 points)* Compute $\frac{dy}{dx}$ for each of the following functions. You do not need to simplify your answers.

$$(a)\ \ y = 2^x \qquad (b)\ \ y = \log_3 x \qquad (c)\ \ y = \frac{(x-1)^2(x-2)^3}{\sqrt{x^2+5}}$$
$$(d)\ \ y = (\sec^2 x)^x \qquad (e)\ \ y = \arcsin\left(x^2+1\right) \qquad (f)\ \ y = \sinh\left(e^x\right)$$

Answer: The first two are applications of the formulas in the text: $\frac{d}{dx}2^x = 2^x \ln 2$, and $\frac{d}{dx}\log_3 x = \frac{1}{x \ln 3}$.

Part (c) is best done using logarithmic differentiation:

$$\begin{aligned}
y &= \frac{(x-1)^2(x-2)^3}{\sqrt{x^2+5}} \\
\ln y &= 2\ln(x-1) + 3\ln(x-3) - \frac{1}{2}\ln(x^2+5) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{2}{x-1} + \frac{3}{x-3} - \frac{1}{2}\frac{2x}{x^2+5} \\
\frac{dy}{dx} &= y\left(\frac{2}{x-1} + \frac{3}{x-3} - \frac{x}{x^2+5}\right) \\
&= \left(\frac{(x-1)^2(x-2)^3}{\sqrt{x^2+5}}\right)\left(\frac{2}{x-1} + \frac{3}{x-3} - \frac{x}{x^2+5}\right)
\end{aligned}$$

Part (d) is also probably best done by taking logarithms:

$$\begin{aligned}
y &= (\sec^2 x)^x \\
\ln y &= x\ln(\sec^2 x) = 2x\ln(\sec x) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{2x}{\sec x}\sec x \tan x + 2\ln(\sec x) \\
&= 2x\tan x + 2\ln(\sec x) \\
\frac{dy}{dx} &= y(2x\tan x + 2\ln(\sec x)) \\
&= (\sec^2 x)^x(2x\tan x + 2\ln(\sec x))
\end{aligned}$$

We have $\frac{d}{dx}\arcsin(x^2+1) = \frac{2x}{\sqrt{1-(x^2+1)^2}}$, which certainly can be simplified a bit further.

Finally, $\frac{d}{dx}\sinh(e^x) = \cosh(e^x)e^x$, by the chain rule.

3. *(18 points)* Compute the following limits. If a limit does not exist, but is equal to $+\infty$ or $-\infty$, you must state that in order to receive full credit.

$$\lim_{x\to 0^+} \ln(\tan x) \qquad \lim_{x\to 0^+} \ln(\cos x) \qquad \lim_{x\to\infty} \arctan(2x+1)$$

Answer: We have $\lim_{x\to 0^+} \ln(\tan x) = -\infty$, because $\lim_{x\to 0^+} \tan x = 0$, and $\lim_{y\to 0^+} \ln y = -\infty$.

We have $\lim_{x\to 0^+} \ln(\cos x) = \ln 1 = 0$.

Finally, $\lim_{x\to\infty} \arctan(2x+1) = \frac{\pi}{2}$.

4. *(10 points)* Let $f(x) = x^2 - 4$. Starting with the value $x_1 = 1$, perform two iterations (steps) of Newton's method to find a solution to the equation $f(x) = 0$. Leave your answer in fractional form.

Answer: The formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, which in this case becomes $x_{n+1} = x_n - \frac{x^2-4}{2x}$. We have, starting with $x_1 = 1$:

$$x_2 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-3}{2} = \frac{5}{2}$$

$$x_3 = \frac{5}{2} - \frac{f(\frac{5}{2})}{f'(\frac{5}{2})} = \frac{5}{2} - \frac{9/4}{5} = \frac{5}{2} - \frac{9}{20} = \frac{41}{20}$$

5. *(10 points)* Suppose that $g(x) = f^{-1}(x)$, and further that $f(2) = 3$, $f(3) = 4$, $f'(2) = 11$, $f'(3) = 5$, and $f'(4) = 6$. Do you have enough information to compute $g'(3)$? If so, what is $g'(3)$? If not, what additional information do you need to compute $g'(3)$?

Answer: This follows from the formula $g'(x) = 1/f'(g(x))$: $g'(3) = 1/f'(g(3)) = 1/f'(2) = 1/11$.

6. *(10 points)* Prove the identity

$$2 \arcsin x = \arccos(1 - 2x^2)$$

if $x \geq 0$. Be sure to point out where in your answer you use the assumption that $x \geq 0$. (The equation is false if $x < 0$, because the left-hand side of the equation will be negative, while the right-hand side is always positive.)

Answer: Let $F(x) = 2 \arcsin x$, and $G(x) = \arccos(1 - 2x^2)$. We start by showing that $F'(x) = G'(x)$. We have $F'(x) = \frac{2}{\sqrt{1-x^2}}$. The harder part is $G'(x)$:

$$G'(x) = \frac{-1}{\sqrt{1-(1-2x^2)^2}}(-4x) = \frac{4x}{\sqrt{1-(1-2x^2)^2}} = \frac{4x}{\sqrt{4x^2-4x^4}} = \frac{4x}{2x\sqrt{1-x^2}} = \frac{2}{\sqrt{1-x^2}}.$$

Therefore, $F'(x) = G'(x)$. Incidentally, we used the fact that $x > 0$ when we simplified $\sqrt{4x^2 - 4x^4}$ to $2x\sqrt{1-x^2}$.

This means that $F(x)$ and $G(x)$ differ by a constant. To verify that the constant is 0, substitute in any value of x that you choose, such as 0. We have $F(0) = 2 \arcsin 0 = 0$, while $G(0) = \arccos(1) = 0$.

Grade	Number of people
91	1
90	1
87	1
82	1
80	1
76	2
73	1
64	1
63	1
61	2
54	1
49	1
48	2
41	1
21	1

Mean: 64.72
Standard deviation: 18.42