Mathematics 102
Final Examination
Answers

1. *(5 points)* Using only the definition of the derivative and basic facts about limits, compute the derivative of $f(x) = x^{-1/2}$.

Answer: We know that the definition of the derivative is $f'(x) = \lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$. In this case, we get

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(x+h)^{-1/2} - x^{-1/2}}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h\to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x} - \sqrt{x+h}}{h\sqrt{x+h}\sqrt{x}}\right)\left(\frac{\sqrt{x} + \sqrt{x+h}}{\sqrt{x} + \sqrt{x+h}}\right) = \lim_{h\to 0} \frac{-h}{h\sqrt{x+h}\sqrt{x}\left(\sqrt{x} + \sqrt{x+h}\right)} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x+h}\sqrt{x}\left(\sqrt{x} + \sqrt{x+h}\right)} = \frac{-1}{\sqrt{x}\cdot\sqrt{x}\cdot 2\sqrt{x}} = \frac{-1}{2x^{3/2}}
\end{aligned}$$

2. *(5 points)* State the Intermediate Value Theorem

Answer: If $f(x)$ is continuous on the closed interval $[a, b]$, with $f(a) \neq f(b)$, and N is a number between $f(a)$ and $f(b)$, then there is a number c between a and b so that $f(c) = N$.

3. *(12 points)* Compute $\frac{dy}{dx}$ for each of the following functions. You do not need to simplify your answers.

$(a)\ \ y = \tan(e^x)$	$(b)\ \ y = \log_x 2$	$(c)\ \ y = \sqrt{\dfrac{x^2+1}{(x-1)^2}}$
$(d)\ \ y = x^x$	$(e)\ \ y = \arctan(x^2+1)$	$(f)\ \ y = e^{\cosh x}$

Answer: In (a), we have $y' = e^x \sec^2(e^x)$.

Part (b) is a bit trickier. The simplest thing to do is to rewrite $\log_x 2 = \frac{\ln 2}{\ln x}$, and then $y' = \frac{-\ln 2}{x(\ln x)^2}$.

You can do part (c) without taking logarithms, but it's much simpler if you do:

$$\begin{aligned}
y &= \sqrt{\frac{x^2+1}{(x-1)^2}} \\
\ln y &= \frac{1}{2}\ln(x^2+1) - \ln(x-1) \\
\frac{1}{y}\frac{dy}{dx} &= \frac{x}{x^2+1} - \frac{1}{x-1} \\
\frac{dy}{dx} &= y\left(\frac{x}{x^2+1} - \frac{1}{x-1}\right) \\
\frac{dy}{dx} &= \sqrt{\frac{x^2+1}{(x-1)^2}}\left(\frac{x}{x^2+1} - \frac{1}{x-1}\right)
\end{aligned}$$

You should not try to do part (d) without taking logarithms:

$$\begin{aligned}
y &= x^x \\
\ln y &= x\ln x \\
\frac{1}{y}\frac{dy}{dx} &= 1 + \ln x \\
\frac{dy}{dx} &= y(1+\ln x) = x^x(1+\ln x)
\end{aligned}$$

Part (e) is straightforward: $y' = \frac{2x}{(x^2+1)^2+1}$.
Part (f) is also straightforward: $y' = \sinh x e^{\cosh x}$.

4. *(10 points)* Find the equation of the tangent line to the graph defined by the equation $x^2(x^2+y^2) = y^2$ at the point $(\sqrt{2}/2, \sqrt{2}/2)$. Simplify your answer as much as possible.
Answer: We need to use implicit differentiation to get started. Take $\frac{d}{dx}$ of both sides:

$$\begin{aligned} x^2(x^2+y^2) &= y^2 \\ \frac{d}{dx}\left(x^2(x^2+y^2)\right) &= \frac{d}{dx}\left(y^2\right) \\ 2x(x^2+y^2) + x^2(2x + 2y\frac{dy}{dx}) &= 2y\frac{dy}{dx} \end{aligned}$$

At this point, it is simplest to substitute $x = y = \sqrt{2}/2$ immediately, yielding $\sqrt{2} + \frac{1}{2}\left(\sqrt{2} + \sqrt{2}\frac{dy}{dx}\right) = \sqrt{2}\frac{dy}{dx}$. Divide through by $\sqrt{2}$, and we have $1 + \frac{1}{2}\left(1 + \frac{dy}{dx}\right) = \frac{dy}{dx}$. Multiply by 2, and we have $3 + \frac{dy}{dx} = 2\frac{dy}{dx}$, or $\frac{dy}{dx} = 3$.

Therefore, the equation of the tangent line is $y - \sqrt{2}/2 = 3(x - \sqrt{2}/2)$, which becomes $y = 3x - \sqrt{2}$.

5. *(10 points)* (a) State the Mean Value Theorem.
(b) Suppose that $-\frac{\pi}{4} < a < b < \frac{\pi}{4}$. Show that

$$\tan b - \tan a < 2(b-a).$$

Answer: If $f(x)$ is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there is a c in the open interval (a, b) so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Now apply the result with $f(x) = \tan x$. We get the inequality

$$\frac{\tan b - \tan a}{b - a} = \sec^2 c,$$

where $a < c < b$. This means that $-\frac{\pi}{4} < c < \frac{\pi}{4}$, so $\sec^2 c < 2$, and therefore $\tan b - \tan a < 2(b - a)$.

6. *(5 points)* Find the maximum and minimum values of the function $f(x) = x^3 + 3x + 4$ if $-1 \le x \le 1$.
Answer: We first compute $f'(x) = 3x^2 + 3$. There are no solutions to the equation $f'(x) = 0$, and therefore the minimum and maximum values of the function must occur at the end points of the interval, -1 and 1. We find $f(-1) = 0$ is the minimum value, and $f(1) = 8$ is the maximum value.

7. *(8 points)* Compute the following limits. If a limit does not exist, but is ∞ or $-\infty$, you must say to in order to receive full credit.

$$\lim_{x\to 0} \cos\left(\frac{1}{x}\right) \qquad \lim_{x\to\infty} \cos\left(\frac{1}{x}\right) \qquad \lim_{x\to-\infty} \arctan\left(\frac{x}{3}\right) \qquad \lim_{x\to 3} \left[\frac{9x}{5}\right]$$

In the last limit, $\left[\frac{9x}{5}\right]$ refers to the greatest-integer function.
Answer: The first limit does not exist, while the second is 1. The third is $-\frac{\pi}{2}$, and the fourth is 5.

8. *(5 points)* Use a linear approximation to estimate $\sqrt[4]{15}$.

Answer: Let $f(x) = \sqrt[4]{x} = x^{1/4}$, and then $f'(x) = x^{-3/4}/4$. Therefore, $f(16) = 2$, and $f'(16) = 1/32$. We use the approximation $f(x) \approx f(a) + f'(a)(x-a)$, with $a = 16$ and $x = 15$, and we get $\sqrt[4]{15} = f(15) \approx 2 + \frac{1}{32}(-1) = 1\frac{31}{32} \approx 1.9688$. The correct answer to 4 decimal places is 1.9680.

9. *(15 points)* Let $f(x) = 3x + \frac{9}{(x-3)^2}$. Find all vertical, horizontal, and slant asymptotes. Use the first derivative to classify the intervals on which the function is increasing or decreasing, and locate all local extrema. Use the second derivative to classify the intervals on which the graph of the function is concave up and concave down, and locate all inflection points.

Answer: Because $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = -\infty$, there are no horizontal asymptotes.

There is a vertical asymptote at $x = 3$, and we compute $\lim_{x\to 3} f(x) = \infty$.

Finally, $y = 3x$ is a slant asymptote, because

$$\lim_{x\to\infty} f(x) - y = \lim_{x\to\infty} \frac{9}{(x-3)^2} = 0$$

and similarly for $x \to -\infty$.

We compute $f'(x) = 3 - \frac{18}{(x-3)^3}$, and solving for $f'(x) = 0$ yields the equation $3(x-3)^3 = 18$, or $x - 3 = \sqrt[3]{6}$. Therefore, there is a critical point at $3 + \sqrt[3]{6}$. We see that if $x < 3$, then $f'(x)$ is positive. If $3 < x < 3+\sqrt[3]{6}$, then $f'(x)$ is negative. If $3+\sqrt[3]{6} < x$, then $f'(x)$ is positive. We also compute $f''(x) = \frac{54}{(x-3)^4}$, which is always positive.

We can summarize all of this information:

Interval	Behavior
$x < 3$	Increasing, concave up
$3 < x < 3 + \sqrt[3]{6}$	Decreasing, concave up
$3 + \sqrt[3]{6} < x$	Increasing, concave up

10. *(5 points)* The sum of 2 non-negative numbers is 20. Find the numbers if the product of one number and the square root of the other is to be as large as possible.

Answer: Let x be one number, and let $20 - x$ be the other. We need to maximize $f(x) = x\sqrt{20-x}$. We compute the derivative: $f'(x) = \sqrt{20-x} - \frac{x}{2\sqrt{20-x}}$. Setting it equal to 0 gives the equation $2(20-x) = x$, or $3x = 40$, or $x = 13\frac{1}{3}$. The two numbers are therefore $6\frac{2}{3}$ and $13\frac{1}{3}$.

11. *(10 points)* Suppose that $g(x) = f^{-1}(x)$. Derive formulas for $g'(x)$ and $g''(x)$ in terms of g, f, f', and f''.

Answer: We start with $f(g(x)) = x$, and differentiate with respect to x. The chain rule yields $f'(g(x))g'(x) = 1$, and therefore $g'(x) = 1/f'(g(x))$.

The second derivative is a bit trickier. Take the last equation, and differentiate again:

$$g''(x) = \frac{-\frac{d}{dx}f'(g(x))}{f'(g(x))^2} = \frac{-f''(g(x))g'(x)}{f'(g(x))^2} = \frac{-f''(g(x))\frac{1}{f'(g(x))}}{f'(g(x))^2} = -\frac{f''(g(x))}{f'(g(x))^3}.$$

12. *(10 points)* Prove the identity

$$\arcsin\frac{x-1}{x+1} = 2\arctan(\sqrt{x}) - \frac{\pi}{2}.$$

Answer: Let $f(x) = \arcsin\frac{x-1}{x+1}$, and let $g(x) = 2\arctan(\sqrt{x}) - \frac{\pi}{2}$. Our strategy is to show that $f'(x) = g'(x)$, and also that $f(x)$ and $g(x)$ agree at a particular x-value.

This latter job is much simpler than the former. We can check that $f(0) = \arcsin -1 = -\frac{\pi}{2}$, while $g(0) = 2\arctan(0) - \frac{\pi}{2} = 0 - \frac{\pi}{2} = -\frac{\pi}{2}$.

Now for the harder part. We have

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\left(\frac{x-1}{x+1}\right)' = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\left(\frac{(x+1)-(x-1)}{(x+1)^2}\right) = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}\left(\frac{2}{(x+1)^2}\right)$$
$$= \frac{2}{(x+1)\sqrt{(x+1)^2-(x-1)^2}} = \frac{2}{(x+1)\sqrt{4x}} = \frac{1}{(x+1)\sqrt{x}}$$

On the other hand, we also have

$$g'(x) = \frac{2}{1+(\sqrt{x})^2} \cdot \frac{1}{2\sqrt{x}} = \frac{2}{1+x} \cdot \frac{1}{2\sqrt{x}} = \frac{1}{(1+x)\sqrt{x}}.$$