Please label your answers clearly, as I will not have time to perform extensive searches for answers. No credit will be given for answers without explanations. Cheating will result in a failing grade.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

You may not use a calculator on this examination.

1. (5 points) Find an equation for the tangent plane to the graph of

\[ z = x^3 y^2 + 11 \]

at the point (1, 2, 15).

2. (5 points) Find an equation for the plane that passes through the points (0, 0, 0), (4, 3, 2), and (5, 6, 8).

3. (5 points) If \( f(t) \) is a \( C^1 \) function, and \( z = f(ax + by) \) where \( a \) and \( b \) are constants, show that

\[ b \frac{\partial z}{\partial x} = a \frac{\partial z}{\partial y}. \]

4. (10 points) Let \( \mathbf{F}(x, y, z) = (z^3 + 2xy) \mathbf{i} + x^2 \mathbf{j} + 3xz^2 \mathbf{k} = (z^3 + 2xy, x^2, 3xz^2) \).

(a) Show that \( \mathbf{F} = \nabla f \) for some scalar-valued function \( f(x, y, z) \).

(b) Let \( C \) be any path from (0, 0, 0) to (3, 4, 2). Evaluate

\[ \int_C (z^3 + 2xy) \, dx + x^2 \, dy + 3xz^2 \, dz. \]

5. (15 points) Let \( f(x, y) = y^2 + x^2y + x^2 - 2y \). Find all critical points for this function, and use the second derivative test to decide if each critical point is a relative minimum, a relative maximum, or a saddle point.

6. (5 points) Evaluate

\[ \int_0^\infty \int_0^y xe^{-y^3} \, dx \, dy. \]

7. (15 points) The plane \( x + y + z = 12 \) intersects the paraboloid \( z = x^2 + y^2 \) in an ellipse. Using Lagrange multipliers, find the point on this ellipse that is closest to the origin.
8. (10 points) Evaluate
\[ \int_{-1}^{1} \int_{0}^{\sqrt{1-x^2}} x^2(y^2 + z^2) \, dy \, dx \]
by switching to polar coordinates.

9. (5 points) Suppose that \( g(t) \) is a \( C^1 \) function, and
\[ f(x, y) = \int_{0}^{xy} g(t) \, dt. \]
What is \( \frac{\partial f}{\partial x} \)? Your answer might have an integral in it, and obviously will be in terms of the function \( g \).

10. (15 points) Let \( S \) be the unit sphere \( x^2 + y^2 + z^2 = 1 \).
(a) If \( S \) is parametrized by
\begin{align*}
x &= \sin \varphi \cos \theta \\
y &= \sin \varphi \sin \theta \\
z &= \cos \varphi
\end{align*}
show that the outward normal vector \( \mathbf{N} = \sin \varphi (x, y, z) \).
(b) Let \( \mathbf{F}(x, y, z) = xi + yj + zk = (x, y, z) \). Compute
\[ \int \int_{S} \mathbf{F} \cdot dS \]
directly.
(c) Check your answer to the integration in part (b) by applying Gauss’s theorem.

11. (10 points) Let \( T \) be the triangle with vertices \((0, 0, 0)\), \((2, 0, 6)\) and \((0, 3, 6)\). This triangle lies on the plane \( 3x + 2y - z = 0 \). Verify Stokes’s Theorem for this triangle and the function \( \mathbf{F}(x, y, z) = zi + yk = (z, 0, y) \).