These problems are designed to show two methods (completely different from the one used in class) to show that
\[ \int_{-\infty}^{\infty} e^{-x^2} \, dx = \sqrt{\pi}. \]

However, we will actually show that
\[ \int_{0}^{\infty} e^{-x^2} \, dx = \frac{\sqrt{\pi}}{2}. \]

Why are these two equations equivalent?

The following facts will be useful in doing some of these problems.

- If \( F(x) = \int_{a}^{b} G(x, y) \, dy \), then \( F'(x) = \int_{a}^{b} G_x(x, y) \, dy \). This is called differentiating under the integral sign.
- If \( n \) is any positive integer, then
  \[ \lim_{x \to \infty} \frac{x^n}{e^x} = 0. \]
- John Wallis’s product for \( \pi \):
  \[ \frac{\pi}{2} = \lim_{n \to \infty} \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{2n}{2n-1} \cdot \frac{2n}{2n+1}. \]

In order to write this more succinctly, we use the notation \( n!! \) to represent the product of every other positive integer less than or equal to \( n \). In other words, \( 6!! = 6 \cdot 4 \cdot 2 \), and \( 7!! = 7 \cdot 5 \cdot 3 \cdot 1 \). (For simplicity, set \( 0!! = 1 \) and \( 1!! = 1 \).) Using this notation, we can write the above formula as
\[ \frac{\pi}{2} = \lim_{n \to \infty} \frac{(2n)!!}{(2n-1)!!(2n+1)!!} = \lim_{n \to \infty} \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{1}{2n+1}. \]

- If \( At^2 + Bt + C \) is a polynomial that is positive for every value of \( t \), then \( B^2 - 4AC < 0 \).

**Method 1**

Let
\[ f(x) = \left( \int_{0}^{x} e^{-t^2} \, dt \right)^2 \]
and
\[ g(x) = \int_{0}^{1} e^{-x^2(1+u^2)} \, du. \]

Notice that \( \lim_{x \to \infty} f(x) \) is related to the integral that we are trying to compute.

Let \( h(x) = f(x) + g(x) \).

1. Compute \( f(0) \).
2. Show that \( g(0) = \pi/4 \).
3. Compute $h(0)$.

4. Compute $f'(x)$.

5. Compute $g'(x)$ by differentiating under the integral sign.

6. Make some substitutions to show that $f'(x) + g'(x) = 0$.

7. Show that $h(x)$ is a constant.

8. What is $\lim_{x \to \infty} h(x)$?

9. Show that $\lim_{x \to \infty} g(x) = 0$.

10. Show that $\lim_{x \to \infty} f(x) = \pi/4$.

11. Show that

$$\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}.$$ 

**Method 2**

If $n$ is a positive integer, define

$$I_n = \int_{0}^{\infty} x^n e^{-x^2} dx.$$ 

Notice that $I_0$ is related to the integral that we are trying to compute.

12. Show that $I_1 = \frac{1}{2}$.

13. Use integration by parts to show that $I_n = \frac{n-1}{2} I_{n-2}$.

14. Using the previous problem, show that

$$I_{2n+1} = \frac{(2n)!!}{2^n} \cdot \frac{1}{2}.$$ 

15. Similarly, show that

$$I_{2n} = \frac{(2n-1)!!}{2^n} I_0.$$ 

16. Now, we need to relate $I_n$ and $I_{n+1}$. Let $t$ be a variable, and define the function

$$F_n(t) = \int_{0}^{\infty} (x + t)^2 x^n e^{-x^2} dx.$$ 

Using only the definition of $F_n(t)$, show that $F_n(t) > 0$ for any value of $t$.

17. Show that $F_n(t) = t^2 I_n + 2t I_{n+1} + I_{n+2}$.
18. Using the fact that \( F_n(t) > 0 \), show that \( I_{n+1}^2 < I_n I_{n+2} \). This is almost the relationship that we need, but it must be re-written. Use an earlier problem to show that

\[
I_{n+1}^2 < \frac{n+1}{2} I_n^2.
\]

19. The previous problem shows that

\[
I_{2n}^2 < n I_{2n-1}^2.
\]

We can now plug in the values for \( I_{2n} \) and \( I_{2n-1} \) from above, and get the inequality

\[
\left( \frac{(2n-1)!!}{2^n} I_0 \right)^2 < n \left( \frac{(2n-2)!!}{2^{n-1}} \cdot \frac{1}{2} \right)^2.
\]

Simplify this to get the inequality

\[
I_0^2 < \left( \frac{(2n-2)!!}{(2n-1)!!} \right)^2 n = \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{1}{(2n+1)} \cdot \frac{n(2n+1)}{(2n)^2}.
\]

Now take a limit as \( n \to \infty \), and conclude that

\[
I_0^2 \leq \lim_{n \to \infty} \left[ \left( \frac{(2n)!!}{(2n-1)!!} \right)^2 \cdot \frac{1}{(2n+1)} \cdot \frac{n(2n+1)}{(2n)^2} \right] = \pi \cdot \frac{1}{2} = \frac{\pi}{4}.
\]

Therefore, \( I_0 \leq \sqrt{\pi}/2 \).

20. We can also write

\[
I_{2n+1}^2 < \frac{2n+1}{2} I_{2n}^2.
\]

Now, we substitute in the values that we have for \( I_{2n+1} \) and \( I_{2n} \), and we get the inequality.

\[
\left( \frac{(2n)!!}{2^n} \right)^2 1 \cdot \frac{1}{4} < \frac{2n+1}{2} \left( \frac{(2n-1)!!}{2^n} \right)^2 I_0^2.
\]

Rearrange this to get

\[
\frac{(2n)!!^2}{((2n-1)!!)^2(2n+1)} \cdot \frac{1}{2} < I_0^2.
\]

Now, take a limit as \( n \to \infty \), and we get

\[
\lim_{n \to \infty} \frac{(2n)!!^2}{((2n-1)!!)^2(2n+1)} \cdot \frac{1}{2} \leq I_0^2.
\]

Therefore, \( I_0 = \sqrt{\pi}/2 \).