31. Find the absolute maximum and minimum of

\[ f(x, y, z) = x^2 + xz - y^2 + 2z^2 + xy + 5x \]

on the block \( \{(x, y, z) \mid -5 \leq x \leq 0, 0 \leq y \leq 3, 0 \leq z \leq 2\} \).

**Answer:** We start by finding all unrestricted critical points, and after that, we’ll work our way around the boundary. We have

\[
\begin{align*}
\frac{\partial f}{\partial x} &= 2x + z + y + 5 \\
\frac{\partial f}{\partial y} &= -2y + x \\
\frac{\partial f}{\partial z} &= x + 4z
\end{align*}
\]

so we must solve

\[
\begin{align*}
2x + y + z &= -5 \\
x - 2y &= 0 \\
x + 4z &= 0
\end{align*}
\]

The third equation says that \( z = -\frac{1}{4}x \), and the second says that \( y = \frac{1}{2}x \), so the first equation becomes \( 2x - \frac{1}{2}x + \frac{1}{4}x = -5 \), or \( \frac{9}{4}x = -5 \), and so \( x = -\frac{20}{9} \), meaning that \( y = -\frac{10}{9} \), and \( z = \frac{5}{9} \). Because the \( y \)-value is not in the given box, we know that the absolute maximum and minimum must lie on the boundary of the box.

We must work our way around the six faces of the box. Start with \( x = -5 \), and then we must find the maximum and minimum of \( f(-5, y, z) = g(y, z) = 25 - 5z - y^2 + 2z^2 - 5y - 25 = -y^2 - 5y + 2z^2 - 5z \) with the constraints \( 0 \leq y \leq 3 \) and \( 0 \leq z \leq 2 \). We compute

\[
\begin{align*}
\frac{\partial g}{\partial y} &= -2y - 5 \\
\frac{\partial g}{\partial z} &= 4z - 5
\end{align*}
\]

and then we see that the critical point has \( y = -\frac{5}{2} \) and \( z = \frac{5}{4} \), which again is not in the region under consideration.

So again, the maximum and minimum on the face \( x = -5 \) must occur on the boundary. Start with \( y = 0 \), and we must find the maximum and minimum of \( f(-5, 0, z) = h(z) = 2z^2 - 5z \) with \( 0 \leq z \leq 2 \). We have \( h'(z) = 4z - 5 \), giving a critical point at \( z = \frac{5}{4} \), and we also must check the endpoints \( z = 0 \) and \( z = 2 \). We compute \( h(5/4) = -\frac{25}{8} \), and \( h(0) = 0 \), and \( h(2) = -2 \). We finally have a tentative maximum of 0, at \((-5, 0, 0)\), and a tentative minimum of \(-\frac{25}{4}\) at \((-5, 0, 5/4)\).

Next, we take \( y = 3 \), and we must find the maximum and minimum of \( f(-5, 3, z) = h(z) = 2z^2 - 5z - 24 \). Again, \( h'(z) = 4z - 5 \), giving a critical point of \( z = \frac{5}{4} \), and again we must check the endpoints \( z = 0 \) and \( z = 2 \). We have \( h(5/4) = -\frac{121}{8} \), \( h(0) = -24 \), and \( h(2) = -26 \). So our tentative maximum is still 0, and now our tentative minimum is \( f(-5, 3, 5/4) = -\frac{121}{8} = -27.125 \).

Next, we take \( z = 0 \), and we must find the maximum and minimum of \( f(-5, y, 0) = h(y) = -5y - y^2 \) with \( 0 \leq y \leq 3 \). The critical point occurs when \( 2y = -5 \), which is not in the region. We also have \( h(0) = 0 \), and \( h(3) = -24 \).

We finish off the face \( x = -5 \) by taking \( z = 2 \), and we must find the maximum and minimum of \( f(-5, y, 2) = h(y) = -2 - 5y - y^2 \) with \( 0 \leq y \leq 3 \). Again, the equation \( h'(y) = 0 \) leads to a solution outside of the region, so we check the endpoints, leading to \( f(-5, 0, 2) = -2 \) and \( f(-5, 3, 2) = -26 \).
Next, we move onto the face \( x = 0 \), and we must find the maximum and minimum value of \( f(0, y, z) = g(y, z) = -y^2 + 2z^2 \) with \( 0 \leq y \leq 3 \) and \( 0 \leq z \leq 2 \). We have

\[
\frac{\partial g}{\partial y} = -2y \\
\frac{\partial g}{\partial z} = 4z
\]

meaning that there is a critical point at \( y = 0 \) and \( z = 0 \). We compute \( f(0,0,0) = 0 \), which is still our tentative maximum and minimum values.

We also have to check the boundaries of the face. Start with \( y = 0 \), and we must find the maximum and minimum values of \( f(0,0,z) = h(z) = 2z^2 \) with \( 0 \leq z \leq 2 \). We solve \( h'(z) = 0 \) and find that the critical point is \( z = 0 \). Also checking the other endpoint, we find \( h(0) = 0 \) and \( h(2) = 8 \). Now, our tentative maximum value is \( f(0,0,2) = 8 \).

Next, we try \( y = 3 \), and we must find the maximum and minimum values of \( f(0,3,z) = h'(z) = 2z^2 - 9 \) with \( 0 \leq z \leq 2 \). Again, solving \( h'(z) = 0 \) gives \( z = 0 \), and we also check \( z = 2 \). We have \( h(0) = -9 \) and \( h(2) = -1 \).

Next, we take \( z = 0 \), and we must find the maximum and minimum values of \( f(0,y,0) = h(y) = -y^2 \) with \( 0 \leq y \leq 3 \). We have the maximum value when \( y = 0 \), and the minimum when \( y = 3 \), and neither exceeds the previous tentative values.

Finally, we take \( z = 2 \), and we must find the maximum and minimum values of \( f(0, y, 2) = h(y) = -y^2 + 8 \) with \( 0 \leq y \leq 3 \). The maximum again occurs when \( y = 0 \), and the minimum when \( y = 3 \), with values 8 and -1 respectively.

Moving along, we take the face defined by \( y = 0 \), and we must find the maximum and minimum values of \( f(x,0,z) = g(x,z) = x^2 + 5x + 2z^2 + xz \). We start by finding critical points:

\[
\frac{\partial g}{\partial x} = 2x + 5 + z \\
\frac{\partial g}{\partial z} = 4z + x
\]

Solving \( 2x + z = -5 \) and \( x + 4z = 0 \) yields \( x = -\frac{20}{7} \) and \( z = \frac{5}{7} \). We compute \( f(-20/7,0,5/7) = -\frac{50}{7} \), which doesn’t alter our tentative minimum and maximum values.

We also must check the boundaries. We start with \( x = 0 \), and then we must find the maximum and minimum values of \( f(0,0,z) = h(z) = 2z^2 \) with \( 0 \leq z \leq 2 \). Fortunately, we have already done this.

Similarly, we take \( x = -5 \), and we must find the maximum and minimum values of \( f(-5,0,z) = 2z^2 - 5z \) with \( 0 \leq z \leq 2 \), which also was already done.

Next, we take \( z = 0 \), and we must find the maximum and minimum values of \( f(x,0,0) = h(x) = x^2 + 5x \) with \( -5 \leq x \leq 0 \). We have \( h'(x) = 2x + 5 \), so there is a critical point at \( -\frac{5}{2} \). We also must check the endpoints \( x = 0 \) and \( x = -5 \). We have \( f(-5/2,0,0) = -\frac{25}{4} \), and \( f(0,0,0) = 0 \) and \( f(-5,0,0) = 0 \).

Finally, we take \( z = 2 \), and we must find the maximum and minimum values of \( f(x,0,2) = h(x) = x^2 + 7x + 8 \). We have \( f'(x) = 2x + 7 \), so there is a critical point at \( -\frac{7}{2} \). We also must check the endpoints \( x = 0 \) and \( x = -5 \). We have \( f(-7/2,0,2) = -\frac{17}{4} \), \( f(0,0,2) = 8 \), and \( f(-5,0,2) = -2 \).

Next, we take the face defined by \( y = 3 \), and we must find the maximum and minimum values of \( f(x,3,z) = h(x,z) = x^2 + 8x + xz + 2z^2 - 9 \) with \( -5 \leq x \leq 0 \) and \( 0 \leq z \leq 2 \). We have

\[
\frac{\partial h}{\partial x} = 2x + 8 + z \\
\frac{\partial h}{\partial z} = x + 4z
\]

and solving \( 2x + z = -8 \) and \( x + 4z = 0 \) yields the solution \( x = -\frac{32}{7} \) and \( z = \frac{8}{7} \). We compute \( f(-32/7,3,8/7) = -\frac{191}{7} = -27\frac{2}{7} \). This is now the minimum value for the function.

We must check the edges. Start with \( x = 0 \), and then we are looking for the maximum and minimum values of \( f(0,3,z) = h(z) = 2z^2 - 9 \) with \( 0 \leq z \leq 2 \). We have already found these. Next, we take \( x = -5 \),
and we are looking for the maximum and minimum values of $f(-5, 3, z) = h(z) = 2z^2 - 5z - 24$, with $0 \leq z \leq 2$. We have already done this also.

Next, take $z = 0$, and we are looking for the maximum and minimum values of $f(x, 3, 0) = h(x) = x^2 + 8x + 9$ with $-5 \leq x \leq 0$. Solving $h'(x) = 0$ yields $x = -4$, and we also must check the endpoints $x = -5$ and $x = 0$. We have $f(-4, 3, 0) = -25$, $f(-5, 3, 0) = -24$, and $f(0, 3, 0) = -9$.

Finally, we take $z = 2$, and we are looking for the maximum and minimum values of $f(x, 3, 2) = h(x) = x^2 + 10x - 1$ with $-5 \leq x \leq 0$. This has a critical point at $x = -5$, and we also must check the other endpoint $x = 0$. We have $f(-5, 3, 2) = -26$ and $f(0, 3, 2) = -1$.

We next take the face defined by $z = 0$, and we look for the maximum and minimum values of $f(x, y, 0) = g(x, y) = x^2 + xy - y^2 + 5x$. We have

\[
\frac{\partial g}{\partial x} = 2x + y + 5 \\
\frac{\partial g}{\partial y} = x - 2y
\]

and solving $2x + y = -5$ and $x - 2y = 0$ yields $x = -2$ and $y = -1$, which is not in the region under consideration.

We also must check the edges. We start with $x = -5$, and we are looking for the maximum and minimum values of $f(-5, y, 0) = h(y) = -y^2 - 5y$ with $0 \leq y \leq 3$; we have already done this. Next, we take $x = 0$, and we are looking for the maximum and minimum values of $f(0, y, 0) = h(y) = -y^2$ with $0 \leq y \leq 3$; we also have done this already.

Next, we take $y = 0$, and then we are looking for the maximum and minimum values of $f(x, 0, 0) = h(x) = x^2 + 5x$ with $-5 \leq x \leq 0$; we have already done this. Finally, we take $y = 3$, and then we are looking for the maximum and minimum values of $f(x, 3, 0) = h(x) = x^2 + 8x - 9$ with $-5 \leq x \leq 0$; we have already done this also.

The sixth and last face is defined by $z = 2$, and we must find the maximum and minimum values of $f(x, y, 2) = g(x, y) = x^2 + xy - y^2 + 7x + 8$ with $-5 \leq x \leq 0$ and $0 \leq y \leq 3$. We have

\[
\frac{\partial g}{\partial x} = 2x + y + 7 \\
\frac{\partial g}{\partial y} = x - 2y
\]

yielding the equations $2x + y = -7$ and $x - 2y = 0$. The solution is $x = -\frac{14}{5}$ and $y = -\frac{7}{5}$, which is not in the region under consideration.

We also have to check the edges. Start with $x = -5$, and then we need the maximum and minimum values of $f(-5, y, 2) = h(y) = -y^2 - 5y - 2$ with $0 \leq y \leq 3$; we have already done this computation. Next, take $x = 0$, and we need the maximum and minimum values of $f(0, y, 2) = h(y) = -y^2 + 8$ with $0 \leq y \leq 3$; we have already done this computation. Next, take $y = 0$, and we must find the maximum and minimum values of $f(x, 0, 2) = h(x) = x^2 + 7x + 8$ with $-5 \leq x \leq 0$; we have already done this. Last, take $y = 3$, and we must find the maximum and minimum values of $f(x, 3, 2) = x^2 + 10x - 1$ with $-5 \leq x \leq 0$; we have already done this as well.

The maximum value of the function is $-27\frac{2}{5}$ at the point $x = -\frac{12}{7}$, $y = 3$, and $z = \frac{6}{7}$. The maximum value is 8, which occurs when $x = 0$, $y = 0$, and $z = 2$. 

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