Mathematics 210
Examination 1
Answers

1. *(15 points)* Finish this definition:

When we say that a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is *linearly independent*, we mean that...

Answer: ...the only solution to the equation $a_1\mathbf{x}_2 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n = \mathbf{0}$ is the trivial solution $a_1 = a_2 = \cdots = a_n = 0$.

2. *(20 points)* (a) Let

$$A = \begin{bmatrix} 2 & 1 & 3 & -2 \\ 2 & 1 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

What is the reduced row echelon matrix to which A is row equivalent?

(b) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be defined by $T(\mathbf{x}) = A\mathbf{x}$. Use your answer to part (a) to determine if T is surjective, injective, both, or neither.

Answer: Using elementary matrices to indicate the row operations (to save space), and starting by switching the first and third rows to save time, we have

$$\begin{bmatrix} 2 & 1 & 3 & -2 \\ 2 & 1 & 5 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 5 & 2 \\ 2 & 1 & 3 & -2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 3 & 0 \\ 0 & -1 & 1 & -4 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 1 & -4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & -2 & -4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

This last matrix is the (unique) reduced row echelon matrix row-equivalent to A.

Because there is a pivot in every row of A, we know that the linear transformation T is surjective. Because there is not a pivot in every column of A, we know that the linear transformation T is not injective.

3. *(10 points)* Let

$$A = \begin{bmatrix} 0 & 3 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 0 & 4 \end{bmatrix}.$$

Are the columns of A linearly independent vectors?

Answer: You do not need to do any work to answer this question. Because the third column of A is the 0-vector, we know that the columns of A are not linearly independent vectors.

4. *(20 points)* Suppose that the set $\{\mathbf{u}, \mathbf{v}\}$ contains 2 linearly independent vectors. Show that the set $\{\mathbf{u} + \mathbf{v}, \mathbf{u} - \mathbf{v}\}$ also contains 2 linearly independent vectors.

Answer: We write down the equation $a(\mathbf{u}+\mathbf{v}) + b(\mathbf{u}-\mathbf{v}) = \mathbf{0}$. We need to show that the only solution to this equation is the trivial solution $a = b = 0$. Simplifying, we get the equation $(a+b)\mathbf{u} + (a-b)\mathbf{v} = \mathbf{0}$. Because $\mathbf{u}$ and $\mathbf{v}$ are linearly independent, we know that the only solution to this equation is the trivial solution $a+b = a-b = 0$. The only solution to that pair of equations is $a = b = 0$, so we can conclude that the set $\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}\}$ contains 2 linearly independent vectors.

5. *(20 points)* Consider this homogeneous system of equations:

$$\begin{array}{rcrcrcrcl} 3x_1 & - & 2x_2 & - & x_3 & - & 4x_4 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & - & 3x_4 & = & 0 \end{array}$$

Find vectors $\mathbf{u}, \mathbf{v} \in \mathbf{R}^4$ so that the solution can be written in the form $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = a\mathbf{u} + b\mathbf{v}$ where a and b are arbitrary real numbers.

Answer: Again, we proceed by row reduction to reduced echelon form, and again it is much simpler to reverse the order of the rows before doing anything more:

$$\begin{bmatrix} 3 & -2 & -1 & -4 \\ 1 & 1 & -2 & -3 \end{bmatrix} \overset{\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]}{\longrightarrow} \begin{bmatrix} 1 & 1 & -2 & -3 \\ 3 & -2 & -1 & -4 \end{bmatrix} \overset{\left[\begin{smallmatrix} 1 & 0 \\ -3 & 1 \end{smallmatrix}\right]}{\longrightarrow} \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & -5 & 5 & 5 \end{bmatrix}$$
$$\overset{\left[\begin{smallmatrix} 1 & 0 \\ 0 & -\frac{1}{5} \end{smallmatrix}\right]}{\longrightarrow} \begin{bmatrix} 1 & 1 & -2 & -3 \\ 0 & 1 & -1 & -1 \end{bmatrix} \overset{\left[\begin{smallmatrix} 1 & -1 \\ 0 & 1 \end{smallmatrix}\right]}{\longrightarrow} \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -1 \end{bmatrix}$$

We can conclude that $x_1 - x_3 - 2x_4 = 0$, or $x_1 = x_3 + 2x_4$, and that $x_2 - x_3 - x_4 = 0$, or that $x_2 = x_3 + x_4$. Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 + 2x_4 \\ x_3 + x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

Among many possible answers, one is $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

6. *(15 points)* Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbf{R}^n$. Suppose that $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$. Show that the span of the set $\{\mathbf{u}, \mathbf{v}\}$ is the same as the span of the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Answer: Any vector $\mathbf{x}$ in the span of $\{\mathbf{u}, \mathbf{v}\}$ can be definition be written in the form $\mathbf{x} = a\mathbf{u} + b\mathbf{v}$. Because this is the same as $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + 0\mathbf{w}$, we can conclude that $\mathbf{x}$ is in the span of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

Suppose on the other hand that $\mathbf{x}$ is in the span of $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. This means that we can write $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c\mathbf{w}$. Substitute $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$, and we get $\mathbf{x} = a\mathbf{u} + b\mathbf{v} + c(2\mathbf{u} - 3\mathbf{v}) = (a_2 c)\mathbf{u} + (b - 3c)\mathbf{v}$. This shows that $\mathbf{x}$ is in the span of $\{\mathbf{u}, \mathbf{v}\}$.

Grade	Number of people
100	1
85	2
84	1
79	1
75	6
65	7
60	1
59	2
55	7
50	1
45	1
40	1
35	1

Mean: 64.72
Standard deviation: 13.84