1. (15 points) Finish this definition:
   When we say that a set of vectors \( \{x_1, x_2, \ldots, x_n\} \) is linearly independent, we mean that...
   
   **Answer:** ...the only solution to the equation \( a_1x_1 + a_2x_2 + \cdots + a_nx_n = 0 \) is the trivial solution \( a_1 = a_2 = \cdots = a_n = 0 \).

2. (20 points) (a) Let
   \[
   A = \begin{bmatrix}
   2 & 1 & 3 & -2 \\
   2 & 1 & 5 & 2 \\
   1 & 1 & 1 & 1
   \end{bmatrix}
   \]
   What is the reduced row echelon matrix to which \( A \) is row equivalent?
   
   **Answer:** Using elementary matrices to indicate the row operations (to save space), and starting by switching the first and third rows to save time, we have
   
   \[
   \begin{bmatrix}
   2 & 1 & 3 & -2 \\
   2 & 1 & 5 & 2 \\
   1 & 1 & 1 & 1
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   0 & 0 & 1 \\
   1 & 0 & 0 \\
   0 & 1 & 0
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 1 & 1 & 1 \\
   1 & 1 & 1 \\
   -2 & 1 & 0
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 1 & 1 & 1 \\
   0 & -1 & 3 & 0 \\
   0 & -1 & 1 & -4
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 1 & 1 \\
   1 & 1 \\
   1 & -3 & 0
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 1 & 1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 1 & 1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 1 & 0 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -1 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & -7 \\
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   0 & 1 & 0 \\
   0 & 0 & 1 & 2
   \end{bmatrix}
   .
   
   This last matrix is the (unique) reduced row echelon matrix row-equivalent to \( A \).
   
   Because there is a pivot in every row of \( A \), we know that the linear transformation \( T \) is surjective. Because there is not a pivot in every column of \( A \), we know that the linear transformation \( T \) is not injective.

3. (10 points) Let
   \[
   A = \begin{bmatrix}
   0 & 3 & 0 & 2 \\
   0 & 1 & 0 & 1 \\
   1 & 2 & 0 & 4
   \end{bmatrix}
   \]
   Are the columns of \( A \) linearly independent vectors?
   
   **Answer:** You do not need to do any work to answer this question. Because the third column of \( A \) is the 0-vector, we know that the columns of \( A \) are not linearly independent vectors.

4. (20 points) Suppose that the set \( \{u, v\} \) contains 2 linearly independent vectors. Show that the set \( \{u + v, u - v\} \) also contains 2 linearly independent vectors.
Answer: We write down the equation $a(u + v) + b(u - v) = 0$. We need to show that the only solution to this equation is the trivial solution $a = b = 0$. Simplifying, we get the equation $(a + b)u + (a - b)v = 0$. Because $u$ and $v$ are linearly independent, we know that the only solution to this equation is the trivial solution $a + b = a - b = 0$. The only solution to that pair of equations is $a = b = 0$, so we can conclude that the set $\{u + v, u - v\}$ contains 2 linearly independent vectors.

5. (20 points) Consider this homogeneous system of equations:

\[
\begin{align*}
3x_1 - 2x_2 - x_3 - 4x_4 &= 0 \\
x_1 + x_2 - 2x_3 - 3x_4 &= 0
\end{align*}
\]

Find vectors $u, v \in \mathbb{R}^4$ so that the solution can be written in the form

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = au + bv
\]

where $a$ and $b$ are arbitrary real numbers.

Answer: Again, we proceed by row reduction to reduced echelon form, and again it is much simpler to reverse the order of the rows before doing anything more:

\[
\begin{bmatrix}
3 & -2 & -1 & -4 \\
1 & 1 & -2 & -3
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -2 & -3 \\
3 & -2 & -1 & -4
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 1 \\
-3 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 1 & -2 & -3 \\
0 & 5 & 5 & 0
\end{bmatrix}
\]

We can conclude that $x_1 - x_3 - 2x_4 = 0$, or $x_1 = x_3 + 2x_4$, and that $x_2 - x_3 - x_4 = 0$, or that $x_2 = x_3 + x_4$. Therefore,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
x_3 + 2x_4 \\
x_3 + x_4 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
1 \\
x_3 \\
1 \\
x_4
\end{bmatrix} +
\begin{bmatrix}
x_3 \\
1 \\
0 \\
2
\end{bmatrix}
\]

Among many possible answers, one is $u = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ and $v = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$.

6. (15 points) Let $u$ and $v$ be vectors in $\mathbb{R}^n$. Suppose that $w = 2u - 3v$. Show that the span of the set $\{u, v\}$ is the same as the span of the set $\{u, v, w\}$.

Answer: Any vector $x$ in the span of $\{u, v\}$ can be definition be written in the form $x = au + bv$. Because this is the same as $x = au + bv + 0w$, we can conclude that $x$ is in the span of $\{u, v, w\}$.

Suppose on the other hand that $x$ is in the span of $\{u, v, w\}$. This means that we can write $x = au + bv + cw$. Substitute $w = 2u - 3v$, and we get $x = au + bv + c(2u - 3v) = a_2c)u + (b - 3c)v$. This shows that $x$ is in the span of $\{u, v\}$.
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Mean: 64.72
Standard deviation: 13.84