Mathematics 210
Examination 2
Answers

1. *(5 points)* Finish this definition:

When we say that a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is *linearly independent*, we mean that...

Answer: ...the only solution to the vector equation $a_1\mathbf{x}_1 + \cdots + a_n\mathbf{x}_n = \mathbf{0}$ is $a_1 = \cdots = a_a = 0$.

2. *(15 points)* Compute the matrix product

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 9 & 3 & 2 \end{bmatrix}.$$

Answer: We have $\begin{bmatrix} 2 \\ 3 \end{bmatrix} \begin{bmatrix} 9 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 18 & 6 & 4 \\ 27 & 9 & 6 \end{bmatrix}$.

3. *(10 points)* Let A be an m-by-n matrix. What is the definition of $\operatorname{Nul} A$?

Answer: $\operatorname{Nul} A = \{\mathbf{v} \in \mathbf{R}^n : A\mathbf{v} = \mathbf{0}\}$.

4. *(10 points)* Suppose that A is a square n-by-n matrix, and $\mathbf{x} \in \mathbf{R}^n$ is a non-zero vector so that $A\mathbf{x} = \mathbf{0}$. Find a non-zero square n-by-n matrix B so that $AB = 0$.

Answer: There are many possible answers. One is to let $B = [\mathbf{x}\mathbf{x}\cdots\mathbf{x}]$, a matrix with n columns in which every column consists of the vector $\mathbf{x}$.

5. *(10 points)* Suppose that A is a square $n \times n$ matrix, and $A^2 = 0$. Prove that A is not invertible.

Answer: We have $\det(A^2) = (\det A)^2 = \det 0 = 0$, so $\det A = 0$. Therefore, A is not invertible.

6. *(15 points)* Compute the determinant

$$\begin{vmatrix} 2 & 3 & 4 & 5 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 \\ 1 & 9 & 8 & 0 & 0 \\ 4 & 2 & 3 & 2 & 1 \end{vmatrix}$$

Answer: We have $\begin{vmatrix} 2 & 3 & 4 & 5 & 1 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 3 & 4 & 0 & 0 \\ 1 & 9 & 8 & 0 & 0 \\ 4 & 2 & 3 & 2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 & 5 & 1 \\ 0 & 4 & 0 & 0 \\ 1 & 8 & 0 & 0 \\ 4 & 3 & 2 & 1 \end{vmatrix} = 2\cdot 4\begin{vmatrix} 2 & 5 & 1 \\ 1 & 0 & 0 \\ 4 & 2 & 1 \end{vmatrix} = 2\cdot 4\cdot(-1)\begin{vmatrix} 5 & 1 \\ 2 & 1 \end{vmatrix} = 2\cdot 4\cdot(-1)\cdot 3 = -24$.

7. *(15 points)* Suppose that $T : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear transformation. Suppose also that $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$. Find the matrix of T with respect to the standard basis.

Answer: If $T\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$, then $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. Also, $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Therefore, the matrix of T with respect to the standard basis is $\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.

8. *(20 points)* Let $H = \text{Span}\left\{\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right\}$. Find a matrix A so that $\text{Nul}\, A = H$.

Answer: If $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is an element of H, then $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = k \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, resulting in the equations $x_1 = 3k$, $x_2 = 0$, $x_3 = k$, and $x_4 = 0$. This can be more directly expressed as $x_1 - 3x_3 = 0$, $x_2 = 0$, and $x_4 = 0$. The matrix A corresponding to the left-hand side of those 3 homogeneous equations is $A = \begin{bmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. This matrix is not unique, because you can add as many rows of zeros as you like, as well as performing any row operations on A.

Grade	Number of people
100	1
99	2
96	2
95	1
91	1
90	3
89	2
85	2
83	1
79	1
78	1
75	1
72	1
71	1
70	2
69	1
67	1
64	1
63	1
60	1
50	1
37	1
30	1
25	1

Mean: 76.03
Standard deviation: 19.59