Mathematics 210
Examination 3
Answers

1. *(5 points)* Finish this definition:

When we say that a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is a *basis* of a vector space V, we mean that…

Answer: …the set of vectors is linearly independent and spans V.

2. *(10 points)* Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ is a basis for a vector space V. Let $H = \text{Span}\{\mathbf{x}_1 + \mathbf{x}_2, \mathbf{x}_2 + \mathbf{x}_3, \mathbf{x}_1 - \mathbf{x}_3\}$. What is the dimension of H?

Answer: Because H is spanned by 3 vectors, its dimension is at most 3. However, those three vectors are linearly dependent, because $1(\mathbf{x}_1 + \mathbf{x}_2) + (-1)(\mathbf{x}_2 + \mathbf{x}_3) + (-1)(\mathbf{x}_1 - \mathbf{x}_3) = \mathbf{0}$. The two vectors $\mathbf{x}_1 + \mathbf{x}_2$ and $\mathbf{x}_1 + \mathbf{x}_3$ are linearly independent, and therefore the dimension of H is 2.

3. *(15 points)* Let

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

The matrix B is the reduced echelon form of A. (You are not expected to check this statement.)

(a) Find a basis for Nul A.
(b) Find a basis for Col A.
(c) Find a basis for Row A.

Answer: (a) We write out the 3 equations in B: $x_1 + x_3 + x_5 = 0$, $x_2 - 2x_3 + 3x_5 = 0$, and $x_4 - 5x_5 = 0$. The free variables are x_3 and x_5. Taking $x_3 = 1$ and $x_5 = 0$ results in $x_1 = -1$, $x_2 = 2$, and $x_4 = 0$. Taking $x_3 = 0$ and $x_5 = 1$ results in $x_1 = -1$, $x_2 = -3$, and $x_4 = 5$. A basis for Nul A is given by $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$.

(b) A basis for Col A is given by the columns of A that are in pivot positions: $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$.

(c) A basis for Row A is given by the three non-zero rows of B (which can be written either as row or column vectors, according to your taste): $\{(1,0,1,0,1), (0,1,-2,0,3), (0,0,0,1,-5)\}$.

4. *(10 points)* Suppose that A is a square matrix, and $A^5 = 0$. Show that the only eigenvalue for A is $\lambda = 0$.

Answer: Suppose that $\mathbf{x}$ is an eigenvector. Then $A\mathbf{x} = \lambda\mathbf{x}$, and $A^5\mathbf{x} = \mathbf{0}$, so $\lambda^5\mathbf{x} = \mathbf{0}$. Therefore, $\lambda^5 = 0$, and so $\lambda = 0$.

5. *(10 points)* Finish the following definition:

When we say that a matrix A is *similar* to a matrix B, we mean that…

Answer: …there is a matrix P so that $A = PBP^{-1}$.

6. *(10 points)* Show that I, the identity matrix, is similar only to itself.

Answer: Suppose that $I = PBP^{-1}$. Multiply on the left by P^{-1} and on the right by P, and we have $P^{-1}IP = B$. But $P^{-1}IP = I$, so $B = I$.

7. *(20 points)* Let $A = \begin{bmatrix} 7 & -9 \\ 3 & -5 \end{bmatrix}$. Find matrices P and D so that D is diagonal, P is invertible, and $A = PDP^{-1}$.

Answer: We start by finding the characteristic equation:

$$\det(A - \lambda I) = \begin{vmatrix} 7-\lambda & -9 \\ 3 & -5-\lambda \end{vmatrix} = (7-\lambda)(-5-\lambda) + 27 = \lambda^2 - 2\lambda - 8.$$

We have $\lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2)$, so the two eigenvalues are $\lambda = 4$ and $\lambda = -2$.

Corresponding to $\lambda = 4$ we have

$$A - 4I = \begin{bmatrix} 3 & -9 \\ 3 & -9 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} 3 & -9 \\ 0 & 0 \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

So an eigenvector is given by $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$. Corresponding to $\lambda = -2$, we have

$$A + 2I = \begin{bmatrix} 9 & -9 \\ 3 & -3 \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{9} & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 3 & -3 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

So an eigenvector is given by $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We therefore can set $P = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 \\ 0 & -2 \end{bmatrix}$.

8. *(20 points)* Remember that $\mathbf{P}_3 = \{a + bt + ct^2 + dt^3\}$, the set of all polynomials of degree at most 3. Define $T : \mathbf{P}_3 \to \mathbf{R}^2$ with the formula $T(p) = \begin{bmatrix} p(2) \\ p(3) \end{bmatrix}$. Find a basis for the kernel of T.

Answer: We need to satify $p(2) = 0$ and $p(3) = 0$. That leads to the 2 equations

$$\begin{aligned} a + 2b + 4c + \ \ 8d &= 0 \\ a + 3b + 9c + 27d &= 0 \end{aligned}$$

We row reduce to find a basis for the kernel:

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & 1 & 5 & 19 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & -6 & -30 \\ 0 & 1 & 5 & 19 \end{bmatrix}$$

There are 2 free variables, c and d. Setting $c = 1$ and $d = 0$ results in $a = 6$ and $b = -5$. Setting $c = 0$ and $d = 1$ results in $a = 30$ and $b = -19$. One possible basis is therefore $\{6 - 5t + t^2, 30 - 19t + t^3\}$.

Grade	Number of people
100	1
90	1
88	1
85	2
80	2
75	4
70	6
69	1
65	1
55	2
54	1
50	2
49	1
45	2
40	1
35	1

Mean: 66.90

Standard deviation: 16.02