Mathematics 210
Final Examination
Campion 010 — 9 AM
December 13, 2008

Do all of your work in the blue booklets. Please label your answers clearly, as I will not have time to perform extensive searches for answers. Please leave all rational numbers in fractional form. When you are finished with the exam, please put these pages inside of the blue booklet. No credit will be given for answers without explanations.

You may not use calculators.

Cheating will result in a failing grade.

The problems are not arranged in order of increasing difficulty, so you might want to read all of them before beginning.

1. *(5 points)* Define what it means for a (square) matrix to be *diagonalizable*.

2. *(5 points)* Show that the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is *not* diagonalizable.

3. *(10 points)* Solve the differential equations

$$\begin{aligned} y_1' &= -25y_1 + 24y_2 \\ y_2' &= -40y_1 + 37y_2 \end{aligned}$$

with initial conditions $y_1(0) = 17$ and $y_2(0) = 22$. The equation

$$\begin{bmatrix} -25 & 24 \\ -40 & 37 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}^{-1}$$

should be helpful.

4. *(10 points)* Suppose that a and b are non-zero real numbers. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

5. *(15 points)* Suppose that

$$A = \begin{bmatrix} 1 & 2 & 6 & 10 & 8 \\ 4 & 3 & 9 & 20 & 17 \\ 5 & 6 & 18 & 34 & 28 \\ 2 & 2 & 6 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is the reduced row echelon form of A. (You are not expected to check this statement.)

(*a*) Find a basis for $\operatorname{Col} A$.
(*b*) Find a basis for $\operatorname{Row} A$.
(*c*) Find a basis for $\operatorname{Nul} A$.

6. *(5 points)* Finish this definition:

When we say that a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is *linearly independent*, we mean that...

7. *(10 points)* Recall that the Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. We derived a formula in class for the Fibonacci numbers of the form $F_n = \frac{1}{\gamma}(\alpha^n - \beta^n)$. Give that derivation, complete with determining the values of α, β, and γ.

8. *(10 points)* Suppose that A is a square matrix, and $A = A^T$. (Such a matrix is called *symmetric*.) Suppose also that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for A with unequal eigenvalues λ and μ. In other words, $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \mu\mathbf{w}$, and $\lambda \neq \mu$.

Show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors. In other words, show that $\mathbf{w} \cdot \mathbf{v} = 0$. *Hint:* Use the fact that $\mathbf{w}^T A\mathbf{v} = \mathbf{w}^T A^T \mathbf{v}$, and then evaluate those products.

9. *(20 points)* Recall that $\mathbf{P}_4 = \{a + bt + ct^2 + dt^3 + et^4\}$ is the vector space of all polynomials of degree at most 4. Define a function $T : \mathbf{P}_4 \to \mathbf{R}^2$ with the equation $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$. You may assume that T is a linear transformation.

(*a*) Show that the function T is surjective. In other words, show that if $\begin{bmatrix} x \\ y \end{bmatrix}$ is any vector in $\mathbf{R}^2$, there is some polynomial $p \in \mathbf{P}_4$ so that $T(p) = \begin{bmatrix} x \\ y \end{bmatrix}$.

(*b*) Find a basis for the kernel of T.

10. *(10 points)* Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ is a basis for a vector space V. Let

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{x}_1 + 3\mathbf{x}_2 + 5\mathbf{x}_3 + 2\mathbf{x}_4 + \mathbf{x}_5 \\ \mathbf{v}_2 &= 11\mathbf{x}_1 + 13\mathbf{x}_2 + 19\mathbf{x}_3 + 14\mathbf{x}_4 + 7\mathbf{x}_5 \\ \mathbf{v}_3 &= 16\mathbf{x}_1 + 18\mathbf{x}_2 + 26\mathbf{x}_3 + 20\mathbf{x}_4 + 10\mathbf{x}_5 \end{aligned}$$

and let $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What is the dimension of H? Be sure to explain your answer fully.