Mathematics 210
Final Examination
Answers

1. *(5 points)* Define what it means for a (square) matrix to be *diagonalizable.*

Answer: A matrix A is *diagonalizable* if there is a diagonal matrix D and invertible matrix P so that $A = PDP^{-1}$.

2. *(5 points)* Show that the matrix $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is *not* diagonalizable.

Answer: Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. The characteristic polynomial for A is $(x-1)^2$, so the only eigenvalue is $\lambda = 1$. Therefore, if A were diagonalizable, it would be similar to the identity matrix. However, we showed previously that the identity matrix is similar only to itself.

3. *(10 points)* Solve the differential equations

$$\begin{aligned} y_1' &= -25y_1 + 24y_2 \\ y_2' &= -40y_1 + 37y_2 \end{aligned}$$

with initial conditions $y_1(0) = 17$ and $y_2(0) = 22$. The equation

$$\begin{bmatrix} -25 & 24 \\ -40 & 37 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 7 \end{bmatrix} \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}^{-1}$$

should be helpful.

Answer: We can see from the matrix equation that the matrix $\begin{bmatrix} -25 & 24 \\ -40 & 37 \end{bmatrix}$ has eigenvalue $\lambda = 5$ with eigenvector $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$, and eigenvalue $\lambda = 7$ with eigenvector $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Therefore, the general solution of the differential equation is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = C_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} e^{5t} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix} e^{7t}.$$

To find the values of C_1 and C_2, we substitute in $t = 0$ and get the equation

$$\begin{bmatrix} 17 \\ 22 \end{bmatrix} = C_1 \begin{bmatrix} 4 \\ 5 \end{bmatrix} + C_2 \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

To solve, we use matrix multiplication and invert:

$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 5 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 17 \\ 22 \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 17 \\ 22 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Therefore, the solution is

$$\begin{aligned} y_1 &= 8e^{5t} + 9e^{7t} \\ y_2 &= 10e^{5t} + 12e^{7t} \end{aligned}$$

4. *(10 points)* Suppose that a and b are non-zero real numbers. Find the eigenvalues and eigenvectors of the matrix $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$.

Answer: Let $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. The characteristic polynomial is $\det(A-\lambda I) = (a-\lambda)^2+b^2 = \lambda^2-2a\lambda+a^2+b^2$. The solution is $\lambda = \frac{2a\pm\sqrt{4a^2-4a^2-4b^2}}{2} = \frac{2a\pm\sqrt{-4b^2}}{2} = a \pm bi$. The two eigenvalues are therefore complex numbers.

To find the eigenvector corresponding to $\lambda = a+bi$, we row-reduce:

$$A-(a+bi)I = \begin{bmatrix} -bi & b \\ -b & -bi \end{bmatrix} \xrightarrow{\begin{bmatrix} \frac{1}{b} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}} \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

So if $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is an eigenvector, we have the equation $-ix_1+x_2 = 0$. Take $x_1 = 1$, and then we have $x_2 = i$. So an eigenvector is $\begin{bmatrix} 1 \\ i \end{bmatrix}$. We know that an eigenvector corresponding to $a-bi$ is $\begin{bmatrix} 1 \\ -i \end{bmatrix}$.

5. *(15 points)* Suppose that

$$A = \begin{bmatrix} 1 & 2 & 6 & 10 & 8 \\ 4 & 3 & 9 & 20 & 17 \\ 5 & 6 & 18 & 34 & 28 \\ 2 & 2 & 6 & 13 & 10 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The matrix B is the reduced row echelon form of A. (You are not expected to check this statement.)

(a) Find a basis for $\operatorname{Col} A$.
(b) Find a basis for $\operatorname{Row} A$.
(c) Find a basis for $\operatorname{Nul} A$.

Answer: (a) The pivot columns in B are columns 1, 2, and 4. Therefore, a basis for $\operatorname{Col} A$ is $\left\{ \begin{bmatrix} 1 \\ 4 \\ 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 6 \\ 2 \end{bmatrix}, \begin{bmatrix} 10 \\ 20 \\ 34 \\ 13 \end{bmatrix} \right\}$.

(b) You can give these vectors either in row or column form, but a basis for $\operatorname{Row} A$ is given by the non-zero rows of B: $\{(1,0,0,0,2), (0,1,3,0,3), (0,0,0,1,0)\}$.

(c) Suppose that $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \operatorname{Nul} A$. Then $x_1+2x_5 = 0$, $x_2+3x_3+3x_5 = 0$, and $x_4 = 0$. We therefore have 2 free variables, x_3 and x_5. Taking $x_3 = 1$ and $x_5 = 0$ yields $\begin{bmatrix} 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, while taking $x_3 = 0$ and $x_5 = 1$ yields $\begin{bmatrix} -2 \\ -3 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Those two vectors are a basis for $\operatorname{Nul} A$.

6. *(5 points)* Finish this definition:

When we say that a set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ is *linearly independent*, we mean that…

Answer: …the only solution to the vector equation $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}$ is $c_1 = c_2 = \cdots = c_n = 0$.

7. *(10 points)* Recall that the Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n > 1$. We derived a formula in class for the Fibonacci numbers of the form $F_n = \frac{1}{\gamma}(\alpha^n - \beta^n)$. Give that derivation, complete with determining the values of α, β, and γ.

Answer: We have $\begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} F_{n-1} \\ F_n \end{bmatrix}$. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$. The characteristic polynomial for A is $(0-\lambda)(1-\lambda) - 1 = \lambda^2 - \lambda - 1$, with roots $\lambda = \frac{1 \pm \sqrt{5}}{2}$. Let $\alpha = \frac{1+\sqrt{5}}{2}$, and let $\beta = \frac{1-\sqrt{5}}{2}$.

We compute an eigenvector for α: $A - \alpha I = \begin{bmatrix} -\alpha & 1 \\ 1 & 1-\alpha \end{bmatrix}$ which tells us that an eigenvector is $\begin{bmatrix} 1 \\ \alpha \end{bmatrix}$. Similarly, an eigenvector for β is $\begin{bmatrix} 1 \\ \beta \end{bmatrix}$. So

$$A = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \frac{1}{\beta - \alpha} \begin{bmatrix} \beta & -1 \\ -\alpha & 1 \end{bmatrix}.$$

Recall that $\beta - \alpha = -\sqrt{5}$.

We then have

$$\begin{aligned} \begin{bmatrix} F_n \\ F_{n+1} \end{bmatrix} = A^n \begin{bmatrix} 0 \\ 1 \end{bmatrix} &= \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \frac{-1}{\sqrt{5}} \begin{bmatrix} \beta & -1 \\ -\alpha & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{-1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} \alpha^n & 0 \\ 0 & \beta^n \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ &= \frac{-1}{\sqrt{5}} \begin{bmatrix} 1 & 1 \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} -\alpha^n \\ \beta^n \end{bmatrix} = \frac{-1}{\sqrt{5}} \begin{bmatrix} -\alpha^n + \beta^n \\ -\alpha^{n+1} + \beta^{n+1} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} \alpha^n - \beta^n \\ \alpha^{n+1} - \beta^{n+1} \end{bmatrix} \end{aligned}$$

Therefore, $F_n = \dfrac{\alpha^n - \beta^n}{\sqrt{5}}$.

8. *(10 points)* Suppose that A is a square matrix, and $A = A^T$. (Such a matrix is called *symmetric.*) Suppose also that $\mathbf{v}$ and $\mathbf{w}$ are eigenvectors for A with unequal eigenvalues λ and μ. In other words, $A\mathbf{v} = \lambda\mathbf{v}$ and $A\mathbf{w} = \mu\mathbf{w}$, and $\lambda \neq \mu$.

Show that $\mathbf{v}$ and $\mathbf{w}$ are orthogonal vectors. In other words, show that $\mathbf{w} \cdot \mathbf{v} = 0$. *Hint:* Use the fact that $\mathbf{w}^T A\mathbf{v} = \mathbf{w}^T A^T \mathbf{v}$, and then evaluate those products.

Answer: In fact, this problem is very easy, if you can use all of the given information. On the one hand, $\mathbf{w}^T A\mathbf{v} = \mathbf{w}^T(A\mathbf{v}) = \mathbf{w}^T \lambda\mathbf{v} = \lambda\mathbf{w} \cdot \mathbf{v}$. On the other hand, $\mathbf{w}^T A^T \mathbf{v} = (A\mathbf{w})^T \mathbf{v} = (\mu\mathbf{w}^T)\mathbf{v} = \mu\mathbf{w} \cdot \mathbf{v}$. So we get $\lambda\mathbf{w} \cdot \mathbf{v} = \mu\mathbf{w} \cdot \mathbf{v}$. Because $\mu \neq \lambda$, we must conclude that $\mathbf{w} \cdot \mathbf{v} = 0$.

9. *(20 points)* Recall that $\mathbf{P}_4 = \{a + bt + ct^2 + dt^3 + et^4\}$ is the vector space of all polynomials of degree at most 4. Define a function $T : \mathbf{P}_4 \to \mathbf{R}^2$ with the equation $T(p) = \begin{bmatrix} p(1) \\ p'(1) \end{bmatrix}$. You may assume that T is a linear transformation.

(a) Show that the function T is surjective. In other words, show that if $\begin{bmatrix} x \\ y \end{bmatrix}$ is any vector in $\mathbf{R}^2$, there is some polynomial $p \in \mathbf{P}_4$ so that $T(p) = \begin{bmatrix} x \\ y \end{bmatrix}$.

(b) Find a basis for the kernel of T.

Answer: (a) Let $p_1(t) = t - 1$, and then $T(p_1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Let $p_2(t) = 2t - t^2$, and then $T(p_2) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. If $\begin{bmatrix} x \\ y \end{bmatrix}$ is any vector in $\mathbf{R}_2$, then $T(yp_1 + xp_2) = \begin{bmatrix} x \\ y \end{bmatrix}$.

(b Suppose that $a + bt + ct^2 + dt^3 + et^4$ is in the kernel of T. This means that $a + b + c + d + e = 0$, and also that $b + 2c + 3d + 4e = 0$. We put these equations into a matrix, and row reduce:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \end{bmatrix}$$

There are 3 free variables, c, d, and e. Taking $c = 1$ and $d = e = 0$ yields $a = 1$ and $b = -2$, and the polynomial is $1 - 2t + t^2$. Taking $c = e = 0$ and $d = 1$ yields $a = 2$ and $b = -3$, and the resulting polynomial is $2-3t+t^3$. Taking $c=d=0$ and $e=1$ yields $a=3$ and $b=-4$, and the resulting polynomial is $3-4t+t^4$. These three polynomials are one possible basis for the kernel of T.

10. *(10 points)* Suppose that $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4, \mathbf{x}_5\}$ is a basis for a vector space V. Let

$$\begin{aligned}
\mathbf{v}_1 &= \mathbf{x}_1 + 3\mathbf{x}_2 + 5\mathbf{x}_3 + 2\mathbf{x}_4 + \mathbf{x}_5 \\
\mathbf{v}_2 &= 11\mathbf{x}_1 + 13\mathbf{x}_2 + 19\mathbf{x}_3 + 14\mathbf{x}_4 + 7\mathbf{x}_5 \\
\mathbf{v}_3 &= 16\mathbf{x}_1 + 18\mathbf{x}_2 + 26\mathbf{x}_3 + 20\mathbf{x}_4 + 10\mathbf{x}_5
\end{aligned}$$

and let $H = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. What is the dimension of H? Be sure to explain your answer fully.

Answer: We put the coordinates of those three vectors into a matrix, either as rows or columns, and row reduce to see how many pivots occur. We have

$$\begin{bmatrix} 1 & 3 & 5 & 2 & 1 \\ 11 & 13 & 19 & 14 & 7 \\ 16 & 18 & 26 & 20 & 10 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ -11 & 1 & 0 \\ -16 & 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 3 & 5 & 2 & 1 \\ 0 & -20 & -36 & -8 & -4 \\ 0 & -30 & -54 & -12 & -6 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{6} \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 3 & 5 & 2 & 1 \\ 0 & 5 & 9 & 2 & 1 \\ 0 & 5 & 9 & 2 & 1 \end{bmatrix} \xrightarrow{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 3 & 5 & 2 & 1 \\ 0 & 5 & 9 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are only 2 pivot positions, and in this case that means that $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. One basis for H is $\mathbf{x}_1 + 3\mathbf{x}_2 + 5\mathbf{x}_3 + 2\mathbf{x}_4 + \mathbf{x}_5$ and $5\mathbf{x}_2 + 9\mathbf{x}_3 + 2\mathbf{x}_4 + \mathbf{x}_5$, and therefore the dimension of H is 2.